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Edge Colorings of Planar Graphs Without 6-Cycles with Three Chords

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Abstract A graph G is of class 1 if its edges can be colored with k colors such that adjacent edges receive different colors, where k is the maximum degree of G. It is proved here that every planar graph is of class 1 if its maximum degree is at least 6 and any 6-cycle contains at most two chords.

Keywords Edge coloring · Planar graph · Cycle · Class 1

1 Introduction

All graphs considered here are finite and simple. Let G be a graph with the vertex set $V(G)$ and edge set $E(G)$. We denote the maximum degree of G by $\Delta(G)$. If $v \in V(G)$, then its neighbor set $N_G(v)$ (or simply $N(v)$) is the set of the vertices in G adjacent to v and the degree $d(v)$ of v is $|N_G(v)|$. For $V' \subseteq V(G)$, denote $N(V') = \cup_{u \in V'} N(u)$. A *k-vertex*, *$k^-$-vertex*, or a *$k^+$-vertex* is a vertex of degree k, at most k or at least k, respectively. A k (or k^+)-vertex adjacent to a vertex x is called a k (or k^+)-neighbor of x. Let $d_k(x)$, $d_{k^+}(x)$ denote the number of k-neighbors, k^+- neighbors of x. A k-cycle is a cycle of length k. Two cycles sharing a common edge are said to be adjacent. Given a cycle C of length k in G, an edge $xy \in E(G) \backslash E(C)$ is called a *chord* of C if $x, y \in V(C)$. Such a cycle C is also called a chordal-k-cycle.

Let G be a plane graph, $F(G)$ be the face set of G. A face of an graph is said to be incident with all edges and vertices in its boundary. Two faces sharing an edge e are

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said to be adjacent at e. The degree of a face f, denoted by $d_G(f)$ is the number of edges incident with f where each cut edge is counted twice. A k-, k^+-face is a face of degree k, at least k. A k-face of G is called an $(i_1, i_2, \ldots, i_k)$-face if the vertices in its boundary in clockwise order are of degrees $i_1, i_2, \ldots, i_k$ respectively. A 3-face incident with distinct vertices x, y, z is denoted by (x, y, z), moreover, by $[x, y, z]$ if $d(x) \leq d(y) \leq d(z)$. A 4-face incident with distinct vertices w, v, x, y is denoted by (w, v, x, y), moreover, by $[w, v, x, y]$ if $d(x) = 2$ and v, x, y form a 3-face, we call this 4-face special. For a vertex $v \in V(G)$, we denote by $f_k(v)$ the number of k-faces incident with v.

A graph has an edge k-coloring if its edges can be colored with color set $\{1, 2, \ldots, k\}$ such that adjacent edges receive different colors. A graph is k-edge-colorable if it has an edge k-coloring. The edge chromatic number of a graph G, denoted by $\chi'(G)$, is the smallest integer k such that G is k-edge-colorable. In 1964, Vizing proved that for any simple graph G, $\Delta(G) \leq \chi'(G) \leq \Delta(G) + 1$. A graph G is said to be of class 1 if $\chi'(G) = \Delta(G)$, and of class 2 if $\chi'(G) = \Delta(G) + 1$. A graph G is critical if it is connected and of class 2, and $\chi'(G - e) < \chi'(G)$ for any edge e of G. A critical graph with the maximum degree Δ is called a Δ-critical graph. It is clear that every critical graph is 2-connected.

For planar graphs, Vizing [2] noted that if $\Delta \in \{2, 3, 4, 5\}$, there exist Δ-critical planar graphs, and proved that every planar graph with $\Delta \geq 8$ is of class 1 and then conjectured that every planar graph with maximum degree 6 or 7 is of class 1 (There are more general results, see [3] and [5]). The case $\Delta = 7$ for the conjecture has been verified by Zhang [11] and, independently, by Sanders and Zhao [7]. The case $\Delta = 6$ remains open, but some partial results are obtained. Theorem 16.3 in [2] stated that a planar graph with the maximum degree Δ and the girth g is of class 1 if $\Delta \geq 3$ and $g \geq 8$, or $\Delta \geq 4$ and $g \geq 5$, or $\Delta \geq 5$ and $g \geq 4$. Lam, Liu, Shiu, and Wu [4] proved that a planar graph G is of class 1 if $\Delta \geq 6$ and no two 3-cycles of G sharing a common vertex. Zhou [12] obtained that every planar graph with $\Delta \geq 6$ and without 4 or 5-cycles is of class 1. Bu and Wang [1] proved that every planar graph with $\Delta \geq 6$ and without chordal 5-cycles and chordal 6-cycles is of class 1. Wu and Xue [9] extended the result that every planar graph with $\Delta \geq 6$ and without 5-cycles with two chord is of class 1. Ni [6] proved that every planar graph with $\Delta \geq 6$ and without chordal 6-cycles is of class 1. Recently, Xue and Wu [10] extended the result that every planar graph with $\Delta \geq 6$ and without 6-cycles with two chords is of class 1. In the paper, we shall improve the above result to planar graphs with $\Delta = 6$ and without 6-cycles with three chords.

2 The Main Result and its Proof

Firstly, we introduce some known lemmas.

Lemma 1 [7],[11] *Every planar graph with maximum degree at least* 7 *is of class* 1.

Lemma 2 (Vizing's Adjacency Lemma [2]) *Let G be a Δ-critical graph, and let u and v be adjacent vertices of G with $d(v) = k$.*

(a) If $k < \Delta$, then u is adjacent to at least $\Delta - k + 1$ vertices of degree Δ;

(b) If $k = \Delta$, then u is adjacent to at least two vertices of degree Δ.

From the above Lemma, it is easy to get the following corollary.

Corollary 3 *Let G be a Δ-critical graph. Then*

(a) every vertex is adjacent to at most one 2-vertex and at least two Δ-vertices;
(b) the sum of the degree of any two adjacent vertices is at least $\Delta + 2$;

(c) if $uv \in E(G)$ and $d(u) + d(v) = \Delta + 2$, then every vertex of $N(\{u, v\}) \setminus \{u, v\}$ is a Δ-vertex.

Lemma 4 [11] *Suppose that G is a Δ-critical graph, $uv \in E(G)$ and $d(u)+d(v) = \Delta + 2$. Then*

(a) every vertex of $N(N(\{u, v\})) \setminus \{u, v\}$ is of degree at least $\Delta - 1$;
(b) if $d(u), d(v) < \Delta$, then every vertex of $N(N(\{u, v\})) \setminus \{u, v\}$ is a Δ-vertex.

Lemma 5 [7] *No Δ-critical graph has distinct vertices x, y, z such that x is adjacent to y and z, $d(z) < 2\Delta - d(x) - d(y) + 2$, and xz is in at least $d(x) + d(y) - \Delta - 2$ triangles not containing y.*

Lemma 6 [8] *Let G be a Δ-critical graph with $\Delta(G) \geq 6$ and let x be a 4-vertex. Then the following hold:*

(a) If x is adjacent to a $(\Delta-2)$-vertex, say y, then every vertex of $N_G(N_G(x)) \setminus \{x, y\}$ is a Δ-vertex;
(b) Suppose that x is not adjacent to any $(\Delta-2)$-vertex and y is one neighbor of x. If y is adjacent to $d_G(y) - (\Delta - 3)$ $(\Delta - 2)^-$-vertices, then each of the other three neighbors of x is adjacent to only one $(\Delta - 2)^-$-vertex, which is x;
(c) If x is adjacent to a $(\Delta-1)$-vertex, then there are at least two Δ-vertices in $N_G(x)$ which are adjacent to at most two $(\Delta-2)^-$-vertices. Moreover, if x is adjacent to two $(\Delta-1)$-vertices, then each of the two Δ-neighbors of x is adjacent to exactly one $(\Delta-2)^-$-vertex, which is x.

Let the edges of a graph be colored with colors from $C = \{1, \ldots, k\}$ and let $u \in V$. If an edge incident with u is colored i, we say u sees i. Otherwise, we say u misses i. If a vertex u sees a color i, we use (u, i) to denote the edge incident with u colored i. Given two colors $i, j \in \{1, \ldots, k\}$, an (i, j)-chain is a path whose edges are colored alternatively i and j, and we use $(u, i) \sim (v, j)$ to denote that there is a (i, j)-chain containing (u, i) and (v, j). Let $L_{i,j}(u)$ denote the longest (i, j)-chain passing through u.

The following is the key fact when dealing with a Δ-critical graph G.

Fact 7 Let G be a Δ-critical graph and $xy \in E(G)$. Giving any edge Δ-coloring of $G - xy$, if x misses j and y misses k, then x sees k, y sees j, and $(x, k) \sim (y, j)$.

Proof If x does not see k, then we can color xy with k to obtain an edge Δ-coloring of G, a contradiction. By the same argument, y sees j. If $(x, k) \nsim (y, j)$, then we can swap colors on $L_{k,j}(x)$ and color xy with k to obtain an edge Δ-coloring of G, a contradiction, too. ◻

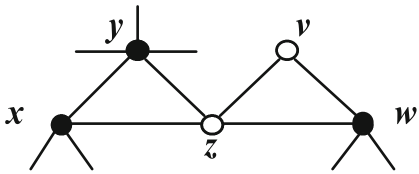

Fig. 1 Black vertices do not have neighbors other than presented in the picture, while white vertices can be adjacent to each other as well as to some other vertices

Lemma 8 *No* 6*-critical graph has distinct vertices* v, w, x, y, z *such that* $d(x) = d(w) = 4$, $d(y) = 5$, *and* vwz *and* xyz *are triangles* (*see Fig.* 1).

Proof Suppose, to be contrary, that a 6-critical graph G contains such vertices v, w, x, y, z. Since G is 6-critical, $G - xy$ has an edge 6-coloring ϕ. By Fact 7, we can assume that $\phi(xz) = 1$, x sees 2, 3 and y sees 4, 5, 6. We consider the following cases.

Case 1 $\phi(yz) \in \{4, 5, 6\}$, With out loss of generality (WLOG), assume that $\phi(yz) = 4$.

Subcase 1.1 y misses 1.

Since $d(y) = 5$, y must miss 2 or 3. WLOG, assume that y misses 2. Then y sees 3. By Fact 7, we have $(x, i) \sim (y, j)$, where $i \in \{1, 2\}$ and $j \in \{4, 5, 6\}$.

Subcase 1.1.1 $\phi(wz) = 2$.

Since $(x, 2) \sim (y, 4)$, w sees 4. If w misses 1, then we can obtain an edge 6-coloring of G by recoloring wz with 1, zx with 4, yz with 2, and coloring xy with 1, a contradiction. So w sees 1. Since $d(w) = 4$, w must miss 5 or 6. WLOG, assume that w misses 5. Since $(x, 1) \sim (y, 5)$, $L_{5,1}(w)$ does not pass x and y. So we swap colors on $L_{1,5}(w)$, recolor wz with 1, zx with 4, yz with 2, and coloring xy with 1 to obtain an edge 6-coloring of G, a contradiction.

Subcase 1.1.2 $\phi(wz) = 3$ and $\phi(vz) = 2$.

Suppose that $\phi(vw) = 1$. Then w sees 2, for otherwise, we can obtain an edge 6-coloring of G by recoloring zv with 1, yz and vw with 2, zx with 4, and coloring xy with 1, a contradiction. If w misses $i \in \{4, 5, 6\}$, then we can swap colors on $L_{2,i}(w)$ to satisfy that w misses 2. So w sees 4, 5, and 6. It is impossible. If $\phi(vw) = 4$, then w sees 2 according to $(x, 2) \sim (y, 4)$, and it follows that w sees 5, 6, it is also impossible. Suppose that $\phi(vw) = 5$. Then w sees 1, for otherwise we swap colors on $L_{5,1}(w)$ to go back to the previous case that $\phi(vw) = 1$. By w seeing 1, we can induce that w also sees 6. Since $d(w) = 4$ and w sees 1, 3, 5, 6, w misses 4. Thus we swap colors on $L_{4,1}(w)$ to satisfy that w misses 1 to go back to the above case. It is similar to settle the case $\phi(vw) = 6$.

Subcase 1.1.3 $\phi(wz) = 3$ and $\phi(vz) \in \{5, 6\}$. WLOG, assume that $\phi(vz) = 5$.

Suppose that $\phi(vw) = 1$. Since $(x, 1) \sim (y, 5)$, w sees 5. Consecutively, it is easy to check that w sees 2, 4, and 6, it is impossible. If $\phi(vw) = 4$, Then w sees 1, for otherwise, we just need to swap colors on $L_{4,1}(w)$ to go back the above case. Consecutively, w sees 2, 5, and 6, it is also impossible. Suppose that $\phi(vw) = 2$. Then w sees 5, for otherwise, we just need to swap colors on $L_{2,5}(w)$ to go back to Subcase

1.1.2. Consecutively, w sees 1 and 6, it is also impossible. It is similar to settle the case that $\phi(vw) = 6$.

Subcase 1.1.4 $\phi(wz) \in \{5, 6\}$. WLOG, assume that $\phi(wz) = 5$.

Since $(x, 1) \sim (y, 5)$, w sees 1. If w misses 4, then we can obtain an edge 6-coloring of G by recoloring wz with 4, zx with 5, yz with 1, and coloring xy with 4, a contradiction. So w sees 4. w also sees 6, for otherwise, we swap colors on $L_{1,6}(w)$ to obtain the case that w misses 1. Hence w sees 1, 4, 5, 6. It follows from $d(w) = 4$ that w misses 2, and we swap colors on $L_{4,2}(w)$ to obtain that w misses 4, a contradiction.

Subcase 1.2 y sees 1.

Since $d(y) = 5$, y misses 2 and 3. By Fact 7, we have $(x, i) \sim (y, j)$, where $i \in \{2, 3\}$ and $j \in \{4, 5, 6\}$. Suppose that $\phi(wz) \in \{2, 3\}$. WLOG, assume that $\phi(wz) = 2$. Since $(x, 2) \sim (y, 4)$, w sees 4. If w misses 3, then we can swap colors on $L_{4,3}(w)$ to obtain a contradiction. So w sees 3. Also w sees 5, for otherwise, we swap colors on $L_{3,5}(w)$ to get a contradiction. It is similar to check that w sees 6. That is impossible. Suppose that $\phi(wz) \in \{5, 6\}$. WLOG, assume that $\phi(wz) = 5$. If w misses 2, then we swap colors on $L_{2,5}(w)$ and go back to the above case. So w sees 2. In the same way, w sees 3. Consecutively, we have that w sees 4 and 6, contrary to that $d(w) = 4$.

Case 2 $\phi(yz) \in \{2, 3\}$, WLOG, assume that $\phi(yz) = 3$.

By Fact 7, we have $(x, i) \sim (y, j)$, where $i \in \{1, 2\}$ and $j \in \{4, 5, 6\}$. Suppose that $\phi(wz) \in \{4, 5, 6\}$. WLOG, assume that $\phi(wz) = 4$. Since $(x, 1) \sim (y, 4)$, w sees 1. By the similar argument, we have that w sees 5 and 6. Since $d(w) = 4$, w misses 2. After swapping colors on $L_{6,2}(w)$, w misses 6, a contradiction. Suppose that $\phi(wz) = 2$. Then w sees 4, 5, and 6. Since $d(w) = 4$, w misses 1. After swapping colors on $L_{6,1}(w)$, w misses 6, a contradiction. □

Now, we begin to prove our main result.

Theorem 9 *Let G be a planar graph with $\Delta \geq 6$. If any 6-cycle contains at most two chords, then G is of class 1.*

Proof Suppose that G is a counterexample to our theorem with the minimum number of edges and suppose that G is embedded in the plane. Then G is a 6-critical graph by Lemma 1, and it is 2-connected. By Euler's formula $|V(G)| - |E(G)| + |F(G)| = 2$, we have

$$\sum_{x \in V(G)} (d(x) - 4) + \sum_{x \in F(G)} (d(x) - 4) = -8 < 0.$$

We define ch to be the initial charge. Let $ch(x) = d(x) - 4$ for each $x \in V \cup F$. So $\sum_{x \in V \cup F} ch(x) < 0$. In the following, we will reassign a new charge denoted by $ch^{'}(x)$ to each $x \in V \cup F$ according to the discharging rules, since our rules only move charges around, and do not affect the sum. If we show that $ch^{'}(x) \geq 0$ for each $x \in V \cup F$, then we get an obvious contradiction $0 \leq \sum_{x \in V \cup F} ch^{'}(x) = \sum_{x \in V \cup F} ch(x) < 0$, which completes our proof.

R1 Let v be a 2-vertex. If v is incident with a 5^+-face f, then v receives 1 from f, $\frac{1}{2}$ from each adjacent vertex; If v is incident with a special 4-face f, then v receives $\frac{1}{3}$ from f, $\frac{5}{6}$ from each adjacent vertex; Otherwise v receives 1 from each adjacent vertex.
R2 Every 3-vertex receives $\frac{1}{3}$ from each adjacent vertex.
R3 Let f be a 3-face $[x, y, z]$ with $d(x) \le d(y) \le d(z)$. If $d(x) \le 4$ and $d(y) \ge 5$, then f receives $\frac{1}{2}$ from y, $\frac{1}{2}$ from z; If $d(x) = d(y) = 4$ and $d(z) = 6$, then f receives 1 from z; If $d(x) \ge 5$, then f receives $\frac{1}{3}$ from x, y, z, respectively.
R4 Let v be a 5-vertex.
R4.1 If v is adjacent to a 3-vertex u and $N(u) = \{v, w, x\}$, then v receives $\frac{1}{3}$ from w and $\frac{1}{3}$ from x;
R4.2 If v is incident with a (4, 5, 5)-face $[u, v, w]$ and $N(u) = \{v, w, x, y\}$, then v receives $\frac{1}{6}$ from x, and $\frac{1}{6}$ from y;
R4.3 If v is incident with a (4, 5, 6)-face $[u, v, w]$, then v receives $\frac{1}{6}$ from w.
R5 Let v be a 6-vertex.
R5.1 If v is incident with a special 4-face $f = [w, v, x, y]$ such that $d(y) = 2$, then v sends $\frac{1}{3}$ to f;
R5.2 If v is incident with two adjacent 3-faces (u, v, x), (v, x, y), and $d(u) = d(x) = 4$, then v receives $\frac{1}{6}$ from y.

Now, let's began to check $ch'(x) \ge 0$ for all $x \in V \cup F$. Let $f \in F(G)$. If $d(f) \ge 5$, then f is incident with at most $d(f) - 4$ 2-vertices by Corollary 3(c), so $ch'(f) \ge ch(f) - (d(f) - 4) = 0$ by R1. Suppose $d(f) = 4$. If f is special, then $ch'(f) = 0 + \frac{1}{3} - \frac{1}{3} = 0$ by R1 and R5.1; Otherwise, $ch'(f) = ch(f) = 0$. Suppose $d(f) = 3$. Since $\Delta = 6$, f must be the $(2^+, 6, 6)$-face, $(3, 5^+, 6)$-face, $(4, 4, 6)$-face, or $(4^+, 5^+, 5^+)$-face by Lemma 2. Hence $ch'(f) = ch(f) + \min\{2 \times \frac{1}{2}, 1, 3 \times \frac{1}{3}\} = 0$ by R3.

Let $w \in V(G)$. Then $d(w) \ge 2$. If $d(w) = 2$, then w is adjacent to two 6-vertices by Corollary 3(a), so $ch'(w) = ch(w) + \min\{1 + 2 \times \frac{1}{2}, \frac{1}{3} + 2 \times \frac{5}{6}, 2 \times 1\} = 0$ by R1. If $d(w) = 3$, then w is adjacent to three 5^+-vertices by Corollary 3(b), and it follows that $ch'(w) = -1 + 3 \times \frac{1}{3} = 0$ by R2. If $d(w) = 4$, then $ch'(w) = ch(w) = 0$.

Suppose that $d(w) = 5$. Then $ch(w) = 1$, $\min\{d(u) | u \in N(w)\} \ge 3$, $d_3(w) \le 1$,and $d_6(w) \ge 2$ by Corollary 3, and $f_3(w) \le 3$ since all 6-cycles in G contain at most two chords. If all neighbors of w are 5^+-vertices, then $ch'(w) \ge 1 - 3 \times \frac{1}{3} = 0$ by R3. If w is adjacent to a 3-vertex, say w_1, then w receives $\frac{1}{3}$ from each of neighbors of w_1 except w by R4.1, and it follows that $ch'(w) \ge 1 + 2 \times \frac{1}{3} - \frac{1}{3} - 2 \times \frac{1}{2} - \frac{1}{3} = 0$ by R2 and R3. Suppose that w is adjacent to a 4-vertex. Then $d_6(w) \ge 3$ by Lemma 2a. If $f_3(w) \le 2$, then $ch'(w) \ge 1 - 2 \times \frac{1}{2} = 0$ by R3; Otherwise, $f_3(w) = 3$, w is incident with at most one $(4, 5, 5^+)$-face by Lemma 5. So $ch'(w) \ge 1 - \max\{2 \times \frac{1}{2} + \frac{1}{3} - 2 \times \frac{1}{6}, \frac{1}{2} + 2 \times \frac{1}{3} - \frac{1}{6}\} = 0$ by R3 and R4.

Suppose that $d(w) = 6$. Then $ch(w) = 2$, $d_2(w) \le 1$, and $d_6(w) \ge 2$ by Lemma 2, and $f_3(w) \le 4$ since all 6-cycles in G contain at most two chords. Note that every special 4-face is adjacent to at most two 3-faces since all 6-cycles contain at most two chords. We denote by $f_{s4}(w)$ the number of special 4-faces incident with w and $f_{34}(w) = f_3(w) + f_{s4}(w)$. It is easy to check that $f_{s4}(w) \le 3$ and $f_{34}(w) \le 4$.

Case 1 w sends some charge to a 5-vertex v (see R4).

Suppose that v is adjacent to a 3-vertex u. Then w sends $\frac{1}{3}$ to v by R4.1. By Lemma 4b, w is adjacent to five 6-vertices, that is, $d_6(w) = 5$. Since $f_{34}(w) \le 4$, $ch'(w) \ge 2 - \frac{1}{3} - 2 \times \frac{1}{2} - 2 \times \frac{1}{3} = 0$ by R3 and R4.1.

Suppose that v is incident with a $(4, 5, 5)$-face $[u, v, x]$ such that $N(u) = \{v, w, x, y\}$. Then v receives $\frac{1}{6}$ from w by R4.2. By Lemma 6c, w is adjacent to one 4^--vertex, which is u. Since $f_{34}(w) \le 4$, $ch'(w) = 2 - 2 \times \frac{1}{6} - 2 \times \frac{1}{2} - 2 \times \frac{1}{3} = 0$.

Suppose that v is incident with a $(4, 5, 6)$-face $[u, v, w]$. Then v receives $\frac{1}{6}$ from w by R4.3. If $d_4(w) = 1$, then $ch'(w) \ge 2 - 2 \times \frac{1}{2} - 2 \times \frac{1}{3} - 2 \times \frac{1}{6} = 0$; otherwise $d_4(w) = 2$. By Lemma 8, w is incident a $(4, 5, 6)$-face and at most one $(4, 6, 6)$-face, then $ch'(w) \ge 2 - 2 \times \frac{1}{2} - 2 \times \frac{1}{3} - \frac{1}{6} = \frac{1}{6} > 0$ by R3 and R4.3.

Case 2 w sends $\frac{1}{6}$ to some 6-vertex v (see R5).

Suppose v is incident with two 3-faces $[w, v, x]$ and $[v, x, y]$ such that $d(x) = d(y) = 4$. Then $f_{34}(w) \le 4$ and $d_6(w) = 5$ by Lemma 4b. So $ch'(w) = 2 - \frac{1}{6} - 2 \times \frac{1}{2} - 2 \times \frac{1}{3} > 0$.

Case 3 w sends no charge to its 5^+-vertices.

Let $k = \min\{d(u) | u \in N(w)\}$. If $k \ge 5$, then $ch'(w) \ge 2 - 4 \times \frac{1}{3} > 0$. Suppose that $k = 4$. Then $d_6(w) \ge 3$ by Lemma 2a. If w is incident with two 3-faces $[u, w, x]$ and $[w, x, y]$ such that $d(x) = d(y) = 4$, then $d_6(w) = 4$. If wy is incident with a 4^+-face, then w receives $\frac{1}{6}$ from u, and it follows that $ch'(w) \ge 2 + \frac{1}{6} - 1 - \frac{1}{2} - 2 \times \frac{1}{3} = 0$ since $f_{34}(w) \le 4$; otherwise wy is incident with another 3-face $[w, y, z]$, then w receives $\frac{1}{6}$ from each of u, z, and it follows that $ch'(w) \ge 2 + 2 \times \frac{1}{6} - 1 - 2 \times \frac{1}{2} - \frac{1}{3} = 0$ since $f_{34}(w) \le 4$; otherwise, $ch'(w) \ge 2 - \max\{1 + 3 \times \frac{1}{3}, 4 \times \frac{1}{2}\} = 0$.

Suppose that $k = 3$. Then $d_6(w) \ge 4$ by Lemma 2a. If $d_3(w) = 1$ and $d_{5^+}(w) \ge 5$, then $ch'(w) \ge 2 - \frac{1}{3} - 2 \times (\frac{1}{2} + \frac{1}{3}) = 0$; otherwise, w is incident with two 4^--vertices u, v, then u and v are incident with at most one 3-face by Lemma 5 since $d(u) + d(v) + d(w) \le 3 + 4 + 6 < 14$. So $f_{34}(w) \le 4$, and it follows that $ch'(w) \ge 2 - \frac{1}{3} - \max\{\frac{1}{2} + 2 \times \frac{1}{3}, \frac{1}{3} + 3 \times \frac{1}{3}\} > 0$ by R1 and R3.

Suppose that $k = 2$, that is, w is adjacent to a 2-vertex v. Then $d_6(w) = 5$ by Lemma 2a. If v is incident with a special 4-face $f = [u, v, w, x]$, then $f_3(v) \le 3$ and w sends $\frac{5}{6}$ to v, and it follows that $ch'(w) \ge 2 - \frac{5}{6} - \frac{1}{2} - 2 \times \frac{1}{3} = 0$; otherwise, v is incident with a 5^+-face or two 4-faces. If v is incident with a 5^+-face, then w sends $\frac{1}{2}$ to v, and it follows that $ch'(w) \ge 2 - \frac{1}{2} - (\frac{1}{2} + 3 \times \frac{1}{3}) = 0$. If v is incident with two 4-faces, then $f_{34} \le 3$, and it follows that $ch'(w) \ge 2 - 1 - 3 \times \frac{1}{3} = 0$. □

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