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Fourier Spectral Approximation to Global Attractor for 2D Convective Cahn–Hilliard Equation

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Abstract In this paper, we consider the long-time behavior of the convective Cahn–Hilliard equation in 2D case. A linear fully discrete Galerkin–Fourier spectral approximation scheme is constructed and the existence of the global attractors of the discrete system is obtained. All results in this paper are obtained without any restriction on the time step size.

Keywords Fourier spectral methods · Convective Cahn–Hilliard equation · Global attractor · Convergence

Mathematics Subject Classification 35K35 · 65M60 · 65N35 · 65N30

1 Introduction

With the development of studies in the field of infinite-dimensional dynamical systems, the long-time behavior of solutions for nonlinear dissipative evolution equations has attracted more and more attention of scientists. As is known to all, the long-time dynamics of a dissipative systems is completely determined by the attractor of the equation. Hence, the existence and structure of the attractor are the most important characteristics for evolution equations. There are many papers related to the global

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attractor for dissipative nonlinear evolution equations, such as Navier–Stokes equation, Kuramoto–Sivashinsky equation, Cahn–Hilliard equation (see [1–6]).

However, the study of long-time behavior for nonlinear evolution equations depended on the results of numerical experimentation to a great extent. For this reason, it is worth studying whether the numerical results are reliable and the calculation schemes are suitable. During the past years, many authors have paid much attention to this problem. For example, in [7], based on Galerkin approximations, Hale, Lin, and Raugle studied the approximate system of a given evolutionary equation as a compact attractor which converges to the original one as the approximation is refined. Furthermore, using finite element approximation, Marion and Temam [8], Elliott and Larsson [9] studied the numerical approximation to attractor for some nonlinear evolution equations. Eden, Michaux, and Rakotoson [10] considered a time discretization of a doubly nonlinear parabolic type equations by the Euler forward scheme. They proved the existence of a compact attractor and estimated its Hausdorff dimension using CFT theory. Lü and Lu [11,12] studied the dynamical properties of the discrete systems of Ginzburg–Landau type equation and generalized KdV–Burgers equation. They constructed the fully discrete scheme, proved the existence, convergence of global attractors of the discrete systems. There is much literature concerned with the approximation to global attractor for evolution equations, for more recent results we refer the reader to [13–20] and the references therein.

The convective Cahn–Hilliard equation, which arises naturally as a continuous model for the formation of facets and corners in crystal growth (see [21,22]), is a kind of important nonlinear equations. For this reason, the research of it is of theoretical and practice significance. In this article, based on Fourier spectral approximation, we study the periodic initial value problem of the 2D convective Cahn–Hilliard equation

$$\frac{\partial u}{\partial t} + \gamma \Delta^2 u - \Delta f(u) - \nabla \cdot g(u) = 0, \quad x = (x_1, x_2) \in \mathbb{R}^2, \quad t > 0, \tag{1.1}$$

$$u(x_1 + 2\pi, x_2, t) = u(x_1, x_2 + 2\pi, t) = u(x_1, x_2, t), \quad x \in \mathbb{R}^2, \quad t \geq 0, \tag{1.2}$$

$$u(x, 0) = u_0(x), \quad x \in \mathbb{R}^2. \tag{1.3}$$

In the following, we first construct a fully discrete Fourier spectral approximation scheme, which is a linear scheme. Then the existence and the convergence of approximate attractors, as well as the stability of discrete scheme, are proved. Throughout this paper, we use the following notation: $\Omega = [0, 2\pi] \times [0, 2\pi]$; $(\cdot, \cdot)$ denotes the inner product of $L^2(\Omega)$, $\|\cdot\|_m$ the norm of $L^m(\Omega)$, and $\|\cdot\| = \|\cdot\|_2$, $\|\cdot\|_\infty = \|\cdot\|_{L^\infty(\Omega)}$. On the other hand, we point out one basic fact about problem (1.1)–(1.3): the spatial average of any solution u is preserved. Indeed,

$$\frac{\partial}{\partial t} \int_\Omega u(x, t) dx = 0, \quad \int_\Omega u(x, t) dx = \int_\Omega u_0 dx, \quad \forall t > 0.$$

We assume that the initial function satisfies $\int_\Omega u_0 dx = m_0$.

For any given positive integer N, $j = (j_1, j_2)$, $j \cdot x = j_1 x_1 + j_2 x_2$, let $S_N = \text{span}\{e^{ij\cdot x} : |j| \leq N\}$, where $|j| = \max\{|j_1|, |j_2|\}$. Denote by $P_N : L_p^2(\Omega) \to S_N$

the orthogonal projection operator(see [27]). Let τ be the mesh size in the variable t, $t_k = k\tau$, $u^k = u(x, t_k)$, $\bar{\partial}_t u^k = \frac{1}{\tau}(u^k - u^{k-1})$. The Fourier spectral scheme for solving (1.1)–(1.3) is to find $u_N^k \in S_N$ such that

$$\left(\bar{\partial}_t u_N^k + \gamma \Delta^2 u_N^k, \varphi\right) + \left(f'(u_N^{k-1})\nabla u_N^k, \nabla\varphi\right) - B\left(u_N^{k-1}, u_N^k, \varphi\right) = 0, \quad \forall\varphi \in S_N, \tag{1.4}$$

$$u_N^0 = P_N u_0. \tag{1.5}$$

where

$$B\left(u_N^{k-1}, u_N^k, \varphi\right) = \left(G(u_N^{k-1})\varphi, \nabla \cdot u_N^k\right) - \left(G(u_N^{k-1})u_N^k, \nabla \cdot \varphi\right),$$

and

$$G(u) = \begin{cases} \frac{1}{u^2}\int_0^u s g'(s) ds, & \text{for } u \neq 0, \\ \frac{1}{2} g'(0), & \text{for } u = 0. \end{cases}$$

It is a linear iteration scheme, and then it needs only to solve a class of linear algebraic equations for every iteration.

Remark 1.1 The main tool for studying the numerical approximation to long-time behavior for evolution equations is nonlinear Galerkin methods. The nonlinear Galerkin methods sometimes can be linear schemes, but the dimension of which is many times that of the nonlinear Galerkin ones: consequently, the computation amount is very heavy. Here, we construct a linear full discrete Galerkin spectral scheme. In comparison with the nonlinear Galerkin methods with linear schemes or classical Galerkin methods with nonlinear schemes, the computation amount of this scheme can be greatly reduced. In addition, without any restriction on the time step size, the results about the uniform stability and convergence of this discrete scheme are obtained.

Remark 1.2 To be more specific in this paper, we restrict ourselves to study the Fourier spectral approximation to global attractor for the convective Cahn–Hilliard equation, which is a kind of important nonlinear equations. More generally, similar schemes and analysis are applicable to other higher-order nonlinear evolution equations, for example, molecular beam epitaxy model [23,24], viscous Cahn–Hilliard equation [25,26], and so on.

The following lemmas are useful in our analysis.

Lemma 1.3 (see [27]) *If $u \in H_p^m(\Omega)$, then there exists a constant c independent of u, N such that*

$$\|u - P_N u\|_\mu \leq cN^{\mu-m}\|D^m u\|, \quad \forall 0 \leq \mu \leq m.$$

Lemma 1.4 (Poincaré inequality(see [28])) *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain and $\|\cdot\|$ be the norm of $L^2(\Omega)$, then $\forall v \in H^1(\Omega)$, we have*

$$\|v\|^2 \leq \frac{|\Omega|^2}{2}\|Dv\|^2 + \frac{1}{|\Omega|}\left(\int_\Omega v dx\right)^2, \quad n = 1,$$
$$\|v\|^2 \leq C(\Omega)\left[\|Dv\|^2 + \left(\int_\Omega v(x)dx\right)^2\right], \quad n \geq 2.$$

Lemma 1.5 (Sobolev's interpolation inequality [29]) *Suppose that $u \in L^q(\Omega)$, $D^m u \in L^r(\Omega)$, $\Omega \subset \mathbb{R}^n$, $1 \leq r \leq \infty$, $0 \leq j \leq m$. Then there exists a constant $c = c(j, m, \Omega, p, q, r)$ independent of u such that*

$$\|D^j u\|_{L^p} \leq c\|D^m u\|^a_{W^{m,r}}\|u\|^{1-a}_{L^q},$$

where

$$\frac{1}{p} = \frac{j}{n} + a\left(\frac{1}{r} - \frac{m}{n}\right) + (1-a)\frac{1}{q}, \quad \frac{j}{m} < a < 1.$$

Lemma 1.6 (Discrete Gronwall's inequality [10,12,13]) *Let y^k, g^k, and h^k be three series satisfying*

$$\frac{y^{k+1} - y^k}{\tau} \leq g^k y^k + h^k, \quad k = 0, 1, 2, \ldots$$

Hence, we have

$$y^n \leq y^0 \exp\left(\tau\sum_{k=0}^{n-1} g^k\right) + \tau\sum_{k=0}^{n-1} h^k \exp\left(\tau\sum_{i=k}^{n-1} g^i\right), \quad \forall n \geq 1.$$

Lemma 1.7 (Discrete uniform Gronwall's inequality [12,13]) *Let y^k, g^k, and h^k be three series satisfying*

$$\frac{y^{k+1} - y^k}{\tau} \leq g^k y^k + h^k, \quad \forall k \geq k_0,$$

and

$$\tau\sum_{k=k_1}^{n_0+k_1} g^k \leq \alpha_1, \quad \tau\sum_{k=k_1}^{n_0+k_1} h^k \leq \alpha_2, \quad \tau\sum_{k=k_1}^{n_0+k_1} y^k \leq \alpha_3, \quad \forall k_1 \geq k_0,$$

with $\tau n_0 = r$. Hence, we have

$$y^k \leq \left(\frac{\alpha_3}{r} + \alpha_2\right)e^{\alpha_1}, \quad \forall k \geq n_0 + k_1.$$

The rest of the article is organized as follows. In the next section, the existence of discrete attractors $\mathcal{A}_N^\tau$ is obtained by the t-independent prior estimates of discrete solutions; In Sect. 3, the convergence of $\mathcal{A}_N^\tau$ is proved by the error estimates in $[0, +\infty)$ of the discrete solutions.

2 Existence of Global Attractor

In this section, we prove the existence of the global attractors $\mathcal{A}_N^\tau$ of problem (1.4)–(1.5).

Lemma 2.1 *If $f \in C^1$, $f'(s) > 0$, $u_0 \in L_p^2(\Omega)$. Then for the solution u_N^n of problem (1.4)–(1.5), we have*

$$\|u_N^n\|^2 \le E_0^2, \quad \forall n \ge 1,$$
$$\overline{\lim_{n\to\infty}} \|u_N^k\|^2 \le (\rho_0')^2,$$
$$\tau \sum_{k=1}^{n} \|\bar{\partial}_t u_N^k\|^2 \le c_1(1 + t_n), \quad \forall n \ge 1,$$

where the constant $c_1 = c_1(\|u_0\|)$ is independent of N, n, and τ.

Proof Let $\varphi = u_N^k$ in (1.4), we derive that

$$\frac{1}{2}\bar{\partial}_t \|u_N^k\|^2 + \frac{\tau}{2}\|\bar{\partial}_t u_N^k\|^2 + \gamma\|\Delta u_N^k\|^2 + \left(f'\left(u_N^{k-1}\right)\nabla u_N^k, \nabla u_N^k\right) \\ -B\left(u_N^{k-1}, u_N^k, u_N^k\right) = 0.$$

Note that $f'(s) > 0$, then

$$\left(f'\left(u_N^{k-1}\right)\nabla u_N^k, \nabla u_N^k\right) > 0.$$

On the other hand, we have

$$B\left(u_N^{k-1}, u_N^k, u_N^k\right) = 0.$$

Therefore

$$\bar{\partial}_t \|u_N^k\|^2 + \tau\|\bar{\partial}_t u_N^k\|^2 + 2\gamma\|\Delta u_N^k\|^2 \le 0. \tag{2.1}$$

By Poincaré's inequality, we obtain

$$\|u_N^k\|^2 \le C(\Omega)\left(\|\nabla u_N^k\|^2 + m_0^2\right) \le C(\Omega)\left(\frac{1}{2C(\Omega)}\|u_N^k\|^2 + \frac{C(\Omega)}{2}\|\Delta u_N^k\|^2 + m_0^2\right),$$

that is

$$\|u_N^k\|^2 \le [C(\Omega)]^2\|\Delta u_N^k\|^2 + 2C(\Omega)m_0^2. \tag{2.2}$$

Then

$$\bar{\partial}_t\|u_N^k\|^2+\tau\|\bar{\partial}_t u_N^k\|^2+\gamma\left(\|\Delta u_N^k\|^2+\frac{1}{[C(\Omega)]^2}\|u_N^k\|^2\right)\le\frac{2\gamma m_0^2}{C(\Omega)}. \tag{2.3}$$

Multiplying (2.3) by $[1+\frac{\gamma}{4\pi^4}\tau]^{k-1}$ and summing them for k from 1 to n, we have

$$\begin{aligned}\|u_N^n\|^2\le&\left(1+\frac{\gamma}{[C(\Omega)]^2}\tau\right)^{-n}\left(\|u_N^0\|^2+2m_0^2C(\Omega)\right)+2m_0^2C(\Omega)\\ \le&\|u_N^0\|^2+2m_0^2C(\Omega)\triangleq E_0^2,\end{aligned}$$

that is

$$\overline{\lim_{n\to\infty}}\|u_N^k\|^2\le 2m_0^2C(\Omega)\triangleq(\rho_0')^2.$$

Taking the sum of (2.3) for k from k_0+1 to n, we recover the proof of the lemma. □

Corollary 2.2 *For any given $\rho_0>\rho_0'$ and $R_0>0$, if $\|u_0\|\le R_0$, then*

$$\|u_N^n\|^2\le\rho_0^2,\quad\forall n\ge n_0=\left(\ln\frac{R_0^2}{\rho_0^2-(\rho_0')^2}\right)/\ln\left(1+\frac{\gamma}{[C(\Omega)]^2}\tau\right).$$

Lemma 2.3 *In addition to the conditions of Lemma 2.1, we suppose that $f,g\in C^1$, $|f'(s)|\le A|s|^{\frac{3}{2}}$, $|g'(s)|\le B|s|^2$, $u_0\in H_p^1(\Omega)$, $\|u_0\|_1\le R_0$. Then for the solution u_N^n of problem (1.4)–(1.5), we have*

$$\begin{aligned}&\|\nabla u_N^k\|^2\le\rho_1^2,\quad\forall n\ge n_0+N_0\triangleq n_1,\\ &\|\nabla u_N^k\|^2\le E_1^2,\quad\forall n\ge 1,\\ &\tau^2\sum_{k=1}^n\|\bar{\partial}_t\nabla u_N^k\|^2\le c_2(1+t_n),\quad\forall n\ge 1,\end{aligned}$$

where n_0 is given by Corollary 2.2, N_0 an arbitrary positive integer, r an arbitrary positive number such that $N_0\tau=r$, the constant ρ_1 independent of N, n, τ and $\|u_0\|_1$, $c_2=c_2(\|u_0\|_1)$, and $E_1=E_1(\|u_0\|_1)$ independent of N, n, and τ.

Proof Let $\varphi=\Delta u_N^k$ in (1.4), we derive that

$$\begin{aligned}&\frac{1}{2}\bar{\partial}_t\|\nabla u_N^k\|^2+\frac{\tau}{2}\|\bar{\partial}_t\nabla u_N^k\|^2+\gamma\|\nabla\Delta u_N^k\|^2\\ &=\left(f'(u_N^{k-1})\nabla u_N^k,\nabla\Delta u_N^k\right)-B\left(u_N^{k-1},u_N^k,-\Delta u_N^k\right).\end{aligned} \tag{2.4}$$

By Sobolev's interpolation inequality, we deduce that

$$\begin{aligned}\left(f'\left(u_N^{k-1}\right)\nabla u_N^k, \nabla\Delta u_N^k\right) \le & \frac{\gamma}{8}\|\nabla\Delta u_N^k\|^2 + \frac{2}{\gamma}\|f'\left(u_N^{k-1}\right)\|_\infty^2\|\nabla u_N^k\|^2 \\ \le & \frac{\gamma}{8}\|\nabla\Delta u_N^k\|^2 + \frac{2A^2C}{\gamma}\|u_N^{k-1}\|_\infty^3\|\nabla\Delta u_N^k\|^{\frac{2}{3}}\|u_N^k\|^{\frac{4}{3}} \\ \le & \frac{\gamma}{4}\|\nabla\Delta u_N^k\|^2 + C\|u_N^{k-1}\|_\infty^4\|u_N^k\|^2 \\ \le & \frac{\gamma}{4}\|\nabla\Delta u_N^k\|^2 + C\|\Delta u_N^{k-1}\|^2\|u_N^{k-1}\|^2\|u_N^k\|^2.\end{aligned}$$

We also have

$$-B\left(u_N^{k-1}, u_N^k, -u_{Nxx}^k\right) = 2\left(G\left(u_N^{k-1}\right)\nabla u_N^k, \Delta u_N^k\right) + \left(G'\left(u_N^{k-1}\right)\nabla u_N^{k-1}u_N^k, \Delta u_N^k\right).$$

Using Sobolev's interpolation inequality again, we get

$$\begin{aligned}2\left(G\left(u_N^{k-1}\right)\nabla u_N^k, \Delta u_N^k\right) &= 2\left(\frac{\nabla u_N^k}{\left(u_N^{k-1}\right)^2}\int_0^{u_N^{k-1}} sg'(s)ds, \Delta u_N^k\right) \\ &= \left(g'\left(\theta u_N^{k-1}\right)\nabla u_N^k, \Delta u_N^k\right) \le \|g'\left(\theta u_N^{k-1}\right)\|_\infty\|\nabla u_N^k\|\|\Delta u_N^k\| \\ &\le B\|u_N^{k-1}\|_\infty^2\|\nabla u_N^k\|\|\Delta u_N^k\| \le C\|\Delta u_N^{k-1}\|\|u_N^{k-1}\|\|\nabla\Delta u_N^k\|\|u_N^k\| \\ &\le \frac{\gamma}{8}\|\nabla\Delta u_N^k\|^2 + C\|\Delta u_N^{k-1}\|^2\|u_N^{k-1}\|^2\|u_N^k\|^2,\end{aligned}$$

and

$$\begin{aligned}&\left(G'\left(u_N^{k-1}\right)\nabla u_N^{k-1}u_N^k, \Delta u_N^k\right) \\ &\quad= \left(\left[\frac{g'\left(u_N^{k-1}\right)}{u_N^{k-1}} - \frac{g'\left(\theta u_N^{k-1}\right)}{u_N^{k-1}}\right]u_N^k\nabla u_N^{k-1}, \Delta u_N^k\right) \\ &\le B\|u_N^{k-1}\|_\infty\|u_N^k\|_\infty\|\nabla u_N^{k-1}\|\|\Delta u_N^k\| \\ &\quad\le C\|\Delta u_N^{k-1}\|^{\frac{1}{2}}\|u_N^{k-1}\|^{\frac{1}{2}}\|\nabla\Delta u_N^k\|\|u_N^k\|\|\nabla u_N^{k-1}\| \\ &\quad\le \frac{\gamma}{8}\|\nabla\Delta u_N^k\|^2 + C\|\Delta u_N^{k-1}\|\|u_N^{k-1}\|\|u_N^k\|^2\|\nabla u_N^{k-1}\|^2 \\ &\quad\le \frac{\gamma}{8}\|\nabla\Delta u_N^k\|^2 + C\|\nabla u_N^{k-1}\|^2\|u_N^k\|^2\left(\|\Delta u_N^{k-1}\|^2\|u_N^{k-1}\|^2 + 1\right).\end{aligned}$$

Hence, (2.4) can be rewritten as

$$\begin{aligned}&\bar{\partial}_t\|\nabla u_N^k\|^2 + \tau\|\bar{\partial}_t\nabla u_N^k\|^2 + \gamma\|\nabla\Delta u_N^k\|^2 \\ &\quad\le C\|\nabla u_N^{k-1}\|^2\|u_N^k\|^2\left(\|\Delta u_N^{k-1}\|^2\|u_N^{k-1}\|^2 + 1\right) + C\|\Delta u_N^{k-1}\|^2\|u_N^{k-1}\|^2\|u_N^k\|^2.\end{aligned} \tag{2.5}$$

Using (2.3), Lemma 2.1, and Corollary 2.2, we obtain for all $k_0 > n_0$

$$
\begin{aligned}
&C\tau \sum_{k=k_0+1}^{k_0+N_0} \left[\|u_N^k\|^2 \left(\|\Delta u_N^{k-1}\|^2 \|u_N^{k-1}\|^2 + 1 \right) \right] \\
&\quad \leq C \left(\rho_0^4 \tau \sum_{k=k_0+1}^{k_0+N_0} \|\Delta u_N^{k-1}\|^2 + \rho_0^2 r \right) \leq C \left[\rho_0^4 \left(\frac{2m_0^2 r}{C(\Omega)} + \frac{r}{\gamma}\rho_0^2 \right) + \rho_0^2 r \right] \triangleq \alpha_1, \\
&C\tau \sum_{k=k_0+1}^{k_0+N_0} \left[\|\Delta u_N^{k-1}\|^2 \|u_N^{k-1}\|^2 \|u_N^k\|^2 \right] \leq C\rho_0^4 \left(\frac{2m_0^2 r}{C(\Omega)} + \frac{r}{\gamma}\rho_0^2 \right) \triangleq \alpha_2, \\
&\tau \sum_{k=k_0+1}^{k_0+N_0} \|\nabla u_N^{k-1}\|^2 \leq \frac{\tau}{2} \sum_{k=k_0+1}^{k_0+N_0} \left(\|u_N^{k-1}\|^2 + \|\Delta u_N^{k-1}\|^2 \right) \\
&\quad \leq \frac{1}{2} \left(\frac{2m_0^2 r}{C(\Omega)} + \frac{r}{\gamma}\rho_0^2 + \rho_0^2 r \right) \triangleq \alpha_3.
\end{aligned}
$$

Applying discrete uniform Gronwall's inequality to (2.5), we have

$$
\|\nabla u_N^k\|^2 \leq \left(\frac{\alpha_3}{r} + \alpha_2 \right) e^{\alpha_1} \triangleq \rho_1^2, \quad \forall n \geq n_1 = n_0 + N_0.
$$

For $n \leq n_1$, using Discrete Gronwall's inequality to (2.5), we deduce that

$$
\begin{aligned}
\|\nabla u_N^n\|^2 \leq & \left(\|\nabla u_0\|^2 + \frac{2m_0^2}{C(\Omega)} E_0^4 t_n + \frac{E_0^2}{\gamma} \right) \exp \left[E_0^4 \left(\frac{2m_0^2}{C(\Omega)} t_{n_1} + \frac{1}{\gamma} \|u_0\|^2 \right) + \rho_0^2 t_{n_1} \right] \\
& \triangleq (E_1')^2.
\end{aligned}
$$

Set $E_1^2 = \max\{\rho_1^2, (E_1')^2\}$. Hence, the second relation is proved. Taking the sum of (2.5), we complete the proof of this lemma. □

Corollary 2.4 *In addition to the conditions of Lemma 2.3, we have the following estimates*

$$
\begin{aligned}
\|u_N^n\|_{L^p} &\leq C(\rho_0, \rho_1), \quad \forall n \geq n_1, \quad 1 \leq p < \infty, \\
\|u_N^n\|_{L^p} &\leq C(E_0, E_1), \quad \forall n \geq 1, \quad 1 \leq p < \infty.
\end{aligned}
$$

Lemma 2.5 *In addition to the condition of Lemma 2.3, we suppose that $f \in C^2$, $|f''(s)| \leq A'|s|^{\frac{1}{2}}$, $u_0 \in H_p^2(\Omega)$ satisfying $\|u_0\|_2^2 \leq R_0^2$. Then*

$$
\begin{gathered}
\|\Delta u_N^n\|^2 \leq \rho_2^2, \quad \forall n \geq n_2 = n_1 + N_0, \\
\|\Delta u_N^n\|^2 \leq E_2^2, \quad \forall n \geq 1, \\
\tau^2 \sum_{k=1}^{n} \|\bar{\partial}_t \Delta u_N^k\|^2 + \tau \sum_{k=1}^{n} \|\Delta^2 u_N^k\|^2 \leq c_3(1 + t_n), \quad \forall n \geq 1,
\end{gathered}
$$

where n_1 is given by Lemma 2.3, N_0 an arbitrary positive integer, r an arbitrary positive number such that $N_0\tau = r$, the constant ρ_2 independent of N, n, τ, and $\|u_0\|_2$, $E_2 = E_2(\|u_0\|_2)$, and $c_3 = c_3(\|u_0\|_2)$ independent of N, n and τ.

Proof Let $\varphi = \Delta^2 u_N^k$ in (1.4), we have

$$\begin{aligned}&\frac{1}{2}\bar{\partial}_t\|\Delta u_N^k\|^2 + \frac{\tau}{2}\|\bar{\partial}_t\Delta u_N\|^2 + \gamma\|\Delta^2 u_N^k\|^2\\&\quad = \left(\nabla\cdot[f'(u_N^{k-1})\nabla\cdot u_N^k], \Delta^2 u_N^k\right) + B\left(u_N^{k-1}, u_N^k, \Delta^2 u_N^k\right).\end{aligned}$$

When $k \geq n_1$: Based on Lemma 2.1, Corollary 2.2, Lemma 2.3, Corollary 2.6, and Sobolev's interpolation inequality, we deduce that

$$\begin{aligned}&\left(\nabla\cdot\left[f'(u_N^{k-1})\nabla\cdot u_N^k\right], \Delta^2 u_N^k\right)\\&\quad = \left(f'\left(u_N^{k-1}\right)\Delta u_N^k, \Delta^2 u_N^k\right) + \left(f''\left(u_N^{k-1}\right)|\nabla u_N^k|^2, \Delta^2 u_N^k\right)\\&\quad \leq \|f'\left(u_N^{k-1}\right)\|_{L^4}\|\Delta u_N^k\|_{L^4}\|\Delta^2 u_N^k\| + \|f''\left(u_N^{k-1}\right)\|_{L^6}\|\nabla u_N^k\|_{L^6}^2\|\Delta^2 u_N^k\|\\&\quad \leq C\|u_N^{k-1}\|_{L^6}^{\frac{3}{2}}\|u_N^k\|^{\frac{3}{8}}\|\Delta^2 u_N^k\|^{\frac{13}{8}} + C\|u_N^{k-1}\|_{L^3}^{\frac{1}{2}}\|u_N^k\|^{\frac{1}{6}}\|\Delta^2 u_N^k\|^{\frac{11}{6}}\\&\quad \leq \frac{\gamma}{4}\|\Delta^2 u_N^k\|^2 + C(\rho_0, \rho_1).\end{aligned}$$

and

$$\begin{aligned}&B(u_N^{k-1}, u_N^k, \Delta^2 u_N^k)\\&\quad = \left(2G\left(u_N^{k-1}\right)\nabla u_N^k + G'\left(u_N^{k-1}\right)\nabla u_N^{k-1}u_N^k, \Delta^2 u_N^k\right)\\&\quad = \left(2g'\left(\theta u_N^{k-1}\right)\nabla u_N^k + \left[\frac{g'\left(u_N^{k-1}\right)}{u_N^{k-1}} - \frac{g'\left(\theta u_N^{k-1}\right)}{u_N^{k-1}}\right]u_N^k\nabla u_N^{k-1}, \Delta^2 u_N^k\right)\\&\quad \leq C\|g'\theta u_N^{k-1}\|_{L^4}\|\nabla u_N^k\|_{L^4}\|\Delta^2 u_N^k\| + C\|u_N^{k-1}\|_{L^6}\|\nabla u_N^{k-1}\|_{L^6}\|u_N^k\|_{L^6}\|\Delta^2 u_N^k\|\\&\quad \leq C\|g'\theta u_N^{k-1}\|_{L^4}\|\nabla u_N^k\|_{L^4}\|\Delta^2 u_N^k\|\\&\qquad + C\|u_N^{k-1}\|_{L^6}\|\Delta u_N^{k-1}\|^{\frac{5}{6}}\|u_N^{k-1}\|^{\frac{1}{6}}\|u_N^k\|_{L^6}\|\Delta^2 u_N^k\|\\&\quad \leq \frac{\gamma}{4}\|\Delta^2 u_N^k\|^2 + C(\rho_0, \rho_1)\left(\|\Delta u_N^{k-1}\|^2 + 1\right).\end{aligned}$$

Summing up, we derive that

$$\bar{\partial}_t\|\Delta u_N^k\|^2 + \tau\|\bar{\partial}_t\Delta u_N^k\|^2 + \gamma\|\Delta^2 u_N^k\|^2 \leq C(\rho_0, \rho_1)\left(\|\Delta u_N^{k-1}\|^2 + 1\right). \tag{2.6}$$

Note that

$$\tau \sum_{k=k_0+1}^{k_0+N_0} C(\rho_0, \rho_1) = C(\rho_0, \rho_1) r \triangleq \alpha_i, \quad i = 1, 2,$$

$$\tau \sum_{k=k_0+1}^{k_0+N_0} \|\Delta u_N^k\|^2 = \frac{2m_0^2 r}{C(\Omega)} + \frac{r}{\gamma}\rho_0^2 \triangleq \alpha_3.$$

Hence, using discrete uniform Gronwall's inequality to (2.6), we deduce that

$$\|\Delta u_N^n\|^2 \le \left(\frac{\alpha_3}{r} + \alpha_2\right) e^{\alpha_1} \triangleq \rho_2^2, \quad \forall n \ge n_2 = n_1 + N_0. \tag{2.7}$$

When $k \ge 1$: As in the proof of the inequality (2.6), we have

$$\bar{\partial}_t \|\Delta u_N^k\|^2 + \tau \|\bar{\partial}_t \Delta u_N^k\|^2 + \gamma \|\Delta^2 u_N^k\|^2 \le C(E_0, E_1)\Big(\|\Delta u_N^{k-1}\|^2 + 1\Big) \tag{2.8}$$

Using Discrete Gronwall's inequality to (2.8), we have

$$\|\Delta u_N^2\|^2 \le \Big(\|\Delta u_0\|^2 + C(E_0, E_1) t_n\Big) e^{C(E_0, E_1) t_n} \triangleq E_2. \tag{2.9}$$

Taking the sum of (2.8) for k from 1 to n, we deduce that

$$\|\Delta u_N^n\|^2 + \tau^2 \sum_{k=1}^{n} \|\bar{\partial}_t \Delta u_N^n\|^2 + \gamma\tau \sum_{k=1}^{n} \|\Delta^2 u_N^k\|^2 \le C(1 + t_n), \quad \forall n \ge 1. \tag{2.10}$$

Combining (2.7), (2.9), and (2.10), we complete the proof of this lemma. □

Corollary 2.6 *In addition to the conditions of Lemma 2.5, we have the following estimates*

$$\|u_N^n\|_\infty \le C(\rho_0, \rho_1, \rho_2), \quad \forall n \ge n_2,$$
$$\|u_N^n\|_\infty \le C(E_0, E_1, E_2), \quad \forall n \ge 1,$$
$$\|u_N^n\|_{W^{1,p}} \le C(\rho_0, \rho_1, \rho_2), \quad \forall n \ge n_2, \quad 1 < p < \infty,$$
$$\|u_N^n\|_{W^{1,p}} \le C(E_0, E_1, E_2), \quad \forall n \ge 1, \quad 1 < p < \infty.$$

Now, we give the main result in this section.

Theorem 2.7 *If $f \in C^2$, $g \in C^1$, $f'(s) > 0$, $|f'(s)| \le A|s|^{\frac{3}{2}}$, $|f''(s)| \le A'|s|^{\frac{1}{2}}$, $|g'(s)| \le B|s|^2$, and $u_0 \in H_p^2(\Omega)$, then the semigroup of operator $\{S_N^\tau(n)\}_{n \ge 0}$ generated by problem (1.4)–(1.5) has a compact global attractor $\mathcal{A}_N^\tau \subset H_p^2(\Omega) \bigcap S_N$.*

Proof Using Theorem I.1.1 of [30] and Lemmas 2.1, 2.3, 2.5, we complete the proof of this theorem. Since it is classical, we omit it. □

3 Convergence of the Global Attractors

Let $G_N : L^2_p(\Omega) \to S_N$ be the integral projection operator, i.e., for any given $u \in L^2(\Omega)$, we have

$$(\nabla G_N u, \nabla v) + (G_N u, v) = (u, v), \quad \forall v \in S_N. \tag{3.1}$$

Then for any $u, v \in L^2(\Omega)$, we have $(G_N u, v) = (u, G_N v)$.

Lemma 3.1 *(see [12]) For the integral projection operator G_N, we have*

(1) $\|G_N u\|_2 \, |P_N u\|, \quad \forall u \in L^2_p(\Omega);$
(2) $\|G_N \nabla u\| = \|\nabla G_N u\|, \quad \forall u \in H^1_p(\Omega);$
(3) $\|G_N \Delta u\| = \|\nabla[G_N \nabla u]\| = \|\Delta G_N u\|, \quad \forall u \in H^2_p(\Omega);$
(4) $\|G^2_N \Delta u\| = \|\nabla[G^2_N \nabla u]\| = \|\Delta(G^2_N u)\|, \quad \forall u \in H^2_p(\Omega).$

Similar to Lemma 2.5, the following result can be proved easily.

Lemma 3.2 *Under the hypotheses of Lemma 2.5, we have the estimates for the smooth solution $u(x, t)$ of problem (1.1)–(1.3)*

$$\int_0^t \|\nabla u_t\|^2 ds \le C(1 + t),$$
$$t\|u_t\|^2 + \int_0^t s\|\Delta u_t\|^2 ds \le C(1 + t^2),$$
$$t^2\|\nabla u_t\|^2 + \int_0^t s^2\|\nabla\Delta u_t\|^2 ds \le C(1 + t^3),$$
$$t^3\|\Delta u_t\|^2 + \int_0^t s^3(\|u_{tt}\|^2 + \|\Delta^2 u_t\|^2) ds \le C(1 + t^4),$$

where the constant t is independent of t.

Using the same method as [31], we can obtain the following result easily.

Lemma 3.3 *If $f \in C^2$, $g \in C^1$, $f'(s) > 0$, $|f'(s)| \le A|s|^4$, $|g'(s)| \le B|s|^5$, and $u_0 \in H^2_p(\Omega)$, then there exists an unique global solution $u(x, t) \in L^\infty(\mathbb{R}^+; H^2_p(\Omega))$ for the problem (1.1)–(1.3) such that*

$$\int_0^t (\|u\|^2_{H^4} + \|u_t\|^2) \mathrm{d}t \le C(t + 1), \quad \forall t > 0,$$

where the constant c is independent of t. Furthermore, if $f \in C^3$, $g \in C^2$, then $u(x, t)$ satisfies

$$t\|\nabla\Delta u\|^2 \le c(t^2 + 1), \quad \forall t > 0,$$

and there exists a global attractor $\mathcal{A} \subset H_p^2(\Omega)$ of problem (1.1)–(1.3).

Now, we give the main result of this subsection.

Theorem 3.4 *Suppose that the conditions of Theorem 2.7 hold, then*

$$dist(\mathcal{A}_N^\tau, \mathcal{A}) \to 0, \quad as\ \tau \to 0,\ N \to +\infty.$$

Proof Let $\|u_0\|_{H^2} \leq R_0$. On account of Lemma 3.3, this theorem will be proved by taking the error estimates of the solution u_N^n of discrete problem (1.4)–(1.5). Now, we accomplish them through two steps.

Step 1 Take the error estimates of the solution v^n of the linear scheme as follows:

$$\begin{aligned}\Big(\bar{\partial}_t v_N^k + \gamma \Delta^2 v_N^k - 2\gamma \Delta v_N^k + \gamma v_N^k - \Delta f(u_N) - \nabla \cdot g(u_N), \varphi\Big)\\ = \Big(-2\gamma \Delta u_N^k + \gamma u_N^k, \varphi\Big),\end{aligned} \tag{3.2}$$

$$v_N^0 = P_N u_0, \quad \forall \varphi \in S_N. \tag{3.3}$$

Set $u^k - v_N^k = u^k - P_N u^k + P_N u^k - v_N^k = \rho^k + \theta^k$. Hence, θ^k satisfies

$$\Big(\bar{\partial}_t \theta^k + \gamma \Delta^2 \theta^k - 2\gamma \Delta \theta^k + \gamma \theta^k - \Big(\bar{\partial}_t u^k - u_t^k\Big), \varphi\Big) = 0, \quad \forall \varphi \in S_N, \tag{3.4}$$

$$\theta^0 = 0. \tag{3.5}$$

Let $\varphi = \theta^k$ in (3.4). Simple calculations show that

$$\frac{1}{2}\bar{\partial}_t \|\theta^k\|^2 + \frac{\tau}{2}\|\bar{\partial}_t \theta^k\|^2 + \gamma\Big(\|\Delta\theta\|^2 + \|\nabla\theta^k\|^2 + \|\theta^k\|_1^2\Big) = \Big(\bar{\partial}_t u^k - u_t^k, \theta^k\Big).$$

Note that

$$\begin{aligned}\Big(\bar{\partial}_t u^k - u_t^k, \theta^k\Big) &= \Big(\nabla\Big[G_N\Big(\bar{\partial}_t u^k - u_t^k\Big)\Big], \nabla\theta^k\Big) + \Big(G_N\Big(\bar{\partial}_t u^k - u_t^k\Big), \theta^k\Big)\\ &\leq \frac{\gamma}{2}\|\theta^k\|_1^2 + \frac{1}{2\gamma}\|G_N\Big(\bar{\partial}_t u^k - u_t^k\Big)\|_1^2\\ &\leq \frac{\gamma}{2}\|\theta^k\|_1^2 + \frac{1}{2\gamma}\frac{1}{\tau^2}\|\int_{t_{k-1}}^{t_k}(s - t_{k-1})G_N u_{tt} ds\|_1^2\\ &\leq \frac{\gamma}{2}\|\theta^k\|_1^2 + \frac{1}{2\gamma}\frac{1}{\tau^2}\int_{t_{k-1}}^{t_k}\frac{(s - t_{k-1})^2}{s^2} ds \int_{t_{k-1}}^{t_k} s^2\|G_N u_{tt}\|_1^2 ds\\ &\leq \frac{\gamma}{2}\|\theta^k\|_1^2 + \frac{\tau}{2\gamma t_k^2}\int_{t_{k-1}}^{t_k} s^2\|G_N u_{tt}\|_1^2 ds.\end{aligned}$$

Therefore

$$\bar{\partial}_t \|\theta^k\|^2 + \tau\|\bar{\partial}_t \theta^k\|^2 + 2\gamma\|\Delta\theta^k\|^2 + \gamma\|\theta^k\|_1^2 \leq \frac{\tau}{\gamma t_k^2}\int_{t_{k-1}}^{t_k} s^2\|G_N u_{tt}\|_1^2 ds.$$

Multiplying the above inequality by t_k^2, taking the sum for k from 1 to n, using $\|G_N u_{tt}\|_1 \le C\|u_t\|_3$, we get

$$\begin{aligned} t_n^2\|\theta^n\|^2 + \gamma\tau\sum_{k=1}^n t_k^2\left(2\|\Delta\theta^k\|^2 + \|\theta^k\|_1^2\right) + \tau^2\sum_{k=1}^n t_k^2\|\bar{\partial}_t\theta^k\|^2 \\ \le 3\tau\sum_{k=1}^n t_k\|\theta^k\|^2 + \frac{\tau^2}{\gamma}\sum_{k=1}^n\int_{t_{k-1}}^{t_k} s^2\|G_N u_{tt}\|_1^2 ds \\ \le 3\tau\sum_{k=1}^n t_k\|\theta^k\|^2 + C\tau^2\int_0^{t_n} s^2\|u_t\|_3^2 ds \\ \le 3\tau\sum_{k=1}^n t_k\|\theta^k\|^2 + C\tau^2(1+t_n^3). \end{aligned} \tag{3.6}$$

Setting $\varphi = G_N\theta^k$ in (3.4), we have

$$\left(\bar{\partial}_t\theta^k + \gamma\Delta^2\theta^k - 2\gamma\Delta\theta^k + \gamma\theta^k - \left(\bar{\partial}_t u^k - u_t^k\right), G_N\theta^k\right) = 0.$$

Note that

$$\begin{aligned} \left(\bar{\partial}_t\theta^k, G_N\theta^k\right) = \frac{1}{2}\bar{\partial}_t\|G_N\nabla\theta^k\|^2, \\ \left(\gamma\Delta^2\theta^k - 2\gamma\Delta\theta^k + \gamma\theta^k, G_N\theta^k\right) = \gamma\|\nabla\theta^k\|^2 + \gamma\|\theta^k\|^2, \\ \left(\bar{\partial}_t u^k - u_t^k, G_N\theta^k\right) \le \frac{\gamma}{2}\|\theta^k\|^2 + \frac{1}{2\gamma}\|G_N\left(\bar{\partial}_t u^k - u_t^k\right)\|^2. \end{aligned}$$

Hence

$$\bar{\partial}_t\|G_N\nabla\theta^k\|^2 + \gamma\left(\|\nabla\theta^k\|^2 + \|\theta^k\|^2\right) \le \frac{1}{\gamma}\|G_N\left(\bar{\partial}_t u^k - u_t^k\right)\|^2. \tag{3.7}$$

Multiplying (3.7) by τt_k, taking the sum for k from 1 to n, using $\|G_N u_{tt}\| \le C\|u_t\|_2$, we get

$$\begin{aligned} t_n\|G_N\nabla\theta^n\|^2 + \gamma\tau\sum_{k=1}^n t_k\left(\|\nabla\theta^k\|^2 + \|\theta^k\|^2\right) \\ \le \frac{\tau}{\gamma}\sum_{k=1}^n t_k\|G_N\left(\bar{\partial}_t u^k - u_t^k\right)\|^2 + \tau\sum_{k=1}^n\|G_N\nabla\theta^k\|^2 \\ \le \frac{\tau^2}{\gamma}\int_0^{t_n} s\|G_N u_{tt}\|^2 ds + \tau\sum_{k=1}^n\|G_N\nabla\theta^k\|^2 \\ \le C\tau^2(1+t_n^2) + \tau\sum_{k=1}^n\|G_N\nabla\theta^k\|^2. \end{aligned} \tag{3.8}$$

Set $\varphi = G_N^2\theta^k$ in (3.4). Therefore

$$\left(\bar{\partial}_t\theta^k + \gamma\Delta^2\theta^k - 2\gamma\Delta\theta^k + \gamma\theta^k - \left(\bar{\partial}_t u^k - u_t^k\right), G_N^2\theta^k\right) = 0.$$

Note that

$$\left(\bar{\partial}_t\theta^k, G_N^2\theta^k\right) = \frac{1}{2}\bar{\partial}_t\|G_N\theta^k\|^2 + \frac{\tau}{2}\|G_N\nabla\theta^k\|^2,$$
$$\left(\gamma\Delta^2\theta^k - 2\gamma\Delta\theta^k + \gamma\theta^k, G_N^2\theta^k\right) = \gamma\|\Delta G_N\theta^k\|^2 + \gamma\|G_N\theta^k\|_1^2,$$
$$\left(\bar{\partial}_t u^k - u_t^k, G_N^2\theta^k\right) \le \frac{\gamma}{2}\|G_N\nabla\theta^k\|^2 + \frac{1}{2\gamma}\|G_N^2\left(\bar{\partial}_t u^k - u_t^k\right)\|_1^2.$$

Thus

$$\bar{\partial}_t\|G_N\theta^k\|^2 + \gamma\|G_N\theta^k\|_1^2 \le \frac{1}{\gamma}\|G_N^2\left(\bar{\partial}_t u^k - u_t^k\right)\|_1^2. \tag{3.9}$$

Taking the sum of (3.9) for k form 1 to n, applying $\|G_N^2\nabla u_{tt}\| \le C\|\nabla u_t\|$, we obtain

$$\begin{aligned}
&\|G_N\theta^k\|^2 + \gamma\tau\sum_{k=1}^{n}\|G_N\theta^k\|_1^2 \\
&\le \frac{\tau}{\gamma}\sum_{k=1}^{n}\|G_N^2\left(\bar{\partial}_t u^k - u_t^k\right)\|_1^2 \le \frac{\tau^2}{\gamma}\int_0^{t_n}\|G_N^2\left(\bar{\partial}_t u^k - u_t^k\right)\|_1^2 ds \\
&\le C\tau^2(1+t_n).
\end{aligned} \tag{3.10}$$

Adding (3.6), (3.8), and (3.10) together gives

$$\begin{aligned}
&t_n^2\|\theta^k\|^2 + \tau^2\sum_{k=1}^{n}t_k^2\|\bar{\partial}_t\theta^k\|^2 + \gamma\tau\sum_{k=1}^{n}t_k^2\|\theta^k\|_2^2 + \gamma\tau\sum_{k=1}^{n}t_k\|\theta^k\|_2^2 \\
&+ \gamma\tau\sum_{k=1}^{n}\|G_N\theta^k\|_1^2 \le C\tau^2(1+t_n^3).
\end{aligned} \tag{3.11}$$

Multiplying (3.11) by $\frac{1}{\gamma\tau}$, we derive that

$$t_n^2\left(\|\Delta\theta^k\|^2 + \|\theta^k\|^2\right) \le C\tau(1+t_n^3). \tag{3.12}$$

Step 2 Take the error estimates of solution u_N^n of problem (1.4)–(1.5). Set $v_N^k - u_N^k = e^k$. Hence, e^k satisfies

$$\begin{aligned}
&\left(\bar{\partial}_t e^k + \gamma\Delta^2 e^k + \gamma\Delta\theta^k - \gamma\theta^k, \varphi\right) + \left(\nabla f(u^k) - f'\left(u_N^{k-1}\right)\nabla u_N^k, \nabla\varphi\right) \\
&\quad = \left(\nabla\cdot g(u^k), \varphi\right) - B\left(u_N^{k-1}, u_N^k, \varphi\right), \quad \forall\varphi\in S_N, k = 1, 2, \ldots, \qquad (3.13) \\
&\qquad e^0 = 0. \qquad (3.14)
\end{aligned}$$

Setting $\varphi = e^k$ in (3.13), we derive that

$$\frac{1}{2}\bar{\partial}_t\|e^k\|^2 + \frac{\tau}{2}\|\bar{\partial}_t e^k\|^2 + \gamma\|\Delta e^k\|^2 = -2\gamma\Big(\Delta\theta^k, e^k\Big) + \gamma\Big(\theta^k, e^k\Big)$$
$$- \Big(\nabla f(u^k) - f'\Big(u_N^{k-1}\Big)\nabla u_N^k, \nabla e^k\Big) \quad + \Big(\nabla\cdot g(u^k), e^k\Big) - B\Big(u_N^{k-1}, u_N^k, e^k\Big). \tag{3.15}$$

Note that

$$\begin{aligned}
&- \Big(\nabla f(u^k) - f'\Big(u_N^{k-1}\Big)\nabla u_N^k, \nabla e^k\Big) = -\Big(f'(u^k)\nabla u^k - f'\Big(u_N^{k-1}\Big)\nabla u_N^k, \nabla e^k\Big)\\
&\quad = -\Big(f'(u^k)\nabla u^k - f'(u^k)\nabla u_N^k + f'(u^k)\nabla u_N^k - f'(u^{k-1})\nabla u_N^k\\
&\qquad + f'(u^{k-1})\nabla u_N^k - f'\big(u_N^{k-1}\big)\nabla u_N^k, \nabla e^k\Big)\\
&\quad = -\Big(f'(u^k)(\nabla u^k - \nabla u_N^k), \nabla e^k\Big) - \Big(\Big[f'(u^k) - f'(u^{k-1})\Big]\nabla u_N^k, \nabla e^k\Big)\\
&\qquad - \Big(\Big[f'(u^{k-1}) - f'\Big(u_N^{k-1}\Big)\Big]\nabla u_N^k, \nabla e^k\Big)\\
&\triangleq I_1 + I_2 + I_3.\\
&I_1 = \Big(f'(u^k)\Delta e^k + f''(u^k)\nabla u^k\nabla e^k, u^k - u_N^k\Big)\\
&\quad \le \Big(\|f'(u^k)\|_\infty\|\Delta e^k\| + \|f''(u^k)\|_\infty\|\nabla u^k\|_{L^4}\|\nabla e^k\|_{L^4}\Big)\|u^k - u_N^k\|\\
&\quad \le \Big(\|f'(u^k)\|_\infty\|\Delta e^k\| + C\|f''(u^k)\|_\infty\|\nabla u^k\|_{L^4}\|\Delta e^k\|\Big)\|u^k - u_N^k\|\\
&\quad \le C\|u^k - u_N^k\|\Big(\|e^k\|^2 + \|\Delta e^k\|^2\Big)\\
&\quad \le C\Big(\|u^k - u^{k-1}\| + \|u^{k-1} - u_N^{k-1}\| + \|u_N^{k-1} - u_N^k\|\Big)\Big(\|e^k\|^2 + \|\Delta e^k\|^2\Big)\\
&\quad \le C\left(\|e^{k-1}\|^2 + \|\rho^{k-1}\|^2 + \|\theta^{k-1}\|^2 + \tau\int_{t_{k-1}}^{t_k}\|u_t\|^2 ds + \tau^2\|\bar{\partial}_t u_N^k\|^2\right)\\
&\qquad + \varepsilon\Big(\|e^k\|^2 + \|\Delta e^k\|^2\Big),\\
&I_2 = -\Big(f''(\phi_1 u^k + (1-\phi_1)u^{k-1})\nabla u_N^k(u^k - u^{k-1}), \nabla e^k\Big)\\
&\quad \le \|f''\Big(\phi_1 u^k + (1-\phi_1)u^{k-1}\Big)\|_\infty\|\nabla u_N^k\|_{L^4}\|u^k - u^{k-1}\|\|\nabla e^k\|_{L^4}\\
&\quad \le C\|f''\Big(\phi_1 u^k + (1-\phi_1)u^{k-1}\Big)\|_\infty\|\nabla u_N^k\|_{L^4}\|u^k - u^{k-1}\|\Big(\|e^k\|^2 + \|\Delta e^k\|^2\Big)\\
&\quad \le C\Big(\|e^k\|^2 + \|\Delta e^k\|^2\Big)\|u^k - u^{k-1}\| \le \varepsilon\Big(\|e^k\|^2 + \|\Delta e^k\|^2\Big) + C\tau\int_{t_{k-1}}^{t_k}\|u_t\|^2 \mathrm{d}s,\\
&\quad I_3 = -\Big(f''(\phi_2 u^{k-1} + (1-\phi_2)u_N^{k-1}\Big)\nabla u_N^k(u^{k-1} - u_N^{k-1}), \nabla e^k)\\
&\le \|f''\Big(\phi_2 u^{k-1} + (1-\phi_2)u_N^{k-1}\Big)\|_\infty\|\nabla u_N^k\|_{L^4}\|u^{k-1} - u_N^{k-1}\|\|\nabla e^k\|_{L^4}\\
&\quad \le C\Big(\|e^k\|^2 + \|\Delta e^k\|^2\Big)\|u^{k-1} - u_N^{k-1}\|\\
&\quad \le \varepsilon\Big(\|e^k\|^2 + \|\Delta e^k\|^2\Big) + C\Big(\|\rho^{k-1}\|^2 + \|\theta^{k-1}\|^2 + \|e^{k-1}\|^2\Big),
\end{aligned}$$

where $\phi_1, \phi_2 \in (0, 1)$. Hence

$$\left(\nabla f(u^k) - f'\left(u_N^{k-1}\right)\nabla u_N^k, \nabla e^k\right) \leq 3\varepsilon\left(\|e^k\|^2 + \|\Delta e^k\|^2\right)$$
$$+ C\left(\|\rho^{k-1}\|^2 + \|\theta^{k-1}\|^2 + \|e^{k-1}\|^2 + \tau\int_{t_{k-1}}^{t_k}\|u_t\|^2 ds + \tau^2\|\bar{\partial}_t u_N^k\|^2\right).$$

On the other hand, we have

$$\begin{aligned}
&\left(\nabla\cdot g(u^k), e^k\right) - B\left(u_N^{k-1}, u_N^k, e^k\right)\\
&= \left(G(u^k)\nabla\cdot u^k - G\left(u_N^{k-1}\right)\nabla\cdot u_N^k, e^k\right) - \left(G(u^k)u^k - G\left(u_N^{k-1}\right)u_N^k, \nabla\cdot e^k\right)\\
&= \left(G(u^k)\nabla\cdot\left(u^k - u_N^k\right), e^k\right) + \left(G(u^k) - G(u^{k-1}), \nabla\cdot u_N^k e^k\right)\\
&\quad + \left(G(u^{k-1} - G\left(u_N^{k-1}\right), \nabla\cdot u_N^k e^k\right) + \left(u^k(G(u^k) - G(u^{k-1}), \nabla\cdot e^k\right)\\
&\quad + \left(u^k G(u^{k-1}) - G\left(u_N^{k-1}\right), \nabla\cdot e^k\right) + \left(G(u^k)\left(u^k - u_N^k\right), \nabla\cdot e^k\right)\\
&\leq C\left(\|e^{k-1}\|^2 + \|\rho^{k-1}\|^2 + \|\theta^{k-1}\|^2 + \tau^2\|\bar{\partial}_t u_N^k\|^2 + \tau\int_{t_{k-1}}^{t_k}\|u_t\|^2 \mathrm{d}s\right)\\
&\quad + \varepsilon\left(\|\Delta e^k\|^2 + \|e^k\|^2\right).
\end{aligned} \tag{3.16}$$

We also have

$$-2\gamma\left(\Delta\theta^k, e^k\right) + \gamma\left(\theta^k, e^k\right) \leq \varepsilon\left(\|\Delta e^k\|^2 + \|e^k\|^2\right) + C\|\theta^k\|^2.$$

Based on Poincaré's inequality, we have

$$\|e^k\|^2 \leq [C(\Omega)]^2\|\Delta e^k\|^2.$$

Then, a simple calculation shows that

$$\begin{aligned}
&\bar{\partial}_t\|e^k\|^2 + 2\left[\gamma - 5\varepsilon(1 + [C(\Omega)]^2)\right]\|\Delta e^k\|^2 + \tau\|\bar{\partial}_t e^k\|^2\\
&\leq C\left(\|e^{k-1}\|^2 + \|\rho^{k-1}\|^2 + \|\theta^k\|^2 + \|\theta^{k-1}\|^2 + \tau^2\|\bar{\partial}_t u_N^k\|^2 + \tau\int_{t_{k-1}}^{t_k}\|u_t\|^2 \mathrm{d}s\right),
\end{aligned} \tag{3.17}$$

where ε is small enough, it satisfies $\gamma - 5\varepsilon(1 + [C(\Omega)]^2) > 0$. Using discrete Gronwall's inequality to (3.17), we have

$$\begin{aligned}
\|e^n\|^2 &\leq Ce^{Ct_n}\tau\sum_{k=1}^{n}\left(\|\rho^{k-1}\|^2 + \|\theta^k\|^2 + \tau^2\|\bar{\partial}_t u_N^k\|^2 + \tau\int_{t_{k-1}}^{t_k}\|u_t\|^2 \mathrm{d}s\right)\\
&\leq Ce^{Ct_n}(N^{-4} + \tau).
\end{aligned} \tag{3.18}$$

Taking the sum of (3.17) for k from 1 to n and using (3.18), we have

$$\tau \sum_{k=1}^{n} \|\Delta e^k\|^2 + \tau^2 \sum_{k=1}^{n} \|\bar{\partial}_t e^k\|^2 \le C(N^{-4} + \tau). \tag{3.19}$$

Setting $\varphi = \Delta e^k$ in (3.13), we derive that

$$\begin{aligned}
&\bar{\partial}_t \|\nabla e^k\|^2 + \gamma \|\nabla \Delta e^k\|^2 + \tau \|\bar{\partial}_t \nabla e^k\|^2 = \gamma \left(\nabla \theta^k, \nabla \Delta e^k\right) + \gamma \left(\theta^k, \Delta e^k\right) \\
&+ \ \left(\nabla f(u^k) - f'(u_N^{k-1}) \nabla u_N^k, \nabla \Delta e^k\right) - \left(\nabla \cdot g(u^k), \Delta e^k\right) + B\left(u_N^{k-1}, u_N^k, \Delta e^k\right).
\end{aligned}$$

Note that

$$\begin{aligned}
&\gamma \left(\nabla \theta^k, \nabla \Delta e^k\right) + \gamma \left(\theta^k, \Delta e^k\right) \le \frac{\gamma}{10} \left(\|\nabla \Delta e^k\|^2 + \|e^k\|^2\right) + C \|\theta_x^k\|^2, \\
&\left(\nabla f(u^k) - f'(u_N^{k-1}) \nabla u_N^k, \nabla \Delta e^k\right) \\
&= \left(\left[f'(u^k) - f'(u^{k-1})\right] \nabla u^k, \nabla \Delta e^k\right) + \left(f'(u^{k-1}) \nabla \left(u^k - u_N^k\right), \nabla \Delta e_{xxx}^k\right) \\
&\quad + \left(\left[f'(u^{k-1}) - f'\left(u_N^{k-1}\right)\right] \nabla u_N^k, \nabla \Delta e^k\right) \\
&\le \frac{3\gamma}{10} \|\nabla \Delta e^k\|^2 + C \left(\|e^k\|^2 + \|\rho^k\|_1^2 + \|\theta^k\|_1^2 + \tau^2 \|\bar{\partial}_t u_N^k\|_1^2\right), \\
&\quad - \left(\nabla \cdot g(u^k), \Delta e^k\right) + B\left(u_N^{k-1}, u_N^k, \Delta e^k\right) \\
&\le \frac{\gamma}{10} \left(\|\nabla \Delta e^k\|^2 + \|\nabla e^k\|^2\right) + C \left(\|e^k\|^2 + \|\rho^k\|_1^2 + \|\theta^k\|_1^2 + \tau^2 \|\bar{\partial}_t u_N^k\|_1^2\right).
\end{aligned}$$

Summing up, we get

$$\bar{\partial}_t \|\nabla e^k\|^2 + \gamma \|\nabla \Delta e^k\|^2 \le C \left(\|e^k\|^2 + \|\rho^k\|_1^2 + \|\theta^k\|_1^2 + \tau^2 \|\bar{\partial}_t u_N^k\|_1^2\right).$$

Multiplying the above relation by τt_k, summing them for k form 1 to n, and using Lemma 2.3, (3.18), (3.19), we derive that

$$\begin{aligned}
&t_n \|\nabla e^k\|^2 + \gamma \tau \sum_{k=1}^{n} t_k \|\Delta e^k\|^2 \\
&\le \tau \sum_{k=1}^{n} \|\nabla e^k\|^2 + \tau \sum_{k=1}^{n} t_k \left(\|e^k\|^2 + \|\rho^k\|_1^2 + \|\theta^k\|_1^2 + \tau^2 \|\bar{\partial}_t u_N^k\|_1^2\right) \\
&\le C(1 + t_n^4) e^{C t_n} (N^{-2} + \tau).
\end{aligned} \tag{3.20}$$

Setting $\varphi = \Delta^2 e^k$ in (3.13), we derive that

$$\begin{aligned}
&\bar{\partial}_t \|\Delta e^k\|^2 + \gamma \|\Delta^2 e^k\|^2 + \tau \|\bar{\partial}_t \Delta e^k\|^2 \\
&= \gamma\Big(- \Delta\theta^k + \theta^k, \Delta^2 e^k\Big) \quad + \Big(\nabla \cdot \Big[\nabla \cdot f(u^k) - f'\big(u_N^{k-1}\big)\nabla \cdot u_N^k\Big], \Delta^2 e^k\Big) \\
&\quad + B\Big(u^k, u^k, \Delta^2 e^k\Big) - B\Big(u_N^{k-1}, u_N^k, \Delta^2 u^k\Big).
\end{aligned}$$

Note that

$$\gamma\Big(- \Delta\theta^k + \theta^k, \Delta^2 e^k\Big) \le \frac{\gamma}{6}\|\Delta^2 e^k\|^2 + C\|\theta^k\|_2^2.$$

On the other hand, we have

$$\begin{aligned}
&\Big(\nabla \cdot \Big[\nabla \cdot f(u^k) - f'\big(u_N^{k-1}\big)\nabla \cdot u_N^k\Big], \Delta^2 e^k\Big) \\
&= \Big(f'(u^k)\Delta u^k - f'\big(u_N^{k-1}\big)\Delta u_N^k + f''(u^k)|\nabla u^k|^2 - f''\big(u_N^{k-1}\big)\nabla u_N^{k-1}\nabla u_N^k, \Delta^2 e^k\Big) \\
&= \Big(f'(u^k)\Delta u^k - f'(u^{k-1})\Delta u^k, \Delta^2 e^k\Big) + \Big(f'(u^{k-1})\Delta u^k - f'(u^{k-1})\Delta u_N^k, \Delta^2 e^k\Big) \\
&\quad + \Big(f'(u^{k-1})\Delta u_N^k - f'\big(u_N^{k-1}\big)\Delta u_N^k, \Delta^2 e^k\Big) + \Big(\Big[f''(u^k) - f''(u^{k-1})\Big]|\nabla u^k|^2, \Delta^2 e^k\Big) \\
&\quad + \Big(f''(u^{k-1})\nabla u^k \nabla(u^k - u_N^k, \Delta^2 e^k\Big) \\
&\quad + \Big(f''(u^{k-1})\nabla u^k \nabla u_N^k - f''(u^{k-1})\nabla u^{k-1}\nabla u_N^k, \Delta^2 e^k\Big) \\
&\quad + \Big(f''(u^{k-1})\nabla u^{k-1}\nabla u_N^k - f''(u^{k-1})\nabla u_N^{k-1}\nabla u_N^k, \Delta^2 e^k\Big) \\
&\quad + \Big(\Big[f''(u^{k-1}) - f''(u_N^{k-1})\Big]\nabla u_N^{k-1}\nabla u_N^k, \Delta^2 e^k\Big) \\
&\le \frac{\gamma}{6}\|\Delta^2 e^k\|^2 + C\Big(\tau^2\|\bar{\partial}_t u^k\|_2^2 + \|\rho^k\|_2^2 + \|\theta^k\|_2^2 + \|e^k\|^2\Big).
\end{aligned}$$

and

$$\begin{aligned}
&B(u^k, u^k, \Delta^2 e^k) - B(u_N^{k-1}, u_N^k, \Delta^2 e^k) \\
&= 2\Big(G(u^k)\nabla \cdot u^k - G(u^{k-1})\nabla \cdot u^k, \Delta^2 e^k\Big) \\
&+2\Big(G(u^{k-1})\nabla \cdot u^k - G\big(u_N^{k-1}\big)\nabla \cdot u^k, \Delta^2 e^k\Big) \\
&+2\Big(G\big(u_N^{k-1}\big)\nabla \cdot u^k - G\big(u_N^{k-1}\big)\nabla \cdot u_N^k, \Delta^2 e^k\Big) \\
&+\Big(G'(u^k)u^k\nabla \cdot u^k - G'(u^{k-1}u^k)\nabla \cdot u^k, \Delta^2 e^k\Big) \\
&+\Big(G'(u^{k-1})u^k\nabla \cdot u^k - G'\big(u_N^{k-1}\big)u^k\nabla \cdot u^k, \Delta^2 e^k\Big) \\
&+\Big(G'\big(u_N^{k-1}\big)u^k\nabla \cdot u^k - G'\big(u_N^{k-1}\big)u_N^k\nabla \cdot u^k, \Delta^2 e^k\Big) \\
&+\Big(G'\big(u_N^{k-1}\big)u_N^k\nabla \cdot u^k - G'\big(u_N^{k-1}\big)u_N^k\nabla \cdot u^{k-1}, \Delta^2 e^k\Big)
\end{aligned}$$

$$+\Big(G'\big(u_N^{k-1}\big)u_N^k\nabla\cdot u^{k-1}-G'\big(u_N^{k-1}\big)u_N^k\nabla\cdot u_N^{k-1},\,\Delta^2e^k\Big)$$
$$\le \frac{\gamma}{6}\|\Delta^2e^k\|^2+C\Big(\|\rho^k\|_2^2+\|\theta^k\|_2^2+\|e^k\|^2+\tau^2\|\bar\partial_t u_N^k\|_2^2\Big).$$

Summing up, we get

$$\bar\partial_t\|\Delta e^k\|^2+\gamma\|\Delta^2e^k\|^2\le C\left(\|\rho^k\|_2^2+\|\theta^k\|_2^2+\|e^k\|^2+\tau^2\|\bar\partial_t u_N^k\|_2^2\right). \tag{3.21}$$

By Lemma 1.3 and Lemma 3.3, we have

$$\|\rho^k\|_2^2\le CN^{-2}\|\nabla\Delta u^k\|^2\le C\left(\frac{1}{t_k}+t_k\right)N^{-2}.$$

Therefore, multiplying (3.21) by τt_k^2, summing up, we deduce that

$$t_n^2\|\Delta e^n\|^2+\gamma\tau\sum_{k=1}^n t_k^2\|\Delta^2e^k\|^2$$
$$\le C\tau\sum_{k=1}^n t_k^2\left[\Big(\frac{1}{t_k}+t_k\Big)N^{-2}+\|\theta^k\|_2^2+\|e^k\|^2+\tau^2\|\bar\partial_t u_N^k\|_2^2\right]+2\tau\sum_{k=1}^n t_k\|\Delta e^k\|^2$$
$$\le Ce^{Ct_n}(N^{-2}+\tau).$$

It then follows from Triangle's inequality that

$$\|u^n-u_N^n\|_2^2\le 3\left(\|\rho^n\|_2^2+\|\theta^n\|_2^2+\|e^n\|_2^2\right)\le C(N^{-2}+\tau),\quad \forall t\in(0,+\infty).$$

Then, we complete the proof of the theorem. □

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