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Duality and Its Applications to Optimality Conditions with Nonsolid Cones

Nguyen Le Hoang Anh[1,2]

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Abstract In this paper, we first establish the weak, strong, and converse duality theorems for a pair of primal–dual problems in set-valued optimization concerning Q-efficient solutions. Then, duality theorems for quasi-relative efficient solutions and Henig efficient solutions are implied with Q being appropriately chosen cones. Finally, their applications to optimality conditions in the Kuhn–Tucker type are obtained.

Keywords Set-valued optimization · Duality · Q-efficient solution · Optimality condition · Closely convexlikeness · Quasi-relative interior

Mathematics Subject Classification 49N15 · 54C60 · 90C46

1 Introduction

In optimization, convex minimization problems (primal problems) are frequently associated maximization problems (dual problems). Studying relationships between optimal values/solutions of the primal–dual pair, known as duality, is one of the most important topics of optimization from both the theoretical and practical viewpoint, see [1–3]. In the last several decades, set-valued mappings are involved in optimization,

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✉ Nguyen Le Hoang Anh
nlhanh@hcmus.edu.vn

1 Department of Mathematics, University of Ostrava, 30. Dubna 22, 70103 Ostrava, Czech Republic

2 Department of Optimization and System Theory, University of Science, Vietnam National University Ho Chi Minh City, 227 Nguyen Van Cu, District 5, Ho Chi Minh City, Vietnam

which leads to a new chapter in optimization theory, called set-valued optimization. Recently, many researches on duality in set-valued optimization have been developing, see [4–9]. The above-mentioned results require that ordering cones have nonempty interior. But, in many problems, the interior of a cone may be empty.

To overcome these cases, several generalized concepts of the interior were introduced in [10–13]. One of them is the quasi-relative interior, known as the furthest generalization, proposed by Borwein and Lewis in [13]. This notion has been employed in different ways in optimization. In [14], optimality conditions for subconvex set-valued optimization were established via the quasi-relative interior. In [15], this notion was employed in regularity conditions for duality theorems of a convex vector optimization problem. In [16], the quasi-relative interior was used to define the quasi-relative efficient point, and optimality conditions of these solutions were obtained for set-valued equilibrium problems. However, there are very few results of duality in set-valued optimization concerning the quasi-relative efficient solutions.

The above observations motivate us first to study duality theorems in set-valued optimization with respect to Q-efficient solutions, which subsumes several kinds of properly minimal solutions (see [17]), when the ordering cones have empty interior. Then, we obtain correlative results for quasi-relative efficient solutions and Henig efficient solutions. Applications to optimality conditions are also discussed to show the advantage of our results over than other existing ones.

The layout of the paper is as follows. In Sect. 2, we recall some concepts and their properties required for the paper. In Sect. 3, we establish the weak, strong, and converse duality theorems for a pair of primal–dual problems with respect to Q-efficient solutions. Then, duality results for quasi-relative efficient solutions and Henig efficient solutions are implied. Their applications to optimality conditions are given in Sect. 4.

2 Preliminaries

Let X, Y be normed spaces, and $C \subseteq Y$ be a pointed convex cone. For $A \subseteq X$, the interior, closure, and cone hull of A are denoted by $\mathrm{int}A$, $\mathrm{cl}A$, and $\mathrm{cone}A$, respectively (resp, for short), where $\mathrm{cone}A := \{\lambda a | \lambda \geq 0, a \in A\}$. With the cone C above, its dual and strictly dual cones in the dual space Y^* are denoted by, resp.

$$C^* := \{c^* \in Y^* | \langle c^*, c\rangle \geq 0, \forall c \in C\},$$
$$C^\# := \{c^* \in C^* | \langle c^*, c\rangle > 0, \forall c \in C\backslash\{0\}\}.$$

For a set-valued mapping $F : X \to 2^Y$, the domain, image, graph of F are defined, resp, by

$$\mathrm{dom}F := \{x \in X | F(x) \neq \emptyset\}, \quad \mathrm{im}F := \{y \in Y | y \in F(X)\},$$
$$\mathrm{gr}F := \{(x, y) \in X \times Y | y \in F(x)\}.$$

We denote $F_+(.) := F(.) + C$.

A convex subset B of the convex cone C is said to be a base of C if $C = \text{cone}B$ and $0 \notin \text{cl}B$. Let B be a base of C, and for some $\epsilon \in (0, \delta)$, setting

$$C^{\triangle}(B) := \{c^* \in C^{\#} | \exists t > 0, \text{ s.t. } \langle c^*, b\rangle \geq t, \forall b \in B\},$$
$$C_{\epsilon}(B) := \text{cone}(B_Y(0, \epsilon) + B),$$

where $\delta := \inf\{||b|| | b \in B\}$ and $B_Y(0, \epsilon) := \{y \in Y | ||y|| < \epsilon\}$. Note that, $C_{\epsilon}(B)$ is convex, and $\text{int}C_{\epsilon}(B) \neq \emptyset$ even if $\text{int}C = \emptyset$.

Definition 2.1 Let $S \subseteq X$ be convex.

(i) The quasi-relative interior of S is the set defined by

$$\text{qri}S := \{x \in S | \text{cl}\,\text{cone}(S - x) \text{ is a linear subspace of } X\}.$$

(ii) A point $s_0 \in S$ is said to be a support point of A if there exists $x^* \in X^* \backslash \{0\}$ such that $\langle x^*, s_0\rangle \leq \langle x^*, s\rangle$ for all $s \in S$. The set of nonsupport points of S is denoted by $N(S)$.

If $\text{int}S \neq \emptyset$, then $N(S) = \text{int}S$ (see [18]). The following example gives us a case for which $\text{int}S = \emptyset$, but $N(S) \neq \emptyset$.

Example 2.2 Let l^2 be the space of the real sequences $\{x_n\}_{n\in\mathbb{N}}$ such that $\sum_{n=1}^{\infty} |x_n|^2 < +\infty$, equipped with $||x||_{l^2} := \left(\sum_{n=1}^{\infty} |x_n|^2\right)^{1/2}$ for $x = \{x_n\}_{n\in\mathbb{N}}$. Consider the positive cone

$$l^2_+ := \{\{x_n\}_{n\in\mathbb{N}} \subseteq l^2 | x_n \geq 0, \forall n \in \mathbb{N}\},$$

then $\text{int}l^2_+ = \emptyset$, but $N(l^2_+) = \{\{x_n\}_{n\in\mathbb{N}} \subseteq l^2 | x_n > 0, \forall n \in \mathbb{N}\}$.

Some basic properties of the quasi-relative interior, see [13,19], are collected in the following proposition.

Proposition 2.3 *Let $S \subseteq X$ be convex. Then,*

(i) *if* $\text{int}S \neq \emptyset$, *then* $\text{int}S = \text{qri}S$*;*
(ii) $\lambda\text{qri}S + (1 - \lambda)S \subseteq \text{qri}S$ *for all* $\lambda \in (0, 1]$, *which implies that* $\text{qri}S$ *is convex;*
(iii) $\text{cl}\,\text{qri}S = \text{cl}S$*;*
(iv) *suppose that S is a pointed cone, then* $0 \notin \text{qri}S$ *and* $\text{qri}S \cup \{0\}$ *is a cone;*
(v) *if $U \subseteq X$ is convex, then* $\text{qri}S \times \text{qri}U = \text{qri}(S \times U)$*;*
(vi) $x \in \text{qri}S$ *if and only if $N_S(x)$ is a linear subspace of X^*, where* $N_S(x) := \{x^* \in X^* | \langle x^*, y - x\rangle\} \leq 0, \forall y \in S\}$.

Definition 2.4 Let $Q \subseteq Y$ be an arbitrary nonempty cone, different from Y, and $A \subseteq Y$. A point $a_0 \in A$ is said to be a Q-efficient point of A if

$$(A - a_0) \cap -Q = \emptyset.$$

Definition 2.4 was also mentioned in [17] with respect to an open cone Q. This notion contains as special cases several kinds of properly efficient points in vector optimization. In the paper, a Q-efficient point is defined without the openness of Q.

Remark 2.5 It follows from Definition 2.4 that

(i) if $Q = \text{qri}C$, a_0 is said to be a quasi-relative efficient point of A. When $\text{int}C \neq \emptyset$, it becomes the well-known definition of a weakly efficient point of A.
(ii) if $Q = \text{int}_\epsilon C(B)$, a_0 is said to be a Henig efficient point of A.

Definition 2.6 ([20]) The mth-order contingent cone of a subset $S \subseteq X$ at $x_0 \in \text{cl}S$ with respect to $u_i \in X$, $i = 1, \ldots, m-1$, is defined by

$$T_S^m(x_0, u_1, \ldots, u_{m-1}) := \{u \in X | \exists t_n \to 0^+, \exists \\ u_n \to u, x_0 + t_n u_1 + \ldots + t_n^{m-1} u_{m-1} + t_n^m u_n \in S\}.$$

When $m = 1$, $T_S(x_0)$ has some properties as follows.

Proposition 2.7 ([20]) *Let $S \subseteq X$ and $x_0 \in \text{cl}S$. Then,*

(i) *$T_S(x_0)$ is closed;*
(ii) *$T_S(x_0) = T_{\text{cl}S}(x_0)$;*
(iii) *if S is convex, then $T_S(x_0) = \text{clcone}(S - x_0)$; whence $T_S(x_0)$ is convex;*
(iv) *if S is convex, then $\text{qri}S = \{x \in S |\ T_S(x)$ is a linear subspace of $X\}$ (see [13]).*

Definition 2.8 ([21]) Let $S \subseteq X$, and $F : X \to 2^Y$. The mapping F is said to be closely C-convexlike on S if $\text{cl}(F(S) + C)$ is convex.

3 Duality

Suppose C and D be closed pointed convex cones in normed spaces Y and Z, resp, and $\text{qri}C \times \text{qri}D \neq \emptyset$. For $S \subseteq X$ be nonempty, and $F : X \to 2^Y$ and $G : X \to 2^Z$ with $\text{dom}F \cup \text{dom}G \subseteq S$, we consider the following constrained set-valued optimization problem

$$\text{(P)} \qquad \begin{cases} \text{Minimize } F(x), \\ x \in S, \\ G(x) \cap -D \neq \emptyset. \end{cases}$$

The feasible set of (P) is denoted by $A := \{x \in S : G(x) \cap -D \neq \emptyset\}$.

Let Q be an arbitrary nonempty convex cone, different from Y. A point $(x_0, y_0) \in \text{gr}F$ is said to be a Q-efficient solution of (P) if $x_0 \in A$ and $(F(A) - y_0) \cap -Q = \emptyset$.

Consider the dual problem of (P) formulated as follows

$$\text{(D)} \qquad \begin{cases} \text{Maximize } h(c^*, d^*, y), \\ (c^*, d^*, y) \in H, \end{cases}$$

where $h : Y^* \times Z^* \times Y \to Y$ is defined by $h(c^*, d^*, y) := y$, and

$$H := \left\{(c^*, d^*, y) \in (Q^*\backslash\{0\}) \times D^* \times Y | \langle c^*, y\rangle \leq \inf_{(v,w)\in(F,G)(S)} \left\{\langle c^*, v\rangle + \langle d^*, w\rangle\right\}\right\}.$$

Let

$$\Delta := \{y \in Y | \text{there exists } (c^*, d^*) \in (Q^*\backslash\{0\}) \times D^* \text{ such that } (c^*, d^*, y) \in H\}.$$

A feasible element $(c^*, d^*, y_0) \in H$ is said to be a Q-efficient solution of (D) if $(\Delta - y_0) \cap Q = \emptyset$.

We now establish the weak, strong, and converse duality theorems for Q-efficient solutions of the primal–dual problems (P) and (D).

Theorem 3.1 (Weak duality) *Suppose that* (x_0, y_0) *and* $(c^*, d^*, \overline{y})$ *are feasible elements of* (P) *and* (D)*, resp. Then,* $\langle c^*, \overline{y}\rangle \leq \langle c^*, y_0\rangle$.

Proof Since (x_0, y_0) is a feasible elements of (P), there exists $z_0 \in G(x_0) \cap -D$ with $x_0 \in S$. It follows from the feasibility of $(c^*, d^*, \overline{y})$ that

$$\begin{aligned}\langle c^*, \overline{y}\rangle &\leq \inf_{(v,w)\in(F,G)(S)} \{\langle c^*, v\rangle + \langle d^*, w\rangle\}\\ &\leq \langle c^*, y_0\rangle + \langle d^*, z_0\rangle\\ &\leq \langle c^*, y_0\rangle \;\; (z_0 \in -D \text{ and } d^* \in D^*).\end{aligned}$$

For the next result, we denote $(F_Q, G)_+(.) := (F, G)(.) + Q \times D$.

Theorem 3.2 (Strong duality) *Let* $(x_0, y_0) \in \mathrm{gr}F$*,* $z_0 \in G(x_0) \cap -D$*, and* $N(D) \neq \emptyset$*. Suppose that* (x_0, y_0) *is a* Q*-efficient solution of (P) and the following conditions hold*

- (i) $(F - y_0, G)$ *is closely* $(C \times D)$*-convexlike on* S*, where* $(F - y_0)(x) := F(x) - y_0$*;*
- (ii) $(F_Q)_+(A)$ *is closed whenever* Q *is not open;*
- (iii) $\exists (e, k) \in -(Q \times \mathrm{qri}D)$ *satisfying* $(e, k) \notin T_M(0, 0)$*, where* $M := \{(F_Q, G)_+(x) - (y_0, 0) | x \in S\backslash A\}$*;*
- (iv) $\exists \overline{x} \in S$ *such that* $G(\overline{x}) \cap -N(D) \neq \emptyset$*.*
  If the cone Q *is open, then there exists* $(c^*, d^*) \in (Q^*\backslash\{0\}) \times D^*$ *such that* (c^*, d^*, y_0) *is a* Q*-efficient solution of* (D)*. Otherwise, the efficiency of* (c^*, d^*, y_0) *is fulfilled if, additionally, the following condition hold*
- (v) $I(H) \subseteq Q^\#$*, where* $I : Y^* \times Z^* \times Y \to Y^*$ *is defined by* $I(\overline{c}^*, \overline{d}^*, y) = \overline{c^*}$ *for every* $(\overline{c}^*, \overline{d}^*, y) \in H$*.*

Proof With (e, k) in the condition (iii), we first prove that $(e, k) \notin T_{\hat{M}}(0, 0)$, where $\hat{M} := (F_Q, G)_+(S) - (y_0, 0)$. Suppose that $(e, k) \in T_{\hat{M}}(0, 0)$, i.e., there exist $t_n \to 0^+$, $\{x_n\}_{n\in\mathbb{N}} \subseteq S$, $(y_n, z_n) \in (F, G)(x_n)$ for all $n \in \mathbb{N}$, and $\{(q_n, d_n)\}_{n\in\mathbb{N}} \subseteq Q \times D$ such that

$$\begin{aligned} v_n &:= \frac{y_n + q_n - y_0}{t_n} \to e,\\ w_n &:= \frac{z_n + d_n}{t_n} \to k. \end{aligned} \tag{3.1}$$

If $\{x_n\}_{n\in\mathbb{N}} \subseteq S\backslash A$ for infinitely many $n \in \mathbb{N}$, then $(e,k) \in T_M(0,0)$ (by Definition 2.6), which contradicts (iii). If $\{x_n\}_{n\in\mathbb{N}} \subseteq A$, by the Q-efficiency of (x_0, y_0), we get $(F(A) - y_0) \cap -Q = \emptyset$, which implies that

$$\text{cone}((F_Q)_+(A) - y_0) \cap -Q = \emptyset. \tag{3.2}$$

It follows from (3.1) that $v_n \in \text{cone}((F_Q)_+(A) - y_0)$ and $e \in \text{clcone}((F_Q)_+(A) - y_0)$. If Q is open, it follows from (3.2) that $\text{clcone}((F_Q)_+(A) - y_0) \cap -Q = \emptyset$, which contradicts the fact that $e \in \text{clcone}((F_Q)_+(A) - y_0)$ and $e \in -Q$. When Q is not open, the condition (ii) ensures that $e \in \text{cone}((F_Q)_+(A) - y_0)$, which contradicts (3.2) since $e \in -Q$. Hence, $(e,k) \notin T_{\hat{M}}(0,0)$. □

Moreover, $T_{\hat{M}}(0,0)$ is a closed convex cone [from (i) and Proposition 2.7(i)–(iii)]. Thus, we can separate (e,k) and $T_{\hat{M}}(0,0)$ (see [1]), i.e., there exists $(c^*, d^*) \in (Y^* \times Z^*)\backslash\{(0,0)\}$ such that, for all $(y,z) \in T_{\hat{M}}(0,0)$,

$$\langle c^*, e\rangle + \langle d^*, k\rangle < 0 \le \langle c^*, y\rangle + \langle d^*, z\rangle.$$

By Proposition 2.7(iii), we get $\hat{M} \subseteq T_{\hat{M}}(0,0)$. Therefore, for all $(y,z) \in (F,G)(S)$, $(q,d) \in Q \times D$,

$$\langle c^*, y + q - y_0\rangle + \langle d^*, z + d\rangle \ge 0. \tag{3.3}$$

For given $q_0 \in Q$, taking $y = y_0$, $q = q_0$, and $z = z_0$ in (3.3), we have, for each $d \in D$,

$$\left\langle c^*, \frac{1}{n} q_0 \right\rangle + \langle d^*, z_0 + d\rangle \ge 0,$$

equivalently,

$$\frac{1}{n}\langle c^*, q_0\rangle + \langle d^*, z_0 + d\rangle \ge 0.$$

When $n \to +\infty$, the above inequality implies that, for all $d \in D$,

$$\langle d^*, z_0 + d\rangle \ge 0. \tag{3.4}$$

From (3.4), we have $\langle d^*, z_0\rangle \ge 0$ (with $d = 0$). Moreover, (3.4) shows that $\langle d^*, d\rangle \ge 0$ for all $d \in D$ (since D is a cone), i.e., $d^* \in D^*$. Thus, $\langle d^*, z_0\rangle \le 0$ ($z_0 \in -D$). Hence, $\langle d^*, z_0\rangle = 0$.

Considering (3.3) with $y = y_0$, $z = z_0$, and $d = 0$, we get $\langle c^*, q\rangle \ge 0$ for all $q \in Q$, which implies that $c^* \in Q^*$. Suppose that $c^* = 0$. By (iv), there exists $\overline{z} \in G(\overline{x})$ with $-\overline{z} \in N(D)$. It follows from (3.3) that $\langle d^*, -\overline{z}\rangle \le \langle d^*, d\rangle$. If $d^* \ne 0$, then $-\overline{z}$ is a support point of D, which contradicts the fact that $-\overline{z} \in N(D)$. Thus, $d^* = 0$, which is impossible since $(c^*, d^*) \ne (0,0)$. Hence, $c^* \ne 0$.

On the other hand, (3.3) gives us the following inequality

$$\langle c^*, y_0 \rangle \le \inf_{(v,w)\in(F,G)(S)} \left\{ \left\langle c^*, v \right\rangle + \left\langle d^*, w \right\rangle \right\}.$$

Accordingly, (c^*, d^*, y_0) is a feasible element of (D). Next, we prove that (c^*, d^*, y_0) is a Q-maximal solution of (D). Suppose to the contrary, i.e., there exists $(\hat{c}^*, \hat{d}^*, \hat{y}) \in H$ such that $\hat{y} - y_0 \in Q$ $(\hat{y} \neq y_0)$. There are only two cases as follows.

- *Case* 1 If the cone Q is open, we get that $\langle \hat{c}^*, \hat{y} - y_0 \rangle > 0$. On the other hand, it follows from Theorem 3.1 that $\langle \hat{c}^*, \hat{y} - y_0 \rangle \le 0$, which is a contradiction.
- *Case* 2 If Q is a nonopen cone, from (v), we get that $\langle \hat{c}^*, \hat{y} - y_0 \rangle > 0$, which contradicts Theorem 3.1.

Remark 3.3 (i) The conditions (ii) and (iii) are used to imply that $(e, k) \notin T_{\hat{M}(0,0)}$, i.e., $T_{\hat{M}}(0, 0)$ is not a linear subspace, which means $(0, 0) \notin \text{qri}\hat{M}$ (Proposition 2.7(iv)). Moreover, by Proposition 2.3(vi), $N_{\hat{M}}(0, 0)$ is also not a linear subspace, so there exists $(c^*, d^*) \neq (0, 0)$ such that $(c^*, d^*) \in N_{\hat{M}}(0, 0)$. This is the main idea for the proof of the strong duality.

(ii) The fact that $(0, 0) \notin \text{qri}\hat{M}$ is equivalent to the condition (b) in Theorem 8.1.16 in [20]. Thus, our assumptions can be considered as a sufficient condition for that of Theorem 8.1.16 in [20].

(iii) The condition (v) is necessary for the Q-efficiency of (c^*, d^*, y_0) when the cone Q is not open. In general, the conclusion that $\langle \hat{c}^*, \hat{y} - y_0 \rangle > 0$ with $\hat{y} - y_0 \in Q$ may be not valid in this case. Indeed, let $Y = \mathbb{R}^2$ and $Q = \{(0, a) \in Y | a \ge 0\}$, then Q is not an open cone and $Q^* = \{(y_1, y_2) \in Y | y_1 \in \mathbb{R}, y_2 \ge 0\}$. With $c^* = (1, 0) \in Q^*$, it is easy to see that $\langle c^*, y - y_0 \rangle = 0$ and $y - y_0 \in Q$ for all $y \in Q$.

Theorem 3.4 (Converse duality) *Let* $x_0 \in A$, *and* $(x_0, y_0) \in \text{gr}F$. *Suppose that* (c^*, d^*, y_0) *is a feasible element of* (D). *If* Q *is an open cone, then* (x_0, y_0) *is a* Q-*efficient solution of* (P). *Otherwise, the* Q-*efficiency of* (x_0, y_0) *is satisfied if* $c^* \in Q^{\#}$.

Proof Suppose that (x_0, y_0) is not a Q-efficient solution of (P), i.e., there exists $\overline{x} \in A$ such that $(F(\overline{x}) - y_0) \cap -Q \neq \emptyset$. It means that there is $\overline{y} \in F(\overline{x})$, $\overline{z} \in G(\overline{x}) \cap -D$ with $\overline{y} - y_0 \in -Q$. By the assumption, in the case of Q being a nonopen cone, we get that $\langle c^*, \overline{y} - y_0 \rangle < 0$ (this property is fulfilled for $c^* \in Q^* \backslash \{0\}$ when Q is open). It follows from the feasibility of (c^*, d^*, y_0) that $\langle c^*, y \rangle + \langle d^*, z \rangle \ge \langle c^*, y_0 \rangle$ for all $(y, z) \in (F, G)(S)$. Thus,

$$\langle c^*, y - y_0 \rangle \ge -\langle d^*, z \rangle.$$

With $y = \overline{y}$, $z = \overline{z}$ in the above inequality, we have $\langle c^*, \overline{y} - y_0 \rangle \ge 0$, which is a contradiction. □

The assumption that $c^* \in Q^{\#}$ in Theorem 3.4 is not a necessary condition for the Q-efficiency of (x_0, y_0) by the following example.

Example 3.5 Let $X = Y = \mathbb{R}^2$, $Z = \mathbb{R}$, $S = \mathbb{R}^2_+$, $C = \{0\} \times \mathbb{R}_+$, $D = \mathbb{R}_+$, and $Q = \{0\} \times (\mathbb{R}_+ \backslash \{0\})$. Consider two set-valued map $F : X \to 2^Y$, $G : X \to 2^Z$ defined by $F(x) := \mathbb{R}^2_+$ for all $x \in S$ and

$$G(x) := \begin{cases} D, & \text{if } x \in (\text{int}S) \cup \{(0,0)\}, \\ \{1/n\}_{n \in \mathbb{N}}, & \text{if } x \in \{(x_1, x_2) \in S \backslash \{(0,0)\} | \, x_1 x_2 = 0\}. \end{cases}$$

Then, $A = (\text{int}S) \cup \{(0,0)\}$. Take $x_0 = (0,0)$, $y_0 = (0,0)$, and $z_0 = 0$, it is obvious to see that (c^*, d^*, y_0) (with $c^* = (1,0)$ and $d^* = 0$) is a feasible element of (D) and $c^* \notin Q^\#$, but (x_0, y_0) is a Q-efficient solution of (P).

In the rest of this section, we discuss duality theorems for some kinds of efficient solutions with Q being appropriately chosen cones.

3.1 Quasi-Relative Efficient Solutions

Theorem 3.6 *Let $(x_0, y_0) \in \text{gr}F$, $z_0 \in G(x_0) \cap -D$, and $N(D) \neq \emptyset$.*

(i) (*Strong duality*) *Suppose that (x_0, y_0) is a quasi-relative efficient solution of* (P) *and the conditions* (i)–(iv) *in Theorem* 3.2 *hold with respect to F_+ in* (ii) *and $(F, G)_+$ in* (iii). *Then, there exists $(c^*, d^*) \in (C^* \backslash \{0\}) \times D^*$ such that (c^*, d^*, y_0) is a feasible element of* (D). *If, additionally, the condition* (v) *in Theorem* 3.2 *is satisfied, then (c^*, d^*, y_0) is a quasi-relative efficient solution of* (D).

(ii) (*Converse duality*) *Suppose that $x_0 \in A$, (c^*, d^*, y_0) is a feasible element of* (D), *and $c^* \in C^\#$. Then, (x_0, y_0) is a quasi-relative efficient solution of* (P).

Proof It follows from Theorems 3.2, 3.4 with $Q = \text{qri}C$.

If $\text{int}C \neq \emptyset$, by Remark 2.5(i), we get the following results for weakly efficient solutions when the ordering cone in the constraint space has possibly empty interior. □

Theorem 3.7 *Let $(x_0, y_0) \in \text{gr}F$, $z_0 \in G(x_0) \cap -D$, and $N(D) \neq \emptyset$.*

(i) (*Strong duality*) *Suppose that (x_0, y_0) is a weakly efficient solution of* (P), *and the conditions* (i), (iii), (iv) *in Theorem* 3.2 *hold with respect to $(F, G)_+$ in* (iii). *Then, there exists $(c^*, d^*) \in (C^* \backslash \{0\}) \times D^*$ such that (c^*, d^*, y_0) is a weakly efficient solution of* (D).

(ii) (*Converse duality*) *Suppose that $x_0 \in A$, and (c^*, d^*, y_0) is a feasible solution of* (D). *Then, (x_0, y_0) is a weakly efficient solution of* (P).

Proof It follows from Theorem 3.6. □

If, additionally, $\text{int}D \neq \emptyset$, then Theorem 3.7 (i) can be simplified as follows.

Corollary 3.8 *Let $(x_0, y_0) \in \text{gr}F$, and $z_0 \in G(x_0) \cap -D$. Suppose that (x_0, y_0) is a weakly efficient solution of* (P), *and the conditions* (i), (iv) *in Theorem* 3.2 *hold. Then, there exists $(c^*, d^*) \in (C^* \backslash \{0\}) \times D^*$ such that (c^*, d^*, y_0) is a weakly efficient solution of* (D).

Proof By Theorem 3.7 (i), we need to prove that the condition (iii) in Theorem 3.2 is fulfilled. Indeed, $(e, k) \notin T_M(0, 0)$ for all $(e, k) \in -(\text{int}C \times \text{int}D)$, where $M = \{(F, G)_+(x) - (y_0, 0) | x \in S\backslash A\}$. Suppose to the contrary, i.e., there exists $(e, k) \in -(\text{int}C \times \text{int}D)$ such that $(e, k) \in T_M(0, 0)$. By the definition of the contingent cone, there exist $t_n \to 0^+$, $x_n \in S\backslash A$, $z_n \in G(x_n)$, and $d_n \in D$ such that

$$\frac{z_n + d_n}{t_n} \to k.$$

Since $k \in -\text{int}D$, we have $z_n \in -\text{int}D$ for large enough n. On the other hand, it follows from $x_n \in S\backslash A$ that $G(x_n) \cap -D = \emptyset$, i.e., $z_n \notin -D$ for all n, which is a contradiction. □

3.2 Henig Efficient Solutions

When $Q = \text{int}C_\epsilon(B)$, duality theorems of the problems (P) and (D) with respect to Henig efficient solutions in case of $\text{int}D = \emptyset$ are established as follows.

Theorem 3.9 *Let B be a base of C, $(x_0, y_0) \in \text{gr}F$, $z_0 \in G(x_0) \cap -D$, and $N(D) \neq \emptyset$.*

(i) (*Strong duality*) *Suppose that (x_0, y_0) is a Henig efficient solution of* (P) *and the conditions* (i), (iii), (iv) *in Theorem* 3.2 *hold with respect to $(F, G)_+$ in* (iii). *Then, there exists $(c^*, d^*) \in C^\triangle(B) \times D^*$ such that (c^*, d^*, y_0) is a Henig efficient solution of* (D).
(ii) (*Converse duality*) *Suppose that $x_0 \in A$, (c^*, d^*, y_0) is a feasible element of* (D). *Then, (x_0, y_0) is a Henig efficient solution of* (P).

Proof From Theorems 3.2, 3.4, we only need to prove that $(C_\epsilon(B))^*\backslash\{0\} \subseteq C^\triangle(B)$. Indeed, let $c^* \in (C_\epsilon(B))^*\backslash\{0\}$, then we get $\langle c^*, y\rangle \geq 0$ for all $y \in C_\epsilon(B)$. With a fixed $u \in B_Y(0, \epsilon) : \langle c^*, u\rangle > 0$, we have $B - u \subseteq B + B_Y(0, \epsilon)$ (since $B_Y(0, \epsilon)$ is symmetric). Thus, $\langle c^*, b - u\rangle \geq 0$ for all $b \in B$, which implies that $\langle c^*, b\rangle \geq \langle c^*, u\rangle > 0$. Hence, $c^* \in C^\triangle(B)$. □

Theorem 3.9 (i) can be simplified when $\text{int}D \neq \emptyset$ as follows.

Corollary 3.10 *Let B be a base of C, $(x_0, y_0) \in \text{gr}F$, and $z_0 \in G(x_0) \cap -D$. Suppose that (x_0, y_0) is a Henig efficient solution of* (P) *and the conditions* (i), (iv) *in Theorem* 3.2 *hold. Then, there exists $(c^*, d^*) \in C^\triangle(B) \times D^*$ such that (c^*, d^*, y_0) is a Henig efficient solution of* (D).

4 Applications to Optimality Conditions

From the results in Sect. 3, we obtain optimality conditions in the Kuhn–Tucker type for Q-efficient solutions of the problem (P). Then, results for some kinds of efficient solutions are derived.

A necessary condition for Q-efficient solutions of (P) is implied immediately from Theorem 3.2 as follows.

Theorem 4.1 *Let* $(x_0, y_0) \in \mathrm{gr}F$, $z_0 \in G(x_0) \cap -D$, *and* $N(D) \neq \emptyset$. *Suppose that* (x_0, y_0) *is a* Q*-efficient solution of* (P) *and the conditions* (i)–(iv) *in Theorem* 3.2 *are satisfied. Then, there exist* $c^* \in Q^*\backslash\{0\}$ *and* $d^* \in D^*$ *such that, for all* $(y, z) \in (F, G)(S)$,

$$\langle c^*, y - y_0\rangle + \langle d^*, z - z_0\rangle \geq 0, \tag{4.1}$$

and

$$\langle d^*, z_0\rangle = 0. \tag{4.2}$$

For sufficient conditions, we have the following theorem.

Theorem 4.2 *Let* $(x_0, y_0) \in \mathrm{gr}F$, $x_0 \in A$ *and,* $z_0 \in G(x_0) \cap -D$. *Suppose that either of the following conditions holds*

(i) *there exist* $c^* \in Q^\#$ *and* $d^* \in D^*$ *such that* (4.1) *and* (4.2) *hold for every* $(y, z) \in (F, G)(S)$;
(ii) *there exist* $c^* \in Q^*\backslash\{0\}$ *and* $d^* \in D^*$ *such that* (4.2) *holds, and for every* $(y, z) \in (F, G)(S)\backslash\{(y_0, z_0)\}$,

$$\langle c^*, y - y_0\rangle + \langle d^*, z - z_0\rangle > 0. \tag{4.3}$$

Then, (x_0, y_0) *is a* Q*-efficient solution of* (P).

Proof If (i) holds, then the conclusion follows from Theorem 3.4. When (ii) is fulfilled, suppose that (x_0, y_0) is not a Q-efficient solution of (P), i.e., there exists $\overline{x} \in A$ $(\overline{x} \neq x_0)$ such that $(F(\overline{x}) - y_0) \cap -Q \neq \emptyset$. It means that there is $\overline{y} \in F(\overline{x})$, $\overline{z} \in G(\overline{x}) \cap -D$ and $\overline{y} - y_0 \in -Q$. Since $c^* \in Q^*\backslash\{0\}$, we have $\langle c^*, \overline{y} - y_0\rangle \leq 0$. On the other hand, by the assumption (with $y = \overline{y}$, $z = \overline{z}$), we get $\langle c^*, \overline{y} - y_0\rangle > 0$, which is a contradiction. □

4.1 Optimality Conditions for Quasi-Relative Efficient Solutions

When $Q = \mathrm{qri}C$, necessary and sufficient conditions for quasi-relative efficient solutions of (P) are obtained by

Theorem 4.3 (Necessary condition) *Let* $(x_0, y_0) \in \mathrm{gr}F$, $z_0 \in G(x_0) \cap -D$, *and* $N(D) \neq \emptyset$. *Suppose that* (x_0, y_0) *is a quasi-relative efficient solution of* (P) *and the conditions* (i)–(iv) *in Theorem* 3.2 *are satisfied with respect to* F_+ *in* (ii) *and* $(F, G)_+$ *in* (iii). *Then, there exist* $c^* \in C^*\backslash\{0\}$ *and* $d^* \in D^*$ *such that* (4.1) *and* (4.2) *hold for all* $(y, z) \in (F, G)(S)$.

Proof It follows from Theorem 3.6. □

Theorem 4.4 (Sufficient condition) *Let* $(x_0, y_0) \in \mathrm{gr}F$, $x_0 \in A$ *and,* $z_0 \in G(x_0) \cap -D$. *Suppose that either of the following conditions holds*

(i) *there exist* $c^* \in C^\#$ *and* $d^* \in D^*$ *such that* (4.1) *and* (4.2) *hold for every* $(y, z) \in (F, G)(S)$;

(ii) *there exist* $c^* \in C^*\backslash\{0\}$ *and* $d^* \in D^*$ *such that* (4.3) *and* (4.2) *hold for every* $(y, z) \in (F, G)(S)\backslash\{(y_0, z_0)\}$.
Then, (x_0, y_0) *is a quasi-relative efficient solution of* (P).

Proof It follows from Theorem 4.2. □

If $\mathrm{int}C \neq \emptyset$, Theorem 3.7 implies the following optimality condition for weakly efficient solutions of (P) when the ordering cone D has empty interior.

Theorem 4.5 *Let* $(x_0, y_0) \in \mathrm{gr}F$*,* $z_0 \in G(x_0) \cap -D$*, and* $N(D) \neq \emptyset$*. Suppose the conditions* (i)*,* (iii)*,* (iv) *in Theorem* 3.2 *be satisfied with respect to* $(F, G)_+$ *in* (iii)*. Then,* (x_0, y_0) *is a weakly efficient solution of* (P) *if and only if there exist* $c^* \in C^*\backslash\{0\}$ *and* $d^* \in D^*$ *such that* (4.1) *and* (4.2) *hold for all* $(y, z) \in (F, G)(S)$*.*

The following example shows a case where Theorem 4.5 can be employed, while some existing results cannot.

Example 4.6 Let $X = \mathbb{R}^2$, $Y = \mathbb{R}^2$, $Z = l^2$ be as defined in Example 2.2, $C = \mathbb{R}^2_+$, and $D = \{\{x_n\}_{n\in\mathbb{N}} \subseteq Z | x_n \geq 0, \forall n \in \mathbb{N}\}$. It is easy to see that $\mathrm{int}D = \emptyset$, $\mathrm{qri}D = N(D) = \{\{x_n\}_{n\in\mathbb{N}} \subseteq Z | x_n > 0, \forall n \in \mathbb{N}\}$. Let $S := \{(x_1, x_2) \in X | (x_1 - 1)^2 + (x_2 - 1)^2 \leq 1\}$, $F : X \to 2^Y$ and $G : X \to 2^Z$ be defined by

$$F(x_1, x_2) = \begin{cases} (x_1, x_2), & \text{if } (x_1, x_2) \in S, \\ \emptyset, & \text{otherwise,} \end{cases}$$

$$G(x_1, x_2) = \begin{cases} \{z \in Z | \|z\|_{l^2} \leq x_1 + x_2\}, & \text{if } (x_1, x_2) \in H, \\ \mathrm{qri}D, & \text{if } (x_1, x_2) \in S\backslash H, \\ \emptyset, & \text{otherwise,} \end{cases}$$

where $H := ((1, 1) - C) \cap S$. Then, H is the feasible set of (P).

Let $x_0 = (1, 1) \in H$, $y_0 = (1, 1)$, and $z_0 = \{-1/2^n\}_{n\in\mathbb{N}}$. We can check that all assumptions of Theorem 4.5 are satisfied. By calculating, we see that the necessary condition given by Theorem 4.5 is not fulfilled. Thus, (x_0, y_0) is not a weakly efficient solution of (P), but several other results cannot be used to reject (x_0, y_0) since $\mathrm{int}D = \emptyset$, such as Theorem 4.2 in [6], Theorem 4.7 in [8], Theorem 3.2 in [17], Propositions 3.5, 3.6, 3.8 in [22], and Theorems 3.1, 3.2 in [23].

If, additionally, $\mathrm{int}D \neq \emptyset$, then we can simplify Theorem 4.5 as follows.

Corollary 4.7 *Let* $(x_0, y_0) \in \mathrm{gr}F$*, and* $z_0 \in G(x_0) \cap -D$*. Suppose the conditions* (i)*,* (iv) *in Theorem* 3.2 *be satisfied. Then,* (x_0, y_0) *is a weakly efficient solution of* (P) *if and only if there exist* $c^* \in C^*\backslash\{0\}$ *and* $d^* \in D^*$ *such that* (4.1) *and* (4.2) *hold for all* $(y, z) \in (F, G)(S)$*.*

To illustrate Corollary 4.7, we consider an example as follows.

Example 4.8 Let $X = \mathbb{R}^2$, $Y = Z = \mathbb{R}$, $C = D = \mathbb{R}_+$, $S = \{(x_1, x_2) \in X | x_1 \geq 0\}$, and $F : S \to 2^Y$, $G : S \to 2^Z$ be defined by

$$F(x_1, x_2) := \{y \in Y | y \geq -\sqrt{x_1}\},$$

and

$$G(x_1, x_2) := \{y \in Y | y \geq -x_2\}.$$

Then, $\text{int}C = \text{int}D = \mathbb{R}_+ \backslash \{0\}$. Let $x_0 = (0, 0)$ be a feasible point of (P), and $(y_0, z_0) = (0, 0)$. We can check that $(F - y_0, G)$ is closely $(C \times D)$-convexlike on S. Moreover, the point $(0, 1) \in S$ holds $G(0, 1) \cap -\text{int}D \neq \emptyset$. Then, all assumptions of Corollary 4.7 are satisfied at (x_0, y_0). Let $(c^*, d^*) \in C^* \times D^*$ hold, for all $(y, z) \in (F, G)(S)$,

$$\langle c^*, y\rangle + \langle d^*, z\rangle \geq 0,$$

which implies that, for all $(x_1, x_2) \in S$,

$$c^*(-\sqrt{x_1}) + d^*(-x_2) \geq 0.$$

Since $x_1 \geq$ and $x_2 \in \mathbb{R}$, we must have $(c^*, d^*) = (0, 0)$. By Corollary 4.7, (x_0, y_0) is not a weakly efficient solution of (P).

4.2 Optimality Conditions for Henig Efficient Solutions

When $Q = \text{int}C_\epsilon(B)$, from Theorem 3.9, we get optimality conditions for Henig efficient solutions when $\text{int}D = \emptyset$ as follows.

Theorem 4.9 *Let B be a base of C, $(x_0, y_0) \in \text{gr}F$, $z_0 \in G(x_0) \cap -D$, and $N(D) \neq \emptyset$. Suppose the conditions* (i), (iii), (iv) *in Theorem* 3.2 *be satisfied with respect to* $(F, G)_+$ *in* (iii). *Then,* (x_0, y_0) *is a Henig efficient solution of* (P) *if and only if there exist $c^* \in C^{\triangle}(B)$ and $d^* \in D^*$ such that* (4.1) *and* (4.2) *hold for all* $(y, z) \in (F, G)(S)$.

Corollary 4.10 *Let* $(x_0, y_0) \in \text{gr}F$, $z_0 \in G(x_0) \cap -D$, *and* $\text{int}D \neq \emptyset$. *Suppose the conditions* (i) *and* (iv) *in Theorem* 3.2 *be satisfied. Then,* (x_0, y_0) *is a Henig efficient solution of* (P) *if and only if there exist $c^* \in C^{\triangle}(B)$ and $d^* \in D^*$ such that* (4.1) *and* (4.2) *hold for all* $(y, z) \in (F, G)(S)$.

To compare our results with other existing ones, we recall some notions.

Definition 4.11 (i) ([20]) Let $K \subseteq X$, $x \in \text{cl}K$, and $u_i \in X$, $i = 1, \ldots, m-1$. The mth-order adjacent set of K at x with respect to $(u_1, \ldots, u_{m-1})$ is defined by

$$\begin{aligned} T_K^{\flat(m)}(x, u_1, \ldots, u_{m-1}) := \{&u \in X | \forall t \to 0^+, \\ &\exists u_n \to u, x + t_n u_1 + \ldots + t_n^{m-1} u_{m-1} + t_n^m u_n \in K\}. \end{aligned}$$

(ii) ([20]) Let $F : X \to 2^Y$, $(x_0, y_0) \in \text{gr}F$, and $(u_i, v_i) \in X \times Y$, $i = 1, \ldots, m-1$. The mth-order contingent derivative $D^m F(x_0, y_0, u_1, v_1, \ldots, u_{m-1}, v_{m-1})$ of F at (x_0, y_0) with respect to (u_i, v_i), $i = 1, \ldots, m-1$ is a set-valued mapping from X to Y defined by

$$\begin{aligned}&\mathrm{gr}\,D^m F(x_0, y_0, u_1, v_1, \ldots, u_{m-1}, v_{m-1})\\&\quad := T^m_{\mathrm{gr}F}(x_0, y_0, u_1, v_1, \ldots, u_{m-1}, v_{m-1}).\end{aligned}$$

(iii) ([20]) The mth-order adjacent derivative $D^{\flat(m)} F(x_0, y_0, u_1, v_1, \ldots, u_{m-1}, v_{m-1}) : X \to 2^Y$ of F at (x_0, y_0) with respect to (u_i, v_i), $i = 1, \ldots, m-1$ is defined by

$$\begin{aligned}&\mathrm{gr}\,D^{\flat(m)} F(x_0, y_0, u_1, v_1, \ldots, u_{m-1}, v_{m-1})\\&\quad := T^{\flat(m)}_{\mathrm{gr}F}(x_0, y_0, u_1, v_1, \ldots, u_{m-1}, v_{m-1}).\end{aligned}$$

(iv) ([5]) The mth-order weak adjacent epiderivative $D^{\flat(m)}_w F(x_0, y_0, u_1, v_1, \ldots, u_{m-1}, v_{m-1}) : X \to 2^Y$ of F at (x_0, y_0) with respect to (u_i, v_i), $i = 1, \ldots, m-1$ is defined by

$$D^{\flat(m)}_w F(x_0, y_0, u_1, v_1, \ldots, u_{m-1}, v_{m-1})(x) :=$$

$$\mathrm{WMin}_C\{y \in Y | (x, y) \in T^{\flat(m)}_{\mathrm{epi}F}(x_0, y_0, u_1, v_1, \ldots, u_{m-1}, v_{m-1})\},$$

where $\mathrm{WMin}_C A$ denotes the set of weakly minimal points of A.

(v) ([24]) The mth-order generalized adjacent epiderivative $D^{\flat(m)}_g F(x_0, y_0, u_1, v_1, \ldots, u_{m-1}, v_{m-1}) : X \to 2^Y$ of F at (x_0, y_0) with respect to (u_i, v_i), $i = 1, \ldots, m-1$ is defined by

$$\begin{aligned}&D^{\flat(m)}_g F(x_0, y_0, u_1, v_1, \ldots, u_{m-1}, v_{m-1})(x) :=\\&\quad \mathrm{Min}_C\{y \in Y | (x, y) \in T^{\flat(m)}_{\mathrm{epi}F}(x_0, y_0, u_1, v_1, \ldots, u_{m-1}, v_{m-1})\},\end{aligned}$$

where $\mathrm{Min}_C A := \{a_0 \in A | (A - a_0) \cap (-C\backslash\{0\}) = \emptyset\}$.

(vi) ([4]) The mth-order upper Studniarski derivative $\overline{D}^m F(x_0, y_0) : X \to 2^Y$ of F at (x_0, y_0) is defined by

$$\begin{aligned}\overline{D}^m F(x_0, y_0)(x) := \{y \in Y | \exists t \to 0^+, \exists (x_n, y_n)\\ \to (x, y), y_0 + t_n^m y_n \in F(x_0 + t_n x_n)\}.\end{aligned}$$

Remark 4.11 We have the following relationships of the above-mentioned notions.

$$\overline{D}^1 F_+(x_0, y_0)(x) = D^1 F_+(x_0, y_0)(x),$$

and

$$\begin{aligned}&D^{\flat(m)}_g F_+(x_0, y_0, u_1, v_1, \ldots, u_{m-1}, v_{m-1})(x)\\&\subseteq D^{\flat(m)}_w F_+(x_0, y_0, u_1, v_1, \ldots, u_{m-1}, v_{m-1})(x)\\&\subseteq D^{\flat(m)} F_+(x_0, y_0, u_1, v_1, \ldots, u_{m-1}, v_{m-1})(x)\\&\subseteq D^m F_+(x_0, y_0, u_1, v_1, \ldots, u_{m-1}, v_{m-1})(x).\end{aligned}$$

From Theorems 4.5 and 4.9, we get the following corollaries in terms of the higher-order contingent derivative.

Corollary 4.12 *Let* $(x_0, y_0) \in \mathrm{gr}F$, $z_0 \in G(x_0) \cap -D$, $(u_i, v_i, w_i) \in X \times (-C) \times (-D)$, *and* $N(D) \neq \emptyset$. *Suppose the conditions* (i), (iii), (iv) *in Theorem* 3.2 *be satisfied with respect to* $(F, G)_+$ *in* (iii). *If* (x_0, y_0) *is a weakly efficient solution of* (P), *then there exist* $c^* \in C^*\backslash\{0\}$ *and* $d^* \in D^*$ *such that for all* $(v, w) \in D^m(F, G)_+(x_0, y_0, z_0, u_1, v_1, w_1, \ldots, u_{m-1}, v_{m-1}, w_{m-1})(x)$, $x \in X$.

$$\langle c^*, v\rangle + \langle d^*, w\rangle \geq 0, \tag{4.4}$$

and

$$\langle d^*, z_0\rangle = 0.$$

Proof It follows from Theorem 4.5 that there exist $c^* \in C^*\backslash\{0\}$ and $d^* \in D^*$ such that for all $(y, z) \in (F, G)(S)$

$$\langle c^*, y - y_0\rangle + \langle d^*, z - z_0\rangle \geq 0,$$

and

$$\langle d^*, z_0\rangle = 0.$$

Let $(v, w) \in D^m(F, G)_+(x_0, y_0, z_0, u_1, v_1, w_1, \ldots, u_{m-1}, v_{m-1}, w_{m-1})(x)$, i.e., there are $t_n \to 0^+$, $\{x_n\} \subseteq S$, $(y_n, z_n) \in (F, G)(x_n)$, and $(c_n, d_n) \in C \times D$ for all n such that

$$\frac{x_n - x_0 - t_n u_1 - \ldots - t_n^{m-1} u_{m-1}}{t_n^m} \to x,$$
$$\frac{y_n - y_0 - t_n v_1 - \ldots - t_n^{m-1} v_{m-1}}{t_n^m} \to v,$$
$$\frac{z_n - z_0 - t_n u_1 - \ldots - t_n^{m-1} w_{m-1}}{t_n^m} \to w.$$

Since $(v_i, w_i) \in (-C) \times (-D)$, we get

$$\left\langle \frac{c^*, y_n - y_0 - t_n v_1 - \ldots - t_n^{m-1} v_{m-1}}{t_n^m} \right\rangle + \left\langle \frac{d^*, z_n - z_0 - t_n u_1 - \ldots - t_n^{m-1} w_{m-1}}{t_n^m} \right\rangle \geq 0.$$

Taking $n \to +\infty$, then $\langle c^*, v\rangle + \langle d^*, w\rangle \geq 0$, and the proof is completed. □

By the similar proof, Corollary 4.12 is also valid for all $(v, w) \in \overline{D}^m(F, G)_+(x_0, y_0)(x)$.

Corollary 4.13 *Let B be a base of C, $(x_0, y_0) \in \mathrm{gr}F$, $z_0 \in G(x_0) \cap -D$, and $N(D) \neq \emptyset$. Suppose the conditions* (i)*,* (iii)*,* (iv) *in Theorem 3.2 be satisfied with respect to $(F, G)_+$ in* (iii)*. If (x_0, y_0) is a Henig efficient solution of* (P)*, then there exist $c^* \in C^{\triangle}(B)$ and $d^* \in D^*$ such that* (4.4) *and* (4.2) *hold for all $(v, w) \in D^m(F, G)_+(x_0, y_0, z_0, u_1, v_1, w_1, \ldots, u_{m-1}, v_{m-1}, w_{m-1})(x)$, $x \in X$.*

Proof Based on Theorem 4.9, the proof is similar to that of Corolarry 4.13. □

From Remark 4.11, we can see that Corollaries 4.12 and 4.13 are extensions of Theorem 3.7 in [4], Theorem 6.1 in [25] (for weakly efficient solutions), and Theorem 6.1 in [5], Theorems 5.1, 5.2 in [24] (for Henig efficient solutions), resp, to the case that the ordering cone D in the constraint space has possibly empty interior. Moreover, the convexity assumption in our results is weaker than that in the above-mentioned papers.

5 Conclusions

In the paper, we consider a kind of dual problem (D) (without using generalized derivatives) of a constrained set-valued optimization problem (P). Some duality theorems for the pair of (P)–(D) are established dealing with the concept of Q-efficiency. Then, corresponding results for quasi-relative efficient solutions and Henig efficient solutions are obtained. We also get their applications to optimality conditions in set-valued optimization. Several examples are given to show that our results are extensions of the existing ones in the literature to the case of ordering cones having possibly empty interior.

Recently, Lagrange duality theorems were presented for the same types of solutions of a set-valued optimization problem in [20] (see Subsection 8.1.2). The dual problem in this book is different from our problem here since the dual variables in this subsection are operators, not vectors like in our work. To obtain duality theorems, the authors proposed some regularity assumptions using the quasi-interior of a certain set, see the Theorem 8.1.16 (the condition (c)) in [20]. Thus, for further research, we plan to investigate whether there is any relationship between this assumption and the condition used in our paper or not, and if assumption we considered does hold in general. Moreover, we intend to extend our results from vector approach to set approach, see Subsection 2.6.2 in [20].

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