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Morrey-type Space and Its Köthe Dual Space

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Abstract We define Morrey-type spaces generated by a basis of measurable functions. The main result in this paper gives the description of the Köthe dual of these spaces. As an application, we recover many known cases including Mock Morrey spaces defined by David R. Adams.

Keywords Block space · Fatou property · Komlós theorem · Köthe dual space · Morrey space

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1 Introduction

Inspired by the papers [2,3] due to Adams and Xiao, we introduce Morrey-type spaces generated by the basis of functions. The main aim of this paper is to give a characterization of their Köthe dual space.

Morrey spaces, which were introduced by C. Morrey in order to study regularity problem arising in Calculus of Variations, describe local regularity more precisely than Lebesgue spaces. We note that Morrey spaces are widely used not only in harmonic analysis but also in partial differential equations (cf. [6]).

We shall consider all cubes in $\mathbb{R}^n$ which have their sides parallel to the coordinate axes. We denote by $\mathcal{Q}$ the family of all such cubes. For a cube $Q \in \mathcal{Q}$, we use $l(Q)$ to denote its side length and $|Q|$ to denote its volume.

Let $0 < p < \infty$ and $0 < \lambda < n$. The Morrey space $L^{p,\lambda}(\mathbb{R}^n)$ is defined to be the subset of all $f \in L^p_{\text{loc}}(\mathbb{R}^n)$ such that

$$\|f\|_{L^{p,\lambda}(\mathbb{R}^n)} := \sup_{Q\in\mathcal{Q}} \left(\frac{1}{l(Q)^\lambda}\int_Q |f(x)|^p \,\mathrm{d}x\right)^{1/p} < \infty.$$

Clearly, $\|f\|_{L^{p,\lambda}(\mathbb{R}^n)}$ is a norm (resp., quasi-norm) provided $1 \le p < \infty$ (resp., $0 < p < 1$). The completeness of Morrey spaces follows easily by that of the Lebesgue L^p-spaces.

Before discussing the problem and the contents of the paper, we introduce some definitions and notation.

All measure spaces considered throughout this paper will be complete and σ-finite. Let (Ω, Σ, μ) be a complete σ-finite measure space and let $L^0(\mu)$ (resp., $\widetilde{L}^0(\mu)$) denote the space of all equivalence classes of real-valued (resp., complex-valued) measurable functions on Ω with the topology of convergence in measure on μ-finite sets. A quasi-Banach (function) lattice X on (Ω, Σ, μ) is a subspace of $L^0(\mu)$, which is complete with respect to a quasi-norm $\|\cdot\|_X$ and which has the property: whenever $f \in L^0(\mu)$, $g \in X$ and $|f| \le |g|$ μ-a.e., $f \in X$ and $\|f\|_X \le \|g\|_X$; moreover, we will assume that there exists $u \in X$ with $u > 0$ μ-a.e..

A quasi-Banach lattice X is said to have the *Fatou property* whenever $0 \le f_n \uparrow f$ μ-a.e., $f_n \in X$, and $\sup_{n\ge1} \|f_n\|_X < \infty$ imply that $f \in X$ and $\|f_n\|_X \to \|f\|_X$.

The Köthe dual space X' of a quasi-Banach lattice X on (Ω, Σ, μ) is defined as the space of all $f \in L^0(\mu)$ such that $\int_\Omega |fg|\,\mathrm{d}\mu < \infty$ for every $g \in X$. It is a Banach lattice on (Ω, Σ, μ) when equipped with the norm

$$\|f\|_{X'} = \sup_{\|g\|_X\le1} \int_\Omega |fg|\,\mathrm{d}\mu.$$

In certain cases, X' could be trivial, for instance, if $X = L^p$ on a nonatomic measure space with $0 < p < 1$, then $(L^p)'$ is trivial.

Notice that a Banach lattice X has the Fatou property if and only if $X = X'' := (X')'$ with equality of norms (see, e.g., [10, p. 30]).

We recall that α-dimensional Hausdorff content $H^\alpha(E)$ with $0 < \alpha \le n$ of a set $E \subset \mathbb{R}^n$ is defined by

$$H^\alpha(E) := \inf\left\{\sum_j l(Q_j)^\alpha\right\},$$

where the infimum is taken over all coverings of E by countable families of cubes $\{Q_j\} \subset \mathcal{Q}$.

We also recall that the Choquet integral of a function $\phi\colon \mathbb{R}^n \to [0,\infty)$ with respect to the Hausdorff content H^α is defined by

$$\int_{\mathbb{R}^n} \phi \,\mathrm{d}H^\alpha := \int_0^\infty H^\alpha(\{y \in \mathbb{R}^n;\ \phi(y) > t\})\,\mathrm{d}t.$$

Following [3], for $0 < \lambda < n$, define the class $\overline{\mathcal{B}_\lambda}$ to be the set of all weights $w \in A_1$ such that $\int_{\mathbb{R}^n} w\, dH^\lambda \le 1$. By weights we mean non-negative, locally integrable functions which are positive on a set of positive measure. One says that a weight w on $\mathbb{R}^n$ belongs to the Muckenhoupt class A_1 whenever there exists $C > 0$ such that

$$\frac{1}{|Q|}\int_Q w(x)\,\mathrm{d}x \le C \operatorname*{ess\,inf}_{x\in Q} w(x), \qquad Q \in \mathcal{Q}.$$

The infimum of $C > 0$ satisfying the above condition is denoted by $[w]_{A_1}$.

To discuss the Köthe duality for Morrey spaces, we need to define the space $H^{q,\lambda}(\mathbb{R}^n)$, $1 < q < \infty$, $0 < \lambda < n$, which is made up of all $f \in L^0(dx)$ such that

$$\|f\|_{H^{q,\lambda}(\mathbb{R}^n)} := \inf_{w\in\overline{\mathcal{B}_\lambda}} \left(\int_{\mathbb{R}^n} |f(x)|^q w(x)^{1-q}\,\mathrm{d}x\right)^{1/q} < \infty.$$

As usual if p and q are positive real numbers such that $1/p + 1/q = 1$, then we call p and q a pair of *conjugate exponents*.

In [3], Adams and Xiao established some results on duality between $L^{p,\lambda}(\mathbb{R}^n)$ and $H^{q,\lambda}(\mathbb{R}^n)$. Their result in the language of the Köthe duality can be stated as follows: *Let p and q be conjugate exponents*, $1 < p < \infty$, *and let* $0 < \lambda < n$. *Then the following formulas hold with equality of norms*:

$$L^{p,\lambda}(\mathbb{R}^n)' = H^{q,\lambda}(\mathbb{R}^n), \quad H^{p,\lambda}(R^n)' = L^{q,\lambda}(\mathbb{R}^n).$$

For the preduals of Morrey spaces, we refer to [7,9,16].

The above result of Adams and Xiao was the main motivation for this paper. In Sects. 2 and 3, we define a variant of $L^{p,\lambda}$-spaces and $H^{q,\lambda}$-spaces on a measure space (Ω, Σ, μ) and we will give a characterization of the Köthe dual spaces of these spaces. Our approach is different from those by Adams and Xiao in that it is based on the famous Komlós property. Section 4 is devoted to examples. In the last section, we introduce the class $\overline{\mathcal{B}^1}$ which is smaller than the class $\mathcal{B}_\lambda$ but plays the same role in the description of the Köthe dual of the Morrey space $L^{p,\lambda}(\mathbb{R}^n)$.

Throughout the paper, the letter C will be used for constants that may change from one occurrence to another.

2 The Main Results

Let (Ω, Σ, μ) be a measure space and let $L^0_+(\mu)$ be a cone of all non-negative μ-measurable functions on Ω. The characteristic function of a μ-measurable subset A of Ω will be denoted by 1_A. Throughout this and next sections, we fix a countable subset $\mathcal{B} = \{b_j\} = \{b_j\}_{j\in\mathbb{N}}$ of $L^0_+(\mu)$.

For $1 \le p < \infty$, we denote by $L^{p,\mathcal{B}}(\mu)$ the Morrey-type space of all $f \in L^0(\mu)$ supported in $\bigcup_j \operatorname{supp} b_j$ and equipped with the norm given by

$$\|f\|_{L^{p,\mathcal{B}}(\mu)} := \sup_{j\in\mathbb{N}} \left(\int_\Omega |f|^p b_j \, d\mu \right)^{1/p}.$$

We now define the class $\overline{\mathcal{B}} \subset L^0_+(\mu)$ which will play an essential role in the sequel; it is defined by the minimal set (with respect to inclusion) that satisfies the following conditions:

(i) $\{b_j\} \subset \overline{\mathcal{B}} \subset L^0_+(\mu)$;
(ii) If $\{w_j\} \subset \overline{\mathcal{B}}$, then, for any non-negative sequence $\{c_j\}$ with $\|\{c_j\}\|_{\ell^1} \le 1$, one has $\sum_j c_j w_j \in \overline{\mathcal{B}}$;
(iii) For all $w \in \overline{\mathcal{B}}$,

$$\sup_{\|f\|_{L^{1,\mathcal{B}}} \le 1} \int_\Omega |f w| \, d\mu \le 1;$$

(iv) (the Komlós property) If $\{w_j\} \subset \overline{\mathcal{B}}$, then there exist $w \in \overline{\mathcal{B}}$ and a subsequence $\{v_j\}$ of $\{w_j\}$ such that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} v_j = w \quad \mu-\text{a.e.}.$$

In other words,

$$\overline{\mathcal{B}} = \left\{ \sum_{j=1}^{\infty} c_j b_j \ : \ \|\{c_j\}\|_{\ell^1} \le 1 \right\}$$

as long as there exists a function $v \in L^{1,\mathcal{B}}(\mu)$ such that $v(x) > 0$ for μ-almost all $x \in \Omega$. In fact, defining $\mathcal{B}$ as above, we can readily check (iii) (iv).
In Sect. 4, we present some examples of the class $\overline{\mathcal{B}}$.

To discuss the Köthe duality for Morrey-type spaces, we define the "set" $H^{q,\mathcal{B}}(\mu)$, for $1 < q < \infty$, the space of all $f \in L^0(\mu)$ such that

$$\|f\|_{H^{q,\mathcal{B}}(\mu)} := \inf_{w\in\overline{\mathcal{B}}} \left(\int_\Omega |f|^q w^{1-q} \, d\mu \right)^{1/q} < \infty.$$

We say that a function $\phi \in L^0(\mu)$ is a $(q, \mathcal{B})$-block provided that

$$\|\phi\|_{H^{q,\mathcal{B}}(\mu)} < 1.$$

We also need the definition of the space $B^{q,\mathcal{B}}(\mu)$ which consists of all $f \in L^0(\mu)$ such that

$$\|f\|_{B^{q,\mathcal{B}}(\mu)} := \inf\left\{\left\|\{\lambda_k\}\right\|_{\ell^1};\ f = \sum_k \lambda_k\phi_k\ \mu - \text{a.e.}\right\} < \infty,$$

where each ϕ_k is a $(q, \mathcal{B})$-block and the infimum is taken over all possible decompositions of f. Clearly, $\|\cdot\|_{B^{q,\mathcal{B}}(\mu)}$ is a norm.

We state and prove the following result:

Theorem 2.1 *Suppose that* $\overline{\mathcal{B}}$ *fulfills the condition* (ii). *Then for any* $1 < q < \infty$, *we have* $B^{q,\mathcal{B}}(\mu) = H^{q,\mathcal{B}}(\mu)$ *with equality of norms. In particular,* $H^{q,\mathcal{B}}(\mu)$ *is a Banach space.*

Proof Notice that the functional $\|\cdot\|_{B^{q,\mathcal{B}}(\mu)}$ on $B^{q,\mathcal{B}}(\mu)$ is a norm. Thus, we see that $\|\cdot\|_{H^{q,\mathcal{B}}(\mu)}$ is a norm once we prove their equality. Clearly,

$$\|f\|_{B^{q,\mathcal{B}}(\mu)} \le \|f\|_{H^{q,\mathcal{B}}(\mu)}, \quad f \in H^{q,\mathcal{B}}(\Omega).$$

We now prove the converse is true.

In general, the inclusion map between quasi-normed spaces $\mathrm{id}\colon X \to Y$ satisfies $\|\mathrm{id}\|_{X\to Y} \le 1$ if and only if the following condition holds: $\|x\|_Y \le 1$ whenever $x \in X$ satisfies $\|x\|_X < 1$.

Indeed, fix $0 < c < 1$. Then for any nonzero $x \in X$, we have $z := \frac{cx}{\|x\|_X} \in X$ and $\|z\|_X = c < 1$. By the condition, $\|z\|_Y \le 1$ and so

$$c\|x\|_Y \le \|x\|_X.$$

Since $c \in (0, 1)$ was arbitrary, we get the required inequality. The converse is obvious.

With this observation in mind, we now assume that $\|f\|_{B^{q,\mathcal{B}}(\mu)} < 1$. By the definition of the norm, f can be represented in the form $f = \sum_k \lambda_k\phi_k$, where

$$\|\{\lambda_k\}\|_{\ell^1} \le 1 \tag{2.1}$$

and ϕ_k is a $(q, \mathcal{B})$-block for each $k \in \mathbb{N}$. Assume that each $w_k \in \overline{\mathcal{B}},\ k \in \mathbb{N}$ satisfies

$$\left(\int_\Omega |\phi_k|^q w_k^{1-q}\,\mathrm{d}\mu\right)^{1/q} < 1. \tag{2.2}$$

Define $w := \sum_k |\lambda_k| w_k$. Then, by (2.1) and the condition (ii), we see that $w \in \overline{\mathcal{B}}$. It follows from Hölder's inequality that

$$\begin{aligned} |f|^q &\le \left(\sum_k |\lambda_k|\, |\phi_k| \right)^q \le \left(\sum_k |\lambda_k| w_k \right)^{q-1} \left(\sum_k |\lambda_k| w_k^{1-q} \phi_k^q \right) \\ &= w^{q-1} \left(\sum_k |\lambda_k| \phi_k^q w_k^{1-q} \right), \end{aligned}$$

which implies

$$\int_\Omega |f|^q w^{1-q} \, d\mu \le \sum_k |\lambda_k| \int_\Omega \phi_k^q w_k^{1-q} \, \mathrm{d}\mu \le 1,$$

where we have used (2.1) and (2.2). This means that $\|f\|_{H^{q,\mathcal{B}}(\mu)} \le 1$. Thus, we conclude $\|f\|_{H^{q,\mathcal{B}}(\mu)} \le \|f\|_{B^{q,\mathcal{B}}(\mu)}$. This completes the proof. □

The following theorem generalizes the result due to Izumi, Sato, and Yabuta [8]. For the Fatou property of block spaces, we refer to [14]. As the theorem says, it is (iv) that counts.

Theorem 2.2 *Suppose that $\overline{\mathcal{B}}$ fulfills the condition* (iv). *Then the space $H^{q,\mathcal{B}}(\mu)$ has the Fatou property for every $1 < q < \infty$.*

Proof Let f and f_n, $n \in \mathbb{N}$, belong to $L^0_+(\mu)$ and satisfy $0 \le f_n \uparrow f$ μ-a.e. and

$$\lim_{n \to \infty} \|f_n\|_{H^{q,\mathcal{B}}(\mu)} < 1.$$

Then we can find a sequence $\{w_n\}$ in $\overline{\mathcal{B}}$ such that

$$\left(\int_\Omega f_n^q w_n^{1-q} \, \mathrm{d}\mu \right)^{1/q} < 1.$$

By the condition (iv), extracting a subsequence if necessary and still denoted by $\{w_n\}$, we may assume that

$$\lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^N w_n = w \quad \mu - a.e.\text{and} \quad w \in \overline{\mathcal{B}}. \tag{2.3}$$

Let $k \le n$. Since $0 \le f_k \le f_n$ by assumption, we have

$$\left(\int_\Omega f_k^q w_n^{1-q} \, \mathrm{d}\mu \right)^{1/q} \le \left(\int_\Omega f_n^q w_n^{1-q} \, \mathrm{d}\mu \right)^{1/q} < 1.$$

Since the function t^{1-q} is convex, we have

$$\begin{aligned}\left(\int_\Omega f_k^q\left(\frac{1}{N}\sum_{n=k}^{k+N-1} w_n\right)^{1-q} \mathrm{d}\mu\right)^{1/q} &\le \left(\frac{1}{N}\sum_{n=k}^{k+N-1}\int_\Omega f_k^q w_n^{1-q}\,\mathrm{d}\mu\right)^{1/q}\\ &\le \left(\frac{1}{N}\sum_{n=k}^{k+N-1}\int_\Omega f_n^q w_n^{1-q}\,\mathrm{d}\mu\right)^{1/q}\\ &< 1.\end{aligned}$$

This yields, by the Fatou theorem and (2.3),

$$\left(\int_\Omega f_k^q w^{1-q}\,\mathrm{d}\mu\right)^{1/q} \le 1.$$

Letting $k \to \infty$, we obtain

$$\left(\int_\Omega f^q w^{1-q}\,\mathrm{d}\mu\right)^{1/q} \le 1.$$

Thus, $\|f\|_{H^{q,\mathcal{B}}(\mu)} \le 1$ and this completes the proof. □

We can now state and prove the main theorem on the Köthe duality.

Theorem 2.3 *Suppose that $\overline{\mathcal{B}}$ fulfills the conditions* (i)–(iv). *Let $1 < p < \infty$ and $1 < q < \infty$ be conjugate to each other. Then the following Köthe duality formulas hold with equality of norms*:

$$H^{p,\mathcal{B}}(\mu)' = L^{q,\mathcal{B}}(\mu), \quad \textit{and} \quad L^{p,\mathcal{B}}(\mu)' = H^{q,\mathcal{B}}(\mu).$$

Proof We first observe that for any $f \in L^{q,\mathcal{B}}(\mu)$ and $g \in H^{p,\mathcal{B}}(\mu)$ we have

$$\int_\Omega |fg|\,\mathrm{d}\mu \le \|f\|_{L^{q,\mathcal{B}}(\mu)}\|g\|_{H^{p,\mathcal{B}}(\mu)}. \tag{2.4}$$

Indeed, notice that if $w \in \overline{\mathcal{B}}$ then by the condition (iii), we get

$$\left(\int_\Omega |f|^q w\,\mathrm{d}\mu\right)^{1/q} \le \|f\|_{L^{q,\mathcal{B}}(\mu)}.$$

For a given $\varepsilon > 0$, choose $w \in \overline{\mathcal{B}}$ so that

$$\left(\int_\Omega g^p w^{1-p}\,\mathrm{d}\mu\right)^{1/p} \le \|g\|_{H^{p,\mathcal{B}}(\mu)} + \varepsilon.$$

In this case, $w \neq 0$ μ-a.e. on supp g. Consequently, we obtain

$$\begin{aligned}\int_\Omega |fg|\,\mathrm{d}\mu &= \int_\Omega |f|w^{1/q}\cdot |g|w^{-1/q}\,\mathrm{d}\mu\\ &\le \left(\int_\Omega |f|^q w\,\mathrm{d}\mu\right)^{1/q}\left(\int_\Omega |g|^p w^{1-p}\,\mathrm{d}\mu\right)^{1/p}\\ &\le \|f\|_{L^{q,\mathcal{B}}(\mu)}\left(\|g\|_{H^{p,\mathcal{B}}(\mu)}+\varepsilon\right).\end{aligned}$$

Since ε was arbitrary, (2.4) follows.

Applying the estimate (2.4), we obtain

$$\sup\left\{\int_\Omega |fg|\,\mathrm{d}\mu;\ \|g\|_{H^{p,\mathcal{B}}(\mu)}\le 1\right\}\le \|f\|_{L^{q,\mathcal{B}}(\mu)}.$$

We claim that the converse estimate holds.

By combining the definition of the $L^{q,\mathcal{B}}(\mu)$-norm and the duality we see that, for a given $\varepsilon > 0$, there exist $w_0 \in \mathcal{B}$ and $g_0 \in L^p(\mu)$ with $\|g_0\|_{L^p} \le 1$ such that

$$\|f\|_{L^{q,\mathcal{B}}(\mu)} \le \int_\Omega |fg_0|w_0^{1/q}\,\mathrm{d}\mu + \varepsilon.$$

So we have, by letting $g_1 := |g_0|w_0^{1/q}$,

$$\left(\int_\Omega g_1^p w_0^{1-p}\,\mathrm{d}\mu\right)^{1/p} = \left(\int_\Omega |g_0|^p\,\mathrm{d}\mu\right)^{1/p}\le 1.$$

This means that, by the condition (i), $\|g_1\|_{H^{p,\mathcal{B}}(\mu)} \le 1$ and, hence,

$$\|f\|_{L^{q,\mathcal{B}}(\mu)} \le \sup\left\{\int_\Omega |fg|\,\mathrm{d}\mu;\ \|g\|_{H^{p,\mathcal{B}}(\mu)}\le 1\right\}+\varepsilon,$$

which yields, by letting $\varepsilon \to 0$,

$$\|f\|_{L^{q,\mathcal{B}}(\mu)} \le \sup\left\{\int_\Omega |fg|\,\mathrm{d}\mu;\ \|g\|_{H^{p,\mathcal{B}}(\mu)}\le 1\right\}.$$

This completes the proof of the first formula.

Now observe that the formula we have just proved yields $L^{q,\mathcal{B}}(\mu)' = H^{p,\mathcal{B}}(\mu)''$ with equality of norms. To conclude, it is enough to apply Theorem 2.2, which gives $H^{p,\mathcal{B}}(\mu)'' = H^{p,\mathcal{B}}(\mu)$ with equality of norms (see, e.g., [10, p. 30]). □

3 The End Point Case

In this section, we discuss the endpoint case $q = \infty$. The results hold by the same argument in Sect. 2 with the necessary modifications.

We denote by $H^{\infty,\mathcal{B}}(\mu)$ the space of all $f \in L^0(\mu)$ such that

$$\|f\|_{H^{\infty,\mathcal{B}}(\mu)} := \inf_{w \in \overline{\mathcal{B}}} \|f w^{-1}\|_{L^\infty(\mu)} < \infty.$$

Every function $\phi \in H^{\infty,\mathcal{B}}(\mu)$ with $\|\phi\|_{H^{\infty,\mathcal{B}}(\mu)} < 1$ is said to be an $(\infty, \mathcal{B})$-block.

Define the Banach space $B^{\infty,\mathcal{B}}(\mu)$ which consists of all $f \in L^0(\mu)$ such that

$$\|f\|_{B^{\infty,\mathcal{B}}(\mu)} := \inf \left\{ \left\|\{\lambda_k\}\right\|_{\ell^1};\ f = \sum_k \lambda_k \phi_k \ \mu - \text{a.e.} \right\} < \infty,$$

where each ϕ_k is an $(\infty, \mathcal{B})$-block and the infimum is taken over all possible decompositions of f.

Now we state and prove the following duality result.

Theorem 3.1 *The space $H^{\infty,\mathcal{B}}(\mu)$ is a Banach space with the Fatou property which agrees with $B^{\infty,\mathcal{B}}(\mu)$. Moreover, the following Köthe duality formulas hold with equality of norms*:

$$H^{\infty,\mathcal{B}}(\mu)' = L^{1,\mathcal{B}}(\mu) \quad \textit{and} \quad L^{1,\mathcal{B}}(\mu)' = H^{\infty,\mathcal{B}}(\mu).$$

Proof It is obvious that $B^{\infty,\mathcal{B}}(\mu) \supset H^{\infty,\mathcal{B}}(\mu)$ and $\|f\|_{B^{\infty,\mathcal{B}}(\mu)} \le \|f\|_{H^{\infty,\mathcal{B}}(\mu)}$ for all $f \in H^{\infty,\mathcal{B}}(\mu)$. We prove the converse is true.

Let $f \in B^{\infty,\mathcal{B}}(\mu)$ be such that $\|f\|_{B^{\infty,\mathcal{B}}(\mu)} < 1$. Then we can find a decomposition $f = \sum_k \lambda_k \phi_k$, μ-a.e., where $\{\lambda_k\} \in \ell^1$ satisfies $\|\{\lambda_k\}\|_{\ell^1} < 1$ and each ϕ_k is an $(\infty, \mathcal{B})$-block. By the definition of the block, we can find w_j with $\|\phi_j w_j^{-1}\|_{L^\infty(\mu)} < 1$. Thus, by setting $w = \sum_k |\lambda_k| w_k$, we obtain $w \in \overline{\mathcal{B}}$ from the condition (ii) and

$$|f| w^{-1} \le \sum_k |\lambda_k \phi_k| w^{-1} \le \sum_k |\lambda_k| w_k w^{-1} = 1.$$

Hence, $H^{\infty,\mathcal{B}}(\mu) \supset B^{\infty,\mathcal{B}}(\mu)$ and $\|f\|_{H^{\infty,\mathcal{B}}(\mu)} \le \|f\|_{B^{\infty,\mathcal{B}}(\mu)}$ for all $f \in B^{\infty,\mathcal{B}}(\mu)$.

Suppose that we are given a non-negative increasing sequence $\{f_j\}_{j \in \mathbb{N}}$ which is bounded in $H^{\infty,\mathcal{B}}(\mu)$. Let $A > 0$ be such that

$$\lim_{j \to \infty} \|f_j\|_{H^{\infty,\mathcal{B}}(\mu)} < A.$$

Then, by the definition of the norm, we can find $\{w_j\}$ in $\overline{\mathcal{B}}$ such that

$$\|f_j\|_{H^{\infty,\mathcal{B}}(\mu)} \le \|f_j w_j^{-1}\|_{L^\infty(\mu)} < A.$$

By the condition (iv), we may assume that

$$w = \lim_{N \to \infty} \frac{w_1 + w_2 + \cdots + w_N}{N}$$

exists μ-a.e. and $w \in \overline{\mathcal{B}}$. Since $0 \leq f_j \leq f_{j+1}$, we have

$$\|f_j w_k^{-1}\|_{L^\infty(\mu)} \leq \|f_k w_k^{-1}\|_{L^\infty(\mu)} < A$$

for all $j \leq k$. By using the inequality

$$\frac{N}{a_1 + a_2 + \cdots + a_N} \leq \frac{1}{N}\left(\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_N}\right), \quad a_1, a_2, \ldots, a_N > 0,$$

we obtain

$$\left\| f_j \left(\frac{w_j + w_{j+1} + \cdots + w_{j+N-1}}{N}\right)^{-1} \right\|_{L^\infty(\mu)} \leq \frac{1}{N}\sum_{k=1}^{N} \|f_j w_{j+k-1}^{-1}\|_{L^\infty(\mu)} < A.$$

Letting $N \to \infty$ and then $j \to \infty$, we obtain $\|f w^{-1}\|_{L^\infty(\mu)} \leq A$. This means that $f \in H^{\infty,\mathcal{B}}(\mu)$ and $\|f\|_{H^{\infty,\mathcal{B}}(\mu)} \leq \lim_{j\to\infty} \|f_j\|_{H^{\infty,\mathcal{B}}(\mu)}$. The reverse is obvious.

Now let $f \in L^{1,\mathcal{B}}(\mu)$. We note that for any $g \in H^{\infty,\mathcal{B}}(\mu)$ with the norm less than 1, we can find $w_0 \in \overline{\mathcal{B}}$ such that $|g| \leq w_0$ μ-a.e., thus, we have from the condition (iii)

$$\|fg\|_{L^1(\mu)} \leq \|f w_0\|_{L^1(\mu)} \leq \|f\|_{L^{1,\mathcal{B}}}.$$

This means that

$$\|f\|_{H^{\infty,\mathcal{B}}(\mu)'} \leq \|f\|_{L^{1,\mathcal{B}}(\mu)}.$$

But, we have the reverse inequality, since $b_j \in H^{\infty,\mathcal{B}}(\mu)$ has the norm less than 1 for any $j \in \mathbb{N}$. Thus,

$$\|f\|_{H^{\infty,\mathcal{B}}(\mu)'} = \|f\|_{L^{1,\mathcal{B}}(\mu)}$$

and, hence, we conclude that $H^{\infty,\mathcal{B}}(\mu)' = L^{1,\mathcal{B}}(\mu)$ with norm coincidence. Thanks to the Fatou property of the space $H^{\infty,\mathcal{B}}(\mu)$, we obtain $H^{\infty,\mathcal{B}}(\mu) = H^{\infty,\mathcal{B}}(\mu)'' = L^{1,\mathcal{B}}(\mu)'$. This completes the proof. □

4 Examples of the Class $\overline{\mathcal{B}}$

In this section, we provide examples of the sets $\overline{\mathcal{B}}$. We will need the following Komlós theorem (see [11, Theorem1a]) which states: *If* (Ω, Σ, μ) *is a measure space, then for every bounded sequence* $\{f_n\}$ *in* $L^1(\mu)$ *there are* $f \in L^1(\mu)$ *and a subsequence* $\{g_n\}$ *of* $\{f_n\}$ *such that the sequence of arithmetic means* $\left\{\frac{1}{n}\sum_{k=1}^{n} g_k\right\}_n$ *converges to* f *almost everywhere. Moreover, the conclusion remains true for every subsequence of* $\{g_n\}$.

Since for every Banach lattice X on (Ω, Σ, μ), the inclusion $X \hookrightarrow X''$ has a norm less than or equal to one, it follows that $\int_\Omega xw \, \mathrm{d}\mu \le \|x\|_X$, for any $x \in X$ and $w \in X'$ with $\|w\|_{X'} \le 1$. So, $X \hookrightarrow L^1(\nu)$ with $d\nu = w \, \mathrm{d}\mu$. Here, we a priori assume that there exists $w > 0$ μ-a.e. such that $w \in X'$ and $\|w\|_{X'} \le 1$, which is a consequence of the fact that there exists $u > 0$ μ-a.e. such that $u \in X$ and $\|u\|_X \le 1$.

This simple observation allows us to apply the Komlós theorem for any bounded sequence in X.

We now state a simple example of the class $\overline{\mathcal{B}}$ in the setting of the Lebesgue measure space $(\mathbb{R}^n, \mathrm{d}x)$.

We denote by $\mathcal{D}$ the family of all dyadic cubes of the form $Q = 2^{-k}(m + [0, 1)^n)$, $k \in \mathbb{Z}$, $m \in \mathbb{Z}^n$. Let $1 < p < \infty$, $0 < \lambda < n$. Let $b_Q := 1_Q / l(Q)^\lambda$, $Q \in \mathcal{D}$, and define $\mathcal{B} := \{b_Q\}_{Q \in \mathcal{D}}$. Then one sees that $L^{p,\lambda}(\mathbb{R}^n) = L^{p,\mathcal{B}}(\mathrm{d}x)$ with the equivalence of norms. Let

$$\overline{\mathcal{B}} := \left\{ w = \sum_Q c_Q b_Q ; \; c_Q \ge 0 \text{ and } \|\{c_Q\}\|_{\ell^1} \le 1 \right\}.$$

It is easy to see that the class $\overline{\mathcal{B}}$ satisfies the conditions (i)–(iii). We shall verify that $\overline{\mathcal{B}}$ satisfies the condition (iv) as well.

First of all, we notice that $\|b_Q\|_{L^{n/\lambda}} = 1$ for all $Q \in \mathcal{D}$ and $\|w\|_{L^{n/\lambda}} \le 1$ for all $w \in \overline{\mathcal{B}}$. Suppose that $\{w_j\} \subset \overline{\mathcal{B}}$. Since $L^{n/\lambda}(\mathbb{R}^n)$ is a Banach function space, by the Komlós theorem, extracting a subsequence if necessary and still denoted by $\{w_j\}$, we may assume that

$$\lim_{k \to \infty} \frac{1}{k} \sum_{j=1}^{k} w_j = w \quad \text{a.e.}$$

with $\|w\|_{L^{n/\lambda}} \le 1$. We have to show $w \in \overline{\mathcal{B}}$. To this end, we write

$$v_k := \frac{1}{k} \sum_{j=1}^{k} w_j = \sum_{Q \in \mathcal{D}} c_{k,Q} b_Q$$

with $c_{k,Q} \ge 0$ and $\|\{c_{k,Q}\}_Q\|_{\ell^1} \le 1$. We apply a diagonalization argument and, hence, extracting a subsequence if necessary and still denoted by $\{v_k\}$, we may assume that

$$\lim_{k \to \infty} v_k = w \quad \text{a.e.} \tag{4.1}$$

and, for all $Q \in \mathcal{D}$,

$$\lim_{k \to \infty} c_{k,Q} = c_Q. \tag{4.2}$$

Define

$$w_0 := \sum_{Q\in\mathcal{D}} c_Q b_Q.$$

Then by the Fatou theorem for ℓ^1, we have

$$\|\{c_Q\}_Q\|_{\ell^1} \le \liminf_{k\to\infty} \|\{c_{k,Q}\}_Q\|_{\ell^1} \le 1,$$

which means $w_0 \in \overline{\mathcal{B}}$. Thus, we need only verify that $w = w_0$ a.e.

We show that, for any $Q_0 \in \mathcal{D}$,

$$\lim_{k\to\infty} \int_{Q_0} v_k(x)\,\mathrm{d}x = \int_{Q_0} w_0(x)\,\mathrm{d}x. \tag{4.3}$$

We recall the well-known fact that in the Lebesgue space $L^p(\mathbb{R}^n)$, $1 < p < \infty$, convergence a.e. implies weak convergence if the norms are uniformly bounded (cf. [4]). By the use of this fact, once (4.3) is established, (4.1) and the fact that $v_k \in \overline{\mathcal{B}}$ yield, for all $Q_0 \in \mathcal{D}$,

$$\int_{Q_0} w(x)\,\mathrm{d}x = \int_{Q_0} w_0(x)\,\mathrm{d}x,$$

which implies $w = w_0$ a.e. by virtue of the Lebesgue differentiation theorem.

We notice that, for any $Q \in \mathcal{D}$ with $Q \cap Q_0 \ne \emptyset$,

$$\int_{Q_0} \frac{1_Q(x)}{l(Q)^\lambda}\,\mathrm{d}x = \begin{cases} l(Q)^{n-\lambda}, & \text{when} Q \subset Q_0, \\ l(Q_0)^n l(Q)^{-\lambda}, & \text{otherwise.} \end{cases}$$

Let $\varepsilon > 0$ be given. We set

$$\begin{cases} \mathcal{D}_1(Q_0) := \left\{Q \in \mathcal{D};\ \ Q \cap Q_0 \ne \emptyset,\ \ l(Q)^{n-\lambda} \le \varepsilon/6\right\}, \\ \mathcal{D}_2(Q_0) := \left\{Q \in \mathcal{D};\ \ Q \cap Q_0 \ne \emptyset,\ \ l(Q_0)^n l(Q)^{-\lambda} \le \varepsilon/6\right\}, \\ \mathcal{D}_3(Q_0) := \left\{Q \in \mathcal{D};\ \ Q \cap Q_0 \ne \emptyset\right\} \setminus (\mathcal{D}_1(Q_0) \cup \mathcal{D}_2(Q_0)). \end{cases}$$

It follows that

$$\sum_{Q\in\mathcal{D}_1(Q_0)} \int_{Q_0} |c_{k,Q} - c_Q| b_Q(x)\,\mathrm{d}x \le \sum_{Q\in\mathcal{D}_1(Q_0)} (c_{k,Q} + c_Q) l(Q)^{n-\lambda} \le \frac{\varepsilon}{3}. \tag{4.4}$$

It follows also that

$$\sum_{Q\in\mathcal{D}_2(Q_0)} \int_{Q_0} |c_{k,Q} - c_Q| b_Q(x)\,\mathrm{d}x \le \sum_{Q\in\mathcal{D}_2(Q_0)} (c_{k,Q} + c_Q) \frac{l(Q_0)^n}{l(Q)^\lambda} \le \frac{\varepsilon}{3}. \tag{4.5}$$

Finally,

$$\sum_{Q\in\mathcal{D}_3(Q_0)}\int_{Q_0}|c_{k,Q}-c_Q|b_Q(x)\,dx \le l(Q_0)^{n-\lambda}\sum_{Q\in\mathcal{D}_3(Q_0)}|c_{k,Q}-c_Q|. \quad (4.6)$$

From (4.2) and the fact that $\mathcal{D}_3(Q_0)$ contains the only finite number of dyadic cubes, the right-hand side of (4.6) can be majorized by $\varepsilon/3$ for large k. (4.4)–(4.6) prove (4.3).

We conclude this section with some related examples.

Example 4.1 Let $1 < p < \infty$ and $0 < \lambda < n$. For $f \in L^p_{\mathrm{loc}}(\mathbb{R}^n)$, it define

$$\|f\|_{L^{p,\lambda}_{\mathrm{loc}}} := \sup_{Q\in\mathcal{Q};\ l(Q)\le 1}\left(\frac{1}{l(Q)^\lambda}\int_Q |f(x)|^p\,dx\right)^{1/p}.$$

The local Morrey space $L^{p,\lambda}_{\mathrm{loc}}(\mathbb{R}^n)$ is defined to be the subset of all $L^p(\mathbb{R}^n)$-locally integrable functions f on $\mathbb{R}^n$ for which $\|f\|_{L^{p,\lambda}_{\mathrm{loc}}}$ is finite. Let $b_Q = 1_Q/l(Q)^\lambda$, $Q \in \mathcal{D}$, as before, and define $\mathcal{B} := \{b_Q\}_{Q\in\mathcal{D};\ l(Q)\le 1}$. Then we have a similar conclusion as before.

Example 4.2 Let $1 < p < \infty$. For $f \in L^p_{\mathrm{loc}}(\mathbb{R}^n)$, it define

$$\|f\|_{L^p_{\mathrm{amalgam}}} := \sup_{Q\in\mathcal{Q};\ l(Q)=1}\left(\int_Q |f(x)|^p\,dx\right)^{1/p}.$$

The amalgam space $L^p_{\mathrm{amalgam}}(\mathbb{R}^n)$ is defined to be the subset of all $L^p(\mathbb{R}^n)$-locally integrable functions f on $\mathbb{R}^n$ for which $\|f\|_{L^p_{\mathrm{amalgam}}}$ is finite. Let $\mathcal{D}$ as before, and define

$$\mathcal{B} := \{1_Q\}_{Q\in\mathcal{D};\ l(Q)=1}.$$

Then we have a similar conclusion as before.

Example 4.3 Let $1 < p < \infty$ and $\sigma > 0$. Denote by $B(r) \subset \mathbb{R}^n$ the open ball centered at the origin. For $f \in L^p_{\mathrm{loc}}(\mathbb{R}^n)$, it define

$$\|f\|_{B^\sigma_p} := \sup_{k\in\mathbb{N}}\left(2^{-kp\sigma}\int_{B(2^k)}|f(x)|^p\,dx\right)^{1/p}$$

and

$$\|f\|_{\dot{B}^\sigma_p} := \sup_{k\in\mathbb{Z}}\left(2^{-kp\sigma}\int_{B(2^k)}|f(x)|^p\,dx\right)^{1/p}.$$

The inhomogeneous B^σ space $B^\sigma_p(\mathbb{R}^n)$ and the homogeneous $\dot{B}^\sigma$ space $\dot{B}^\sigma_p(\mathbb{R}^n)$ are defined to be the subsets of all $L^p(\mathbb{R}^n)$-locally integrable functions f on $\mathbb{R}^n$ for which $\|f\|_{B^\sigma_p}$ and $\|f\|_{\dot{B}^\sigma_p}$ are finite, respectively. Let $b_k = 2^{-kp\sigma}1_{B(2^k)}$, $k \in \mathbb{Z}$, and define $\mathcal{B} := \{b_k\}_{k\in\mathbb{N}}$ and $\dot{\mathcal{B}} := \{b_k\}_{k\in\mathbb{Z}}$. Then we have a similar conclusion as before. See [12] for a discussion of $B^\sigma_p(\mathbb{R}^n)$ and $\dot{B}^\sigma_p(\mathbb{R}^n)$.

Example 4.4 Our method will work for Mock Morrey spaces considered in [1]. Let $\mathcal{P}$ be the class of all compact subsets of $\mathbb{R}^n$ and let $S\colon \mathcal{P} \to [0,\infty)$ be a non-negative set function on $\mathcal{P}$ such that, if $Q \in \mathcal{Q}(\subset \mathcal{P})$, $S(Q) \approx l(Q)^\sigma$ for some $\sigma > 0$. Let $1 < p < \infty$ and $0 < \lambda < n$. For $f \in L^p_{\mathrm{loc}}(\mathbb{R}^n)$, define

$$\|f\|_{M^{p,\lambda}_S} := \sup_{E\in\mathcal{P}} \left(\frac{1}{S(E)^{\lambda/\sigma}} \int_E |f(x)|^p \, \mathrm{d}x\right)^{1/p}.$$

The Mock Morrey space $M^{p,\lambda}_S(\mathbb{R}^n)$ is defined to be the subset of all $L^p(\mathbb{R}^n)$-locally integrable functions f on $\mathbb{R}^n$ for which $\|f\|_{M^{p,\lambda}_S}$ is finite. Assume that $\{E_j\}_{j\in\mathbb{N}} \subset \mathcal{P}$ satisfies the following:

(i) For all $E \in \mathcal{P}$, there exists j such that $E \subset E_j$ and $2S(E) \geq S(E_j)$;
(ii) For all $\varepsilon > 0$ and all $Q \in \mathcal{Q}$, $|E_j \cap Q| < \varepsilon S(E_j)^{\lambda/\sigma}$ with the possible exception of a finite number.

Then we have a similar conclusion as before.

5 Concluding Remarks

Let $1 \leq p < \infty$ and $0 < \lambda < n$. In this section, we introduce the class $\overline{\mathcal{B}^1}$ which is smaller than the class $\overline{\mathcal{B}_\lambda}$ defined in Introduction but plays the same role in the definition of the Köthe dual of the Morrey space $L^{p,\lambda}(\mathbb{R}^n)$. We shall need the following two lemmas (see [13,15, Lemmas 1 and 3]).

Lemma 5.1 *Let $0 < \alpha < n$ and $p > \alpha/n$. Then, for some constant C depending only on α, n and p,*

$$\int_{\mathbb{R}^n} M[1_Q]^p \, \mathrm{d}H^\alpha \leq C l(Q)^\alpha.$$

Here M denotes the Hardy-Littlewood maximal operator defined for every $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ by

$$Mf(x) := \sup_{Q\in\mathcal{Q}} \left(\frac{1}{|Q|}\int_Q |f(y)|\, \mathrm{d}y\right) 1_Q(x), \quad x \in \mathbb{R}^n.$$

Lemma 5.2 *For $f \geq 0$, we have*

$$\int_{\mathbb{R}^n} f(x)\,\mathrm{d}x \leq \frac{n}{\alpha}\left(\int_{\mathbb{R}^n} f(x)^{\alpha/n}\,\mathrm{d}H^\alpha(x)\right)^{n/\alpha}.$$

Fix $0 < \lambda < \lambda_0 < n$. For $Q \in \mathcal{D}$, let $b_Q = M[1_Q]^{\lambda_0/n}/l(Q)^\lambda$ and define $\mathcal{B}^1 := \{b_Q\}_{Q\in\mathcal{D}}$. Let

$$\overline{\mathcal{B}^1} := \left\{ w = \sum_Q c_Q b_Q;\ c_Q \geq 0 \text{ and } \|\{c_Q\}\|_{\ell^1} \leq 1 \right\}.$$

It is well-known that for non-negative functions f_j one has

$$\int_{\mathbb{R}^n} \sum_{j=1}^{\infty} f_j \, \mathrm{d}H^\lambda \leq C \sum_{j=1}^{\infty} \int_{\mathbb{R}^n} f_j \, \mathrm{d}H^\lambda \tag{5.1}$$

for some constant C depending only on λ and n (cf. [13]). We recall that, for every dyadic cube $Q \in \mathcal{D}$,

$$\int_{\mathbb{R}^n} \frac{M[1_Q]^{\lambda_0/n}}{l(Q)^\lambda} \, \mathrm{d}H^\lambda \leq C \tag{5.2}$$

by Lemma 5.1. It follows from (5.1) and (5.2) that if $w \in \overline{\mathcal{B}^1}$ then $\int_{\mathbb{R}^n} w \, dH^\lambda \leq C$. We also observe that every b_Q belongs to the Muckenhoupt class A_1 and has the uniform bound of $[b_Q]_{A_1}$, $Q \in \mathcal{D}$, (cf. [5, Chapter II]). Thus, we see that $w \in A_1$ whenever $w \in \overline{\mathcal{B}^1}$. Consequently, we have $c^{-1}\overline{\mathcal{B}^1} \subset \overline{\mathcal{B}_\lambda}$ for some appropriate constant $c > 0$.

We conclude this paper with the following:

Proposition 5.3 *Let* $1 \leq p < \infty$ *and* $0 < \lambda < \lambda_0 < n$.

(1) $L^{p,\lambda}(\mathbb{R}^n) = L^{p,\overline{\mathcal{B}^1}}(dx)$ *with the equivalence of norms.*
(2) $\overline{\mathcal{B}^1}$ *satisfies the conditions (i)–(iv) in Sect. 2.*

Proof We first prove (1). Observe that for any $Q \in \mathcal{Q}$ and any $f \in L^{p,\lambda}(\mathbb{R}^n)$, we have

$$\int_Q |f(x)|^p \, \mathrm{d}x \leq \|f\|^p_{L^{p,\lambda}(\mathbb{R}^n)} l(Q)^\lambda.$$

This implies that for any measurable set E and for any family of countable cubes $\{Q_j\} \subset \mathcal{Q}$ such that $E \subset \bigcup_j Q_j$, we have

$$\int_E |f(x)|^p \, \mathrm{d}x \leq \sum_j \int_{Q_j} |f(x)|^p \, \mathrm{d}x \leq \|f\|^p_{L^{p,\lambda}(\mathbb{R}^n)} \sum_j l(Q_j)^\lambda \tag{5.3}$$

and, hence,

$$\int_E |f(x)|^p \, \mathrm{d}x \leq \|f\|^p_{L^{p,\lambda}(\mathbb{R}^n)} H^\lambda(E). \tag{5.4}$$

Combining (5.3) and (5.4) yields that for any Lebesgue measurable function $\phi \geq 0$ and any $f \in L^{p,\lambda}(\mathbb{R}^n)$,

$$\int_{\mathbb{R}^n} |f(x)|^p \phi(x)\,\mathrm{d}x = \int_0^\infty \left(\int_{\{y\in\mathbb{R}^n;\ \phi(y)>t\}} |f(x)|^p\,\mathrm{d}x \right) \mathrm{d}t \\ \leq \|f\|^p_{L^{p,\lambda}(\mathbb{R}^n)} \int_{\mathbb{R}^n} \phi\,\mathrm{d}H^\lambda.$$

This implies with (5.2) that

$$\|f\|_{L^{p,\mathcal{B}^1}(\mathrm{d}x)} = \sup_{b\in\mathcal{B}^1} \left(\int_{\mathbb{R}^n} |f(x)|^p b(x)\,\mathrm{d}x \right)^{1/p} \leq C\|f\|_{L^{p,\lambda}(\mathbb{R}^n)}.$$

For the reverse inequality, for every cube $Q \in \mathcal{D}$,

$$\frac{1}{l(Q)^\lambda} \int_Q |f(x)|^p\,\mathrm{d}x = \frac{1}{l(Q)^\lambda} \int_{\mathbb{R}^n} |f(x)|^p 1_Q(x)\,\mathrm{d}x \\ \leq \frac{1}{l(Q)^\lambda} \int_{\mathbb{R}^n} |f(x)|^p M[1_Q](x)^{\lambda_0/n}\,\mathrm{d}x \\ \leq \sup_{b\in\mathcal{B}^1} \int_{\mathbb{R}^n} |f(x)|^p b(x)\,\mathrm{d}x.$$

Next we prove (2). It is easy to see that the class $\overline{\mathcal{B}^1}$ satisfies the conditions (i)–(iii). We shall verify that $\overline{\mathcal{B}^1}$ satisfies the condition (iv) as well.

We use the same argument in Sect. 4 and use the same notations as well. First we notice that, by Lemma 5.2, $\|b_Q\|_{L^{n/\lambda}} \leq C$ for all $Q \in \mathcal{D}$ and $\|w\|_{L^{n/\lambda}} \leq C$ for all $w \in \overline{\mathcal{B}^1}$. So, we need only verify that (4.3) holds for any $Q_0 \in \mathcal{D}$.

Let $\lambda_1 = (\lambda + \lambda_0)/2$. Then, for any $Q \in \mathcal{D}$ with $l(Q) \leq l(Q_0)$,

$$\frac{1}{l(Q)^\lambda} \int_{Q_0} M[1_Q](x)^{\lambda_0/n}\,\mathrm{d}x \leq \frac{1}{l(Q)^\lambda} |Q_0|^{1-\lambda_1/n} \left(\int_{\mathbb{R}^n} M[1_Q](x)^{\lambda_0/\lambda_1}\,\mathrm{d}x \right)^{\lambda_1/n} \\ \leq \frac{1}{l(Q)^\lambda} |Q_0|^{1-\lambda_1/n} \big(Cl(Q)^n\big)^{\lambda_1/n} =: c_1 |Q_0|^{1-\lambda_1/n} l(Q)^{\lambda_1-\lambda},$$

where we have used the L^{λ_0/λ_1}-boundedness of M. While, for any $Q \in \mathcal{D}$ with $l(Q) > l(Q_0)$,

$$\frac{1}{l(Q)^\lambda} \int_{Q_0} M[1_Q](x)^{\lambda_0/n}\,\mathrm{d}x \leq l(Q_0)^n l(Q)^{-\lambda},$$

where we have used $M[1_Q] \leq 1$.

Let $\varepsilon > 0$ be given. We set

$$\begin{cases} \mathcal{D}_1(Q_0) := \{Q \in \mathcal{D};\ l(Q) \le l(Q_0),\ c_1|Q_0|^{1-\lambda_1/n} l(Q)^{\lambda_1-\lambda} < \varepsilon/6\}, \\ \mathcal{D}_2(Q_0) := \{Q \in \mathcal{D};\ l(Q) > l(Q_0),\ l(Q_0)^n l(Q)^{-\lambda} < \varepsilon/6\}, \\ \mathcal{D}_3(Q_0) := \mathcal{D} \setminus (\mathcal{D}_1(Q_0) \cup \mathcal{D}_2(Q_0)). \end{cases}$$

We set further

$$\begin{cases} \mathcal{D}_3^-(Q_0) := \{Q \in \mathcal{D}_3(Q_0);\ \int_{Q_0} b_Q(x)\,\mathrm{d}x < \varepsilon/12\}, \\ \mathcal{D}_3^+(Q_0) := \mathcal{D}_3(Q_0) \setminus \mathcal{D}_3^-(Q_0). \end{cases}$$

We claim that $\mathcal{D}_3^+$ is a finite set. Indeed, the quantity $\int_{Q_0} b_Q(x)\,dx$ is uniformly small whenever the dyadic cubes Q are away from the fixed dyadic cube Q_0 and $l(Q)$ are uniformly bounded from above and below.

The estimates for $\mathcal{D}_1$ and $\mathcal{D}_2$ remain unchanged [see (4.4) and (4.5)]. The estimate (4.6) for $\mathcal{D}_3$ is decomposed as follows:

$$\sum_{Q \in \mathcal{D}_3^-(Q_0)} \int_{Q_0} |c_{k,Q} - c_Q| b_Q(x)\,\mathrm{d}x \le \sum_{Q \in \mathcal{D}_3^-(Q_0)} (c_{k,Q} + c_Q) \int_{Q_0} b_Q(x)\,\mathrm{d}x \le \frac{\varepsilon}{6}$$

and

$$\sum_{Q \in \mathcal{D}_3^+(Q_0)} \int_{Q_0} |c_{k,Q} - c_Q| b_Q(x)\,\mathrm{d}x \le c_1 l(Q_0)^{n-\lambda} \sum_{Q \in \mathcal{D}_3^+(Q_0)} |c_{k,Q} - c_Q|. \quad (5.5)$$

From (4.2) and the fact that $\mathcal{D}_3^+(Q_0)$ contains the only finite number of dyadic cubes, the right-hand side of (5.5) can be majorized by $\varepsilon/6$ for large k.

With these modifications, we can check that $w \in \overline{\mathcal{B}^1}$ and finish the proof. □

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