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Two Solutions for a Class of Fractional Boundary Value Problems with Mixed Nonlinearities

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Abstract In this paper we investigate the existence of nontrivial solutions for the following fractional boundary value problem:

$$\begin{cases} \dfrac{d}{dt}\Big(\dfrac{1}{2}{}_0D_t^{-\beta}(u'(t)) + \dfrac{1}{2}{}_tD_T^{-\beta}(u'(t))\Big) + \nabla F(t,u) = 0, \quad a.e.\ t \in [0,T], \\ u(0) = u(T) = 0, \end{cases}$$

where ${}_0D_t^{-\beta}$ and ${}_tD_T^{-\beta}$ are the left and right Riemann-Liouville fractional integrals of order $0 \le \beta < 1$, respectively, and $\nabla F(t,u)$ is the gradient of $F(t,u)$ at u. The novelty of this paper is that, when the nonlinearity $F(t,u)$ involves a combination of superquadratic and subquadratic terms, we present some reasonable assumptions and establish one new criterion to guarantee the existence of at least two nontrivial solutions. Recent results in the literature are generalized and significantly improved.

Keywords Fractional boundary value problems · Critical point · Variational methods · Mountain pass theorem · Minimizing method

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1 Introduction

Mathematical models with fractional differential operators can describe many physical phenomena exhibiting anomalous diffusion and have been applied to many fields such as viscoelasticity, neurons, electrochemistry, control, biomedical physics, porous media, and electromagnetism. There has been significant development in fractional differential equations, one can see the monographs [13,18,21,23,25] and the papers [1,4,11,14,19,27,29] and the references listed therein. In most cases, fractional order models are generalized from traditional models through replacing the classical integer order derivatives by fractional order derivatives to describe the complex real world behaviors. A typical example is the fractional order diffusion equation

$$\frac{\partial^{\alpha}\phi}{\partial t^{\alpha}} = K\frac{\partial^{\beta}\phi}{\partial |x|^{\beta}},$$

where $0 < \alpha \leq 1$ and $1 < \beta \leq 2$, which can model the diffusion process in many complex systems.

For the transport of solute in highly heterogeneous porous media, the dispersive flux does not follow Fick's first law [5,6], and thus, the traditional second-order convection–diffusion equation is not applicable. The argument for a N-dimensional fractional advection–dispersion equation was presented by Fix and Roop [12], and the equation can be expressed by

$$\frac{\partial\phi}{\partial t} = -\nabla\cdot(v\phi) - \nabla\cdot(\nabla^{-\beta}(-k\nabla\phi)) + f, \quad \text{in } \Omega, \tag{1.1}$$

where $\phi(t, x)$ represents the concentration of a solute at a point v at time t in an arbitrary bounded connected set $\Omega \in \mathbb{R}^N$, v is the velocity of the fluid, k is the diffusion constant coefficient, $v\phi$ and $-k\nabla\phi$ are the mass flux due to advection and dispersion, respectively, and f is a source term. The operator $\nabla^{-\beta}$ in Eq. (1.1) is a linear combination of the left and right Riemann-Liouville fractional integral operators, and its ith component is as follows:

$$\nabla^{-\beta}(-k\nabla\phi)_i = (q\,{}_{-\infty}D_{x_i}^{-\beta} + (1-q)\,{}_{x_i}D_{\infty}^{-\beta})\Big(-k\frac{\partial\phi}{\partial x_i}\Big), \quad i = 1, 2, \ldots, N, \tag{1.2}$$

where $q \in [0, 1]$ describes the skewness of the transport process, ${}_{-\infty}D_{x_i}^{-\beta}$ and ${}_{x_i}D_{\infty}^{-\beta}$ are the left and right Riemann-Liouville fractional integral operators, and $\beta \in [0, 1]$ is the order of Riemann-Liouville fractional integral operators on the real line. Physically, Eq. (1.2) can be interpreted as a generalized Fick's law for concentration of particles with strong nonlocal interaction.

If $q = \frac{1}{2}$, the fractional advection–dispersion Eq. (1.1) describes an important case called symmetric transitions, and the fractional order gradient operator $\nabla^{-\beta}$ reduces to the following symmetric operator:

$$(\nabla^{-\beta})_i = \frac{1}{2}\left({}_{-\infty}D_{x_i}^{-\beta} + {}_{x_i}D_{\infty}^{-\beta}\right).$$

Recently, by establishing variational structure and using Ekeland's variational principle and the mountain pass theorem, for the first time the authors in [16] established the existence of at least one nontrivial solution for the following symmetric advection–dispersion equation

$$\begin{cases} \frac{d}{dt}\left(\frac{1}{2}{}_0D_t^{-\beta}(u'(t)) + \frac{1}{2}{}_tD_T^{-\beta}(u'(t))\right) + \nabla F(t,u) = 0, \quad a.e.\ t \in [0,T], \\ u(0) = u(T) = 0, \end{cases} \tag{FADE}$$

where ${}_0D_t^{-\beta}$ and ${}_tD_T^{-\beta}$ are the left and right Riemann-Liouville fractional integrals of order $0 \le \beta < 1$, respectively, and $\nabla F(t,u)$ is the gradient of $F(t,u)$ at u. Explicitly, under the assumption that

(H_1) $|F(t,u)| \le \bar{a}|u|^2 + \bar{b}(t)|u|^{2-\tau} + \bar{c}(t)$ for all $t \in [0,T]$ and $u \in \mathbb{R}^N$,

where $\bar{a} \in [0, \Gamma^2(\alpha+1)/2T^{2\alpha})$, $\tau \in (0,2)$, $\bar{b} \in L^{2/\tau}([0,T],\mathbb{R})$, and $\bar{c} \in L^1([0,T],\mathbb{R})$, combining with some other reasonable hypotheses on $F(t,u)$, the authors showed that (FADE) has at least one nontrivial solution. In addition, assuming that the potential $F(t,u)$ satisfies the following superquadratic condition:

(H_2) there exist $\mu > 2$ and $R > 0$ such that

$$0 < \mu F(t,u) \le (\nabla F(t,u), u)$$

for all $t \in [0,T]$ and $u \in \mathbb{R}^N$ with $|u| \ge R$,

and some other assumptions on $F(t,u)$, they also obtained the existence of at least one nontrivial solution for (FADE). From then on, many authors focused their attention on finding various hypotheses on $F(t,u)$ to guarantee the existence of solutions of (FADE) via critical point theory and variational methods, see, for instance, [3,7,8,30] and the references therein.

As is known to all, (H_2) is the so-called Ambrosetti-Rabinowitz condition due to Ambrosetti and Rabinowitz (see, e.g., [2]), which implies that $F(t,u)$ is superquadratic as $|u| \to +\infty$. Inspired by this fact, in [7–9], the authors investigated (FADE) for the case that $F(t,u)$ is superquadratic, asymptotically quadratic, and subquadratic at infinity, respectively. In the recent paper [30], the authors gave some new subquadratic assumptions on $F(t,u)$ and showed that (FADE) has at least two or infinitely many nontrivial solutions.

Motivated by [7–9,16,30], it is natural to find the existence of solutions for (FADE) when the potential $W(t,u)$ is of the form

$$F(t,u) = F_1(t,u) + F_2(t,u),$$

that is, $F(t,u)$ is a mixed nonlinearity, where $F_1(t,u)$ is superquadratic as $|u| \to +\infty$ and $F_2(t,u)$ is of subquadratic growth at infinity. As far as the authors know there is no literature to consider the mixed nonlinearity associated with (FADE). Consequently, in the present paper, we focus our attention on this problem and give some reasonable assumptions on $F_1(t,u)$ and $F_2(t,u)$ to guarantee the existence of at least two nontrivial solutions for (FADE). For the statement of our main result, the potential $W(t,u)$ is assumed to satisfy the following hypothesis:

$(F)_1$ $F_1 \in C^1([0,T]\times\mathbb{R}^N,\mathbb{R})$ and there exists some constant $\theta > 2$ such that

$$0 < \theta F_1(t,u) \le (\nabla F_1(t,u),u) \quad \text{for all } t \in [0,T] \text{ and } u \in \mathbb{R}^N\backslash\{0\};$$

$(F)_2$ there exists a positive continuous function $a : [0,T] \to \mathbb{R}^+$ such that

$$|\nabla F_1(t,u)| \le a(t)|u|^{\theta-1} \quad \text{for all } (t,u) \in [0,T]\times\mathbb{R}^N;$$

$(F)_3$ $F_2(t,0) \equiv 0$ for all $t \in [0,T]$, $F_2 \in C^1([0,T]\times\mathbb{R}^N,\mathbb{R})$ and there exist a constant $1 < \varrho < 2$ and a positive continuous function $b : [0,T] \to \mathbb{R}^+$ such that

$$F_2(t,u) \ge b(t)|u|^{\varrho}$$

for all $(t,u) \in [0,T]\times\mathbb{R}^N$;

$(F)_4$ for all $t \in [0,T]$ and $u \in \mathbb{R}^N$,

$$|\nabla F_2(t,u)| \le c(t)|u|^{\varrho-1},$$

where $c : [0,T] \to \mathbb{R}^+$ is a positive continuous function.

To guarantee the existence of at least two nontrivial solutions of (FADE), we also need the following estimation on a and c:

$(F)_5$ $\left(\frac{2\bar{c}C_\varrho^\varrho}{\varrho|\cos(\pi\alpha)|}\frac{\theta-\varrho}{\theta-2}\right)^{\theta-2} < \left(\frac{\theta|\cos(\pi\alpha)|}{2\bar{a}C_\theta^\theta}\frac{2-\varrho}{\theta-\varrho}\right)^{2-\varrho}$, where

$$\bar{a} = \max_{t\in[0,T]} a(t), \quad \bar{c} = \max_{t\in[0,T]} c(t),$$

$\alpha = 1-\beta/2 \in (1/2,1]$, $\theta > 2$ and $1 < \varrho < 2$ are defined in $(F)_1$ and $(F)_3$, respectively, C_ϱ and C_θ are defined in (2.5).

Now we in the position to state our main result.

Theorem 1.1 *If* $(F)_1-(F)_5$ *are satisfied, then* (FADE) *possesses at least two nontrivial solutions.*

Remark 1.2 In view of $(F)_1$, we deduce that (see [15, Fact 2.1])

$$F_1(t,u) \le F_1\left(t,\frac{u}{|u|}\right)|u|^\theta \quad \text{for all } t \in [0,T] \text{ and } 0 < |u| \le 1 \tag{1.3}$$

and

$$F_1(t,u) \geq F_1\left(t, \frac{u}{|u|}\right)|u|^{\theta} \quad \text{for all } t \in [0,T] \text{ and } |u| \geq 1. \tag{1.4}$$

According to (F)$_3$ and (F)$_4$, it is obvious that

$$F_2(t,u) \leq \frac{c(t)}{\varrho}|u|^{\varrho} \quad \text{for all } t \in [0,T] \text{ and } u \in \mathbb{R}^N. \tag{1.5}$$

In addition, from (F)$_1$ to (F)$_4$, it is easy to obtain that

$$\begin{aligned} F(t,u) &= \int_0^1 (\nabla F(t,su),u)ds \leq \frac{a(t)}{\theta}|u|^{\theta} + \frac{c(t)}{\varrho}|u|^{\varrho} \\ &\leq \frac{\bar{a}}{\theta}|u|^{\theta} + \frac{\bar{c}}{\varrho}|u|^{\varrho} =: d_1|u|^{\theta} + d_2|u|^{\varrho} \quad \text{for all } t \in [0,T] \text{ and } u \in \mathbb{R}^N. \end{aligned} \tag{1.6}$$

For the reader's convenience, we present one example to illustrate our main result. Let

$$F(t,u) = \frac{a(t)}{3}|u|^3 + \frac{2c(t)}{3}|u|^{\frac{3}{2}},$$

where $a, c : [0,T] \to \mathbb{R}^+$ are positive continuous functions, and then it is easy to check that $F(t,u)$ satisfies (F)$_1$-(F)$_4$. Meanwhile, the additional assumption $2\bar{c}C_{\varrho}^{\varrho}\sqrt{2\bar{a}C_{\theta}^{\theta}} < |\cos(\pi\alpha)|^{\frac{3}{2}}$ guarantees that (F)$_5$ holds.

In addition, in our Theorem 1.1, for the first time we obtain that (FADE) has at least two nontrivial solutions for the case that $F(t,u)$ is a mixed nonlinearity. Therefore, the previous results in [7–9,16,30] are generalized and improved significantly. However, we do not know whether (FADE) possesses infinitely solutions if the potential $F(t,u)$ is even with respect to u as usual.

The remaining part of this paper is organized as follows: Some preliminary results are presented in Sect. 2. In Sect. 3, we are devoted to accomplishing the proof of Theorem 1.1.

2 Preliminary Results

2.1 Fractional Calculus

In this subsection, for the reader's convenience, we introduce some basic definitions of fractional calculus which are used further in this paper, see [18].

Definition 2.1 (*Left and Right Riemann-Liouville fractional integrals*) Let u be a function defined on $[a,b]$. The left and right Riemann-Liouville fractional integrals of order $\alpha > 0$ for function u denoted by ${}_aD_t^{-\alpha}u(t)$ and ${}_tD_b^{-\alpha}u(t)$, respectively, are defined by

$$ {}_aD_t^{-\alpha}u(t) = \frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}u(s)ds, \quad t \in [a,b] $$

and

$$ {}_tD_b^{-\alpha}u(t) = \frac{1}{\Gamma(\alpha)}\int_t^b (s-t)^{\alpha-1}u(s)ds, \quad t \in [a,b], $$

provided the right-hand sides are pointwise defined on $[a,b]$, where $\Gamma > 0$ is the Gamma function.

Definition 2.2 (*Left and Right Riemann-Liouville fractional derivatives*) Let u be a function defined on $[a,b]$. The left and right Riemann-Liouville fractional derivatives of order $\alpha > 0$ for function u denoted by ${}_aD_t^{\alpha}u(t)$ and ${}_tD_b^{\alpha}u(t)$, respectively, are defined by

$$ {}_aD_t^{\alpha}u(t) = \frac{d^n}{dt^n}{}_aD_t^{\alpha-n}u(t) $$

and

$$ {}_tD_b^{\alpha}u(t) = (-1)^n\frac{d^n}{dt^n}{}_tD_b^{\alpha-n}u(t), $$

where $t \in [a,b]$, $n-1 \le \alpha < n$ and $n \in \mathbb{N}$.

The left and right Caputo fractional derivatives are defined via the above Riemann-Liouville fractional derivatives, see [18, p. 91]. In particular, they are defined for the function belonging to the space of absolutely continuous functions, which is denoted by $AC([a,b],\mathbb{R}^N)$. $AC^k([a,b],\mathbb{R}^N)$ $(k = 1,\ldots)$ is the space of functions u such that $u \in C^{k-1}([a,b],\mathbb{R}^N)$ and $u^{(k-1)} \in AC([a,b],\mathbb{R}^N)$. In particular, $AC([a,b],\mathbb{R}^N) = AC^1([a,b],\mathbb{R}^N)$.

Definition 2.3 Let $\alpha \ge 0$ and $n \in \mathbb{N}$. If $\alpha \in (n-1,n)$ and $u \in AC^n([a,b],\mathbb{R}^N)$, then the left and right Caputo fractional derivatives of order α for function u denoted by ${}_a^cD_t^{\alpha}u(t)$ and ${}_t^cD_b^{\alpha}u(t)$, respectively, exist almost everywhere on $[a,b]$. ${}_a^cD_t^{\alpha}u(t)$ and ${}_t^cD_b^{\alpha}u(t)$ are represented by

$$ {}_a^cD_t^{\alpha}u(t) = {}_aD_t^{\alpha-n}u^{(n)}(t) = \frac{1}{\Gamma(n-\alpha)}\int_a^t (t-s)^{n-\alpha-1}u^{(n)}(s)ds $$

and

$$ {}_t^cD_b^{\alpha}u(t) = (-1)^n{}_tD_b^{\alpha-n}u^{(n)}(t) = \frac{(-1)^n}{\Gamma(n-\alpha)}\int_t^b (s-t)^{n-\alpha-1}u^{(n)}(s)ds, $$

respectively, where $t \in [a,b]$. If $\alpha = n-1$, $u \in AC^{n-1}([a,b],\mathbb{R}^N)$, then ${}_a^cD_t^{n-1}u(t)$ and ${}_t^cD_b^{n-1}u(t)$ are represented by

$$ {}_a^cD_t^{n-1}u(t) = u^{(n-1)}(t) \quad \textit{and} \quad {}_t^cD_b^{n-1}u(t) = (-1)^{(n-1)}u^{(n-1)}(t), \quad t \in [a,b]. $$

Proposition 2.4 *The left and right Riemann-Liouville fractional integral operators have the property of a semigroup, that is, for* $\forall\alpha_1, \alpha_2 > 0$,

$$ {}_aD_t^{-\alpha_1}\left({}_aD_t^{-\alpha_2}u(t)\right) = {}_aD_t^{-\alpha_1-\alpha_2}u(t) $$

and

$$ {}_tD_b^{-\alpha_1}\left({}_tD_b^{-\alpha_2}u(t)\right) = {}_tD_b^{-\alpha_1-\alpha_2}u(t), $$

in any point $t \in [a, b]$ *for continuous function* u *and for almost every point in* $[a, b]$ *if the function* $u \in L^1([a, b], \mathbb{R}^N)$.

2.2 Fractional Derivative Spaces

In order to establish the variational structure which enables us to reduce the existence of solutions for (FADE) to find critical points of the corresponding functional, it is necessary to construct appropriate function spaces.

Firstly, we recall some fractional spaces, for more details see [16]. To this end, denote by $L^p([0, T], \mathbb{R}^N)$ $(1 < p < +\infty)$ the Banach spaces of functions on $[0, T]$ with values in $\mathbb{R}^N$ under the norms

$$ \|u\|_p = \left(\int_0^T |u(t)|^p dt\right)^{1/p}, $$

and $C([0, T], \mathbb{R}^N)$ is the Banach space of continuous functions from $[0, T]$ into $\mathbb{R}^N$ equipped with the norm

$$ \|u\|_\infty = \max\{|u(t)| : t \in [0, T]\}. $$

For $0 < \alpha \leq 1$ and $1 < p < +\infty$, the fractional derivative space $E_0^{\alpha,p}$ is defined by

$$ \begin{aligned} E_0^{\alpha,p} &= \{u \in L^p([0, T], \mathbb{R}^N) : {}_0^cD_t^\alpha u \in L^p([0, T], \mathbb{R}^N) \text{ and } u(0) = u(T) = 0\} \\ &= \overline{C_0^\infty([0, T], \mathbb{R}^N)}^{\|\cdot\|_{\alpha,p}}, \end{aligned} $$

where $\|\cdot\|_{\alpha,p}$ is defined as follows:

$$ \|u\|_{\alpha,p} = \left(\int_0^T |u(t)|^p dt + \int_0^T |{}_0^cD_t^\alpha u(t)|^p dt\right)^{1/p}. \tag{2.1} $$

Then $E_0^{\alpha,p}$ is a reflexive and separable Banach space.

Lemma 2.5 ([16, Proposition 3.2]) *Let* $0 < \alpha \leq 1$ *and* $1 < p < +\infty$. *For all* $u \in E_0^{\alpha,p}$, *we have*

$$ \|u\|_p \leq \frac{T^\alpha}{\Gamma(\alpha + 1)}\|{}_0^cD_t^\alpha u\|_p. \tag{2.2} $$

In addition, if $\alpha > \frac{1}{p}$ *and* $\frac{1}{p} + \frac{1}{q} = 1$, *then*

$$\|u\|_\infty \le \frac{T^{\alpha - \frac{1}{p}}}{\Gamma(\alpha)((\alpha - 1)q + 1)^{\frac{1}{q}}} \|{}_0^c D_t^\alpha u\|_p. \tag{2.3}$$

Remark 2.6 According to (2.1) and (2.2), we can consider in $E_0^{\alpha,p}$ the following norm:

$$\|u\|_{\alpha,p} = \|{}_0^c D_t^\alpha u\|_p, \tag{2.4}$$

which is equivalent to (2.1).

In what follows, we denote by $E^\alpha = E_0^{\alpha,2}$. Then it is a Hilbert space with respect to the norm $\|u\|_{\alpha,2}$ given by (2.4). Moreover, from Lemma 2.5 and Remark 2.6, we have

Proposition 2.7 *Let* $0 < \alpha \le 1$. *Then, for any* $p \in (1, +\infty)$, *there exists some constant* $C_p > 0$ *such that*

$$\|u\|_p \le C_p \|u\|_{\alpha,2}, \quad \forall u \in E^\alpha. \tag{2.5}$$

Based on (2.3), for $\alpha > \frac{1}{p}$ with $1 < p < +\infty$, the embedding $E^\alpha \hookrightarrow C([0,T], \mathbb{R}^N)$ is continuous. To verify that the functional corresponding to (FADE) satisfies the (PS) condition, we need the following proposition:

Proposition 2.8 ([16, Proposition 3.3]) *Let* $0 < \alpha \le 1$ *and* $1 < p < +\infty$. *Assume that* $\alpha > \frac{1}{p}$ *and* $u_k \rightharpoonup u$ *in* $E_0^{\alpha,p}$, *then* $u_k \to u$ *in* $C([0,T], \mathbb{R}^N)$, *i.e.,*

$$\|u_k - u\|_\infty \to 0$$

as $k \to +\infty$.

Now we introduce more notations and some necessary definitions. Let $\mathcal{B}$ be a real Banach space, $I \in C^1(\mathcal{B}, \mathbb{R})$ means that I is a continuously Fréchet-differentiable functional defined on $\mathcal{B}$.

Definition 2.9 $I \in C^1(\mathcal{B}, \mathbb{R})$ is said to satisfy the (PS) condition if any sequence $\{u_k\}_{k\in\mathbb{N}} \subset \mathcal{B}$, for which $\{I(u_k)\}_{k\in\mathbb{N}}$ is bounded and $I'(u_k) \to 0$ as $k \to +\infty$, possesses a convergent subsequence in $\mathcal{B}$.

Moreover, let B_r be the open ball in $\mathcal{B}$ with the radius r and centered at 0 and ∂B_r denotes its boundary. Under the conditions of Theorem 1.1, we obtain the existence of the first solution of (FADE) by using the following well-known Mountain Pass Theorem, see [24]:

Lemma 2.10 ([24, Theorem 2.2]) *Let* $\mathcal{B}$ *be a real Banach space and* $I \in C^1(\mathcal{B}, \mathbb{R})$ *satisfying the* (PS) *condition. Suppose that* $I(0) = 0$ *and*

(A1) *there are constants* ρ, $\eta > 0$ *such that* $I|_{\partial B_\rho} \geq \eta$, *and*
(A2) *there is an* $e \in \mathcal{B}\backslash\overline{B}_\rho$ *such that* $I(e) \leq 0$.

Then I *possesses a critical value* $c \geq \eta$. *Moreover,* c *can be characterized as*

$$c = \inf_{g\in\Gamma} \max_{s\in[0,1]} I(g(s)),$$

where

$$\Gamma = \{g \in C([0,1], \mathcal{B}) : g(0) = 0, g(1) = e\}.$$

As far as the second one is concerned, we obtain it by minimizing method, which is contained in a small ball centered at 0, see Step 4 in the proof of Theorem 1.1.

3 Proof of Theorem 1.1

The aim of this section is to give the proof of Theorem 1.1. To do this, we are going to establish the corresponding variational framework of (FADE). Making use of Proposition 2.4 and Definition 2.3, for any $u \in AC([0,T], \mathbb{R}^N)$, (FADE) is equivalent to the following problem:

$$\begin{cases} \dfrac{d}{dt}\Big(\dfrac{1}{2}{}_0D_t^{\alpha-1}({}_0^cD_t^\alpha u(t)) - \dfrac{1}{2}{}_tD_T^{\alpha-1}({}_t^cD_T^\alpha u(t))\Big) + \nabla F(t,u) = 0, \quad a.e.\ t \in [0,T], \\ u(0) = u(T) = 0, \end{cases} \tag{3.1}$$

where $\alpha = 1 - \beta/2 \in (1/2, 1]$. Therefore, we seek a solution u of (3.1), which, of course, corresponds to the solution u of (FADE) provided that $u \in AC([0,T], \mathbb{R}^N)$. In what follows, we treat (3.1) in the Hilbert space $E^\alpha = E_0^{\alpha,2}$ with the corresponding norm $\|u\|_{\alpha,2}$.

Under the conditions of Theorem 1.1, following Theorem 4.1 in [16], it directly yields that

Lemma 3.1 *Let* $L : [0,T] \times \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$ *be defined by*

$$L(t,x,y,z) = -\frac{1}{2}(y,z) - F(t,x),$$

where $F : [0,T] \times \mathbb{R}^N \to \mathbb{R}$ *satisfies the assumptions* $(\mathrm{F})_1 - (\mathrm{F})_4$. *If* $1/2 < \alpha \leq 1$, *then the functional defined by*

$$I(u) = \int_0^T L(t, u(t), {}_0^cD_t^\alpha u(t), {}_t^cD_T^\alpha u(t))dt \tag{3.2}$$

is continuous differentiable on E^α*, and for any* $u, v \in E^\alpha$*, we have*

$$\begin{aligned} I'(u)v &= \int_0^T (D_x L(t, u(t), {}_0^c D_t^\alpha u(t), {}_t^c D_T^\alpha u(t)), v(t))dt \\ &\quad + \int_0^T (D_y L(t, u(t), {}_0^c D_t^\alpha u(t), {}_t^c D_T^\alpha u(t)), v(t))dt \\ &\quad + \int_0^T (D_z L(t, u(t), {}_0^c D_t^\alpha u(t), {}_t^c D_T^\alpha u(t)), v(t))dt. \end{aligned}$$

Let us denote by

$$D^\alpha(u(t)) = \frac{1}{2} {}_0 D_t^{\alpha-1}({}_0^c D_t^\alpha u(t)) - \frac{1}{2} {}_t D_T^{\alpha-1}({}_t^c D_T^\alpha u(t)).$$

We are now in the position to give the definition of the solution of (3.1).

Definition 3.2 ([16, Definition 4.1]) A function $u \in AC([0, T], \mathbb{R}^N)$ is called a solution of (3.1) if

(i) $D^\alpha(u(t))$ is derivative for almost every $t \in [0, T]$, and
(ii) $u(t)$ satisfies (3.1).

The same as Theorem 4.2 in [16], we have the following lemma:

Lemma 3.3 *Let* $1/2 < \alpha \leq 1$ *and* I *be defined in* (3.4)*. If the assumptions* (F)$_1$ – (F)$_4$ *are satisfied and* $u \in E^\alpha$ *is a solution of corresponding Euler equation* $I'(u) = 0$*, then* u *is a solution of* (3.1) *which corresponds to the solution of* (FADE).

In the sequel, we need the following estimate:

Lemma 3.4 ([16, Proposition 4.1]) *If* $1/2 < \alpha \leq 1$*, then for any* $u \in E^\alpha$*, one has*

$$|\cos(\pi\alpha)| \|u\|_{\alpha,2}^2 \leq -\int_0^T ({}_0^c D_t^\alpha u(t), {}_t^c D_T^\alpha u(t))dt \leq \frac{1}{|\cos(\pi\alpha)|} \|u\|_{\alpha,2}^2. \tag{3.3}$$

To consider the problem (3.1), define the functional $I : \mathcal{B} = E^\alpha \to \mathbb{R}$ by

$$I(u) = \int_0^T \left[-\frac{1}{2} ({}_0^c D_t^\alpha u(t), {}_t^c D_T^\alpha u(t)) - F(t, u(t)) \right] dt. \tag{3.4}$$

Then, from Lemma 3.1, I is continuously Fréchet-differentiable on E^α. Moreover, we have

$$\begin{aligned} I'(u)v = &-\int_0^T \frac{1}{2} [({}_0^c D_t^\alpha u(t), {}_t^c D_T^\alpha v(t)) + ({}_t^c D_T^\alpha u(t), {}_0^c D_t^\alpha v(t))] dt \\ &- \int_0^T (\nabla F(t, u(t)), v(t))dt. \end{aligned} \tag{3.5}$$

In order to check that the corresponding functional $I(u)$ satisfies the condition (A1) of Lemma 2.10, the following lemma plays an essential role:

Lemma 3.5 *Let* $1 < \varrho < 2 < \theta$, $A, B > 0$, *and consider the function*

$$\Phi_{A,B}(t) := t^2 - At^{\varrho} - Bt^{\theta}, \quad t \geq 0.$$

Then $\max_{t\geq 0} \Phi_{A,B}(t) > 0$ *if and only if*

$$A^{\theta-2}B^{2-\varrho} < d(\varrho,\theta) := \frac{(\theta-2)^{\theta-2}(2-\varrho)^{2-\varrho}}{(\theta-\varrho)^{\theta-\varrho}}.$$

Furthermore, for $t = t_B := [(2-\varrho)/B(\theta-\varrho)]^{1/(\theta-2)}$, *one has*

$$\max_{t\geq 0} \Phi_{A,B}(t) = \Phi_{A,B}(t_B) = t_B^2\left[\frac{\theta-2}{\theta-\varrho} - AB^{\frac{2-\varrho}{\theta-2}}\left(\frac{\theta-\varrho}{2-\varrho}\right)^{\frac{2-\varrho}{\theta-2}}\right] > 0. \tag{3.6}$$

Proof The proof is essentially the same as that in [10, Lemma 3.2], so we omit its details. □

In what follows, we verify that I satisfies the (PS) condition.

Lemma 3.6 *If* $(F)_1 - (F)_4$ *hold, then* I *satisfies the* (PS) *condition.*

Proof Assume that $\{u_k\}_{k\in\mathbb{N}} \subset E^{\alpha}$ is a sequence such that $\{I(u_k)\}_{k\in\mathbb{N}}$ is bounded and $I'(u_k) \to 0$ as $k \to +\infty$. Then there exists a constant $M > 0$ such that

$$|I(u_k)| \leq M \quad \text{and} \quad \|I'(u_k)\|_{(E^{\alpha})^*} \leq M \tag{3.7}$$

for every $k \in \mathbb{N}$, where $(E^{\alpha})^*$ is the dual space of E^{α}.

We firstly prove that $\{u_k\}_{k\in\mathbb{N}}$ is bounded in E^{α}. From (1.5), (3.4), (3.5), $(F)_1$, $(F)_3$, $(F)_4$ and Proposition 2.7, we obtain that

$$\begin{aligned} M + \frac{M}{\theta}\|u_k\| &\geq I(u_k) - \frac{1}{\theta}I'(u_k)u_k \\ &\geq \left(\frac{1}{2} - \frac{1}{\theta}\right)|\cos(\pi\alpha)|\|u_k\|_{\alpha,2}^2 - \int_0^T [F(t,u_k(t)) \\ &\quad - \frac{1}{\theta}(\nabla F(t,u_k(t)), u_k(t))]dt \\ &\geq \left(\frac{1}{2} - \frac{1}{\theta}\right)|\cos(\pi\alpha)|\|u_k\|_{\alpha,2}^2 - C_{\varrho}^{\varrho}\left(d_2 + \frac{\bar{c}}{\theta}\right)\|u_k\|_{\alpha,2}^{\varrho}. \end{aligned} \tag{3.8}$$

Since $1 < \varrho < 2$, the above inequality (3.8) implies that $\{u_k\}_{k\in\mathbb{N}}$ is bounded in E^{α}. Then the sequence $\{u_k\}_{k\in\mathbb{N}}$ has a subsequence, again denoted by $\{u_k\}_{k\in\mathbb{N}}$, and there exists $u \in E^{\alpha}$ such that

$$u_k \rightharpoonup u \text{ weakly in } E^{\alpha},$$

which yields that

$$(I'(u_k) - I'(u))(u_k - u) \to 0. \tag{3.9}$$

Moreover, due to the fact that $F \in C^1([0, T] \times \mathbb{R}^N, \mathbb{R})$, according to Proposition 2.8, we have

$$\int_0^T (\nabla F(t, u_k(t)) - \nabla F(t, u(t)), u_k(t) - u(t))dt \to 0 \tag{3.10}$$

as $k \to +\infty$. Consequently, combining (3.9), (3.10) with the following equality

$$\begin{aligned}(I'(u_k) - I'(u))(u_k - u) &= -\int_0^T ({}_0^cD_t^\alpha(u_n(t) - u(t)), {}_t^cD_T^\alpha(u_n(t) - u(t)))dt \\ &\quad - \int_0^T (\nabla F(t, u_k(t)) - \nabla F(t, u(t)), u_k(t) - u(t))dt \\ &\geq |\cos(\pi\alpha)| \|u_k - u\|_{\alpha,2}^2 \\ &\quad - \int_0^T (\nabla F(t, u_k(t)) - \nabla F(t, u(t)), u_k(t) - u(t))dt,\end{aligned}$$

we deduce that $\|u_k - u\|_{\alpha,2} \to 0$ as $k \to +\infty$. That is, I satisfies the (PS) condition. □

Now we are in the position to complete the proof of Theorem 1.1. For illumining the ideas, we divide its proof into four steps:

Proof Step 1 It is clear that $I(0) = 0$ and $I \in C^1(E^\alpha, \mathbb{R})$ satisfies the (PS) condition by Lemma 3.6.

Step 2 To show that there exist constants $\rho > 0$ and $\eta > 0$ such that I satisfies $I|_{\partial B_\rho} \geq \eta > 0$, that is, the condition (A1) of Lemma 2.10 holds. To this end, in view of (1.6), we have

$$\begin{aligned}\int_0^T F(t, u)dt &\leq d_1 \int_0^T |u|^\theta dt + d_2 \int_0^T |u|^\varrho dt \\ &= d_1\|u\|_\theta^\theta + d_2\|u\|_\varrho^\varrho,\end{aligned} \tag{3.11}$$

which, combining with Proposition 2.7, yield that

$$\begin{aligned}I(u) &\geq \frac{1}{2}|\cos(\pi\alpha)|\|u\|_{\alpha,2}^2 - \int_0^T F(t, u)dt \\ &\geq \frac{1}{2}|\cos(\pi\alpha)|\|u\|_{\alpha,2}^2 - d_1\|u\|_\theta^\theta - d_2\|u\|_\varrho^\varrho \\ &\geq \frac{1}{2}|\cos(\pi\alpha)|\|u\|_{\alpha,2}^2 - d_1C_\theta^\theta\|u\|_{\alpha,2}^\theta - d_2C_\varrho^\varrho\|u\|_{\alpha,2}^\varrho \quad \text{for all} \quad u \in E^\alpha.\end{aligned} \tag{3.12}$$

Applying Lemma 3.5 with

$$A = \frac{2d_2C_\varrho^\varrho}{|\cos(\pi\alpha)|} \quad \text{and} \quad B = \frac{2d_1C_\theta^\theta}{|\cos(\pi\alpha)|},$$

we obtain that

$$I(u) \geq \frac{|\cos(\pi\alpha)|}{2}\Phi_{A,B}(t_B) > 0,$$

provided that $A^{\theta-2}B^{2-\varrho} < d(\varrho, \theta)$, that is, provided that

$$\Big(\frac{2\bar{c}C_\varrho^\varrho}{\varrho|\cos(\pi\alpha)|}\frac{\theta-\varrho}{\theta-2}\Big)^{\theta-2} < \Big(\frac{\theta|\cos(\pi\alpha)|}{2\bar{a}C_\theta^\theta}\frac{2-\varrho}{\theta-\varrho}\Big)^{2-\varrho}.$$

Let $\rho = t_B = \left[\frac{2-\varrho}{B(\theta-\varrho)}\right]^{\frac{1}{\theta-2}}$ and $\eta = \frac{|\cos(\pi\alpha)|}{2}\Phi_{A,B}(t_B)$, it yields that $I|_{\partial B_\rho} \geq \eta > 0$.

Step 3 To obtain that there exists an $e \in E^\alpha$ such that $I(e) < 0$ with $\|e\|_{\alpha,2} > \rho$, where ρ is defined in Step 2. For this purpose, take $\psi \in E^\alpha$ such that $\psi(t) > 0$ on some closed subset $\Omega \subset (0, T)$. In view of (1.4), (3.4), $(F)_1$ and $(F)_3$, for $l \in (0, \infty)$ such that $|l\psi(t)| \geq 1$ for all $t \in \Omega$, we deduce that

$$\begin{aligned} I(l\psi) &\leq \frac{l^2}{2|\cos(\pi\alpha)|}\|\psi\|_{\alpha,2}^2 - \int_0^T F(t, l\psi(t))dt \\ &\leq \frac{l^2}{2|\cos(\pi\alpha)|}\|\psi\|_{\alpha,2}^2 - \int_\Omega F_1(t, l\psi(t))dt \\ &\leq \frac{l^2}{2|\cos(\pi\alpha)|}\|\psi\|_{\alpha,2}^2 - l^\theta\int_\Omega F_1\Big(t, \frac{\psi(t)}{|\psi(t)|}\Big)|\psi(t)|^\theta dt \\ &\leq \frac{l^2}{2|\cos(\pi\alpha)|}\|\psi\|_{\alpha,2}^2 - ml^\theta\int_\Omega |\psi(t)|^\theta dt, \end{aligned} \tag{3.13}$$

where $m = \min\{F_1(t, u) : t \in \Omega, |u| = 1\}$ (on account of $(F)_1$, it is obvious that $m > 0$). Since $\theta > 2$, (3.13) implies that $I(l\varphi) = I(e) < 0$ for some $l \gg 1$ with $\|l\varphi\|_{\alpha,2} > \rho$, where ρ is defined in Step 2. By Lemma 2.10, I possesses a critical value $c_1 \geq \eta > 0$ given by

$$c_1 = \inf_{g\in\Gamma}\max_{s\in[0,1]} I(g(s)),$$

where

$$\Gamma = \big\{g \in C([0, 1], E^\alpha) : g(0) = 0,\ g(1) = e\big\}.$$

Hence, there is $0 \neq u_1 \in E^\alpha$ such that

$$I(u_1) = c_1 \quad \text{and} \quad I'(u_1) = 0.$$

That is, the first nontrivial solution of (FADE) exists.

Step 4 From (3.12), we see that I is bounded from below on $\overline{B_\rho(0)}$. Therefore, we can denote

$$c_2 = \inf_{\|u\|_{\alpha,2} \le \rho} I(u),$$

where ρ is defined in Step 2. Then there is a minimizing sequence $\{v_k\}_{k\in\mathbb{N}} \subset \overline{B_\rho(0)}$ such that

$$I(v_k) \to c_2 \quad \text{and} \quad I'(v_k) \to 0$$

as $k \to \infty$. That is, $\{v_k\}_{k\in\mathbb{N}}$ is a (PS) sequence. Furthermore, from Lemma 3.6, I satisfies the (PS) condition. Therefore, c_2 is one critical value of I. In what follows, we show that c_2 is one nontrivial critical point. For $0 \not\equiv \varphi \in E^\alpha$, according to $(\mathrm{F})_1$ and $(\mathrm{F})_3$, one deduces that

$$\begin{aligned} I(l\varphi) &\le \frac{l^2}{2|\cos(\pi\alpha)|}\|\varphi\|_{\alpha,2}^2 - \int_0^T F(t, l\varphi(t))dt \\ &\le \frac{l^2}{2|\cos(\pi\alpha)|}\|\varphi\|_{\alpha,2}^2 - \int_0^T F_2(t, l\varphi(t))dt \\ &\le \frac{l^2}{2|\cos(\pi\alpha)|}\|\varphi\|_{\alpha,2}^2 - l^\varrho \int_0^T b(t)|\varphi(t)|^\varrho dt, \quad \forall l \in (0, +\infty). \end{aligned} \tag{3.14}$$

Since $1 < \varrho < 2$, (3.14) implies that $I(l\varphi) < 0$ for l small enough such that $\|l\varphi\|_{\alpha,2} \le \rho$. Therefore, $c_2 < 0 < c_1$. Consequently, there is $0 \ne u_2 \in E^\alpha$ such that

$$I(u_2) = c_2 \quad \text{and} \quad I'(u_2) = 0.$$

That is, (FADE) has another nontrivial solution. □

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