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Algebraical Dependence and Uniqueness Problem for Meromorphic Mappings with Few Moving Targets

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Abstract The purpose of this article is twofold. The first is to prove algebraical dependence of meromorphic mappings from $\mathbb{C}^m$ into $P^n(\mathbb{C})$ sharing few moving hyperplanes. The second is to show uniqueness theorem for meromorphic mappings from the viewpoint of dependence. These results improve and extend some earlier works.

Keywords Meromorphic mappings · Algebraical dependence · Uniqueness theorem · Moving targets

Mathematics Subject Classification 32H30 · 30D35

1 Introduction and Main Results

In 1926, Nevanlinna [6] showed that for two non-constant meromorphic functions f and g on the complex plane $\mathbb{C}$, if they have the same inverse images for five distinct values, then $f = g$, and if they have the same inverse images, counted with multiplicities, for four distinct values, then g is a special type of a linear fractional transformation of f. These results are usually called the five-value theorem and four-value theorem.

In 1929, as an improvement of the above mentioned Nevanlinna's results, Cartan [1] declared that there are at most two meromorphic functions on $\mathbb{C}$ which have the

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same inverse images regardless of multiplicities for four distinct values. However, Steinmetz [14] gave examples which showed that Cartan's declaration is false. In relation to this, Ji [5] obtained algebraic dependence of meromorphic mappings by the use of Cartan's original idea in 1988. Later, Stoll [15] generalized the results of Ji to parabolic covering spaces. Ru [12] studied the case of holomorphic curves for moving targets. In 2010, Thoan and Duc [7] proved some results on algebraic dependence of meromorphic mappings. Recently, Thoan et al. [8] improved their work and proved that

Theorem A (Thoan-Duc-Quang [8]) *Let $f_1, \ldots, f_\lambda : \mathbb{C}^m \to P^n(\mathbb{C})$ be non-constant meromorphic mappings. Let $\{a_j\}_{j=1}^q$ be meromorphic mappings of $\mathbb{C}^m$ into $P^n(\mathbb{C})$ in general position such that $T_{a_j}(r) = o(\max_{1\le i\le\lambda}\{T_{f_i}(r)\})(1 \le j \le q)$ and $(f_i, a_j) \not\equiv 0$ for each $1 \le i \le \lambda,\ 1 \le j \le q$. Assume that the following conditions are satisfied.*

(a) $\min\{\nu_{(f_1,a_j)}, 1\} = \cdots = \min\{\nu_{(f_\lambda,a_j)}, 1\}$ *for each* $1 \le j \le q$.
(b) $\dim\{z : (f, a_i)(z) = (f, a_j)(z) = 0\} \le m - 2\ (1 \le i < j \le q)$.
(c) There exists an integer number l, $2 \le l \le \lambda$, such that for any increasing sequence $1 \le j_1 < \cdots < j_l \le \lambda$, $f_{j_1}(z) \wedge \cdots \wedge f_{j_l}(z) = 0$ on $\cup_{j=1}^q\{z : (f_1, a_j)(z) = 0\}$.

If $q > \frac{n(2n+1)\lambda-(n-1)(\lambda-1)}{\lambda-l+1}$, then $f_1, \ldots, f_\lambda$ are algebraically dependent over $\mathbb{C}$, i.e., $f_1 \wedge \cdots \wedge f_\lambda \equiv 0$ on $\mathbb{C}$.

Let $f : \mathbb{C}^m \to P^n(\mathbb{C})$ be a meromorphic mapping and d be a positive integer or $+\infty$. Let $\{a_j\}_{j=1}^q$ be "small" (with respect to f) meromorphic mappings of $\mathbb{C}^m$ into $P^n(\mathbb{C})$ in general position such that

$$\dim\{z : (f, a_i)(z) = (f, a_j)(z) = 0\} \le m - 2\ (1 \le i < j \le q).$$

Consider the set $\mathscr{G}(f, \{a_j\}_{j=1}^q, d)$ of all meromorphic mappings $g : \mathbb{C}^m \to P^n(\mathbb{C})$ satisfying the conditions:

(i) $\min\{\nu_{(f,a_j)}, d\} = \min\{\nu_{(g,a_j)}, d\}\ (1 \le j \le q)$,
(ii) $f(z) = g(z)$ *on* $\cup_{j=1}^q\{z : (f, a_j)(z) = 0\}$.

For brevity, we will denote by $\sharp S$ the cardinality of set S. In 2013, Quang [10] *proved the following theorem about algebraic dependence of three maps.*

Theorem B (Quang [10]) *Assume $f_1, f_2, f_3 \in \mathscr{G}(f, \{a_j\}_{j=1}^q, 1)$ and $n \ge 2$.*

(a) If $q \ge 3n^2 + 3/2$, then $f_1 \wedge f_2 \wedge f_3 \equiv 0$.
(b) If f is linearly non-degenerate over $\mathscr{R}$ and $q \ge (3n^2 + 3n + 3)/2$, then $f_1 \wedge f_2 \wedge f_3 \equiv 0$.

The first purpose of this article is to give an improvement of Theorem A. Namely, applying the new second main theorems given by Quang [11]*, we will show that*

Theorem 1 *Let $f_1, \ldots, f_\lambda : \mathbb{C}^m \to P^n(\mathbb{C})$ be non-constant meromorphic mappings. Let $\{a_j\}_{j=1}^q$ be meromorphic mappings of $\mathbb{C}^m$ into $P^n(\mathbb{C})$ in general position such that $T_{a_j}(r) = o(\max_{1\le i\le\lambda}\{T_{f_i}(r)\})(1 \le j \le q)$ and $(f_i, a_j) \not\equiv 0$ for each $1 \le i \le \lambda,\ 1 \le j \le q$. Assume that the following conditions are satisfied:*

(a) $\min\{\nu_{(f_1,a_j)}, 1\} = \cdots = \min\{\nu_{(f_\lambda,a_j)}, 1\}$ *for each* $1 \le j \le q$.
(b) $\dim\{z : (f, a_i)(z) = (f, a_j)(z) = 0\} \le m - 2$ $(1 \le i < j \le q)$.
(c) *There exists an integer number* l, $2 \le l \le \lambda$, *such that for any increasing sequence* $1 \le j_1 < \cdots < j_l \le \lambda$, $f_{j_1}(z) \wedge \cdots \wedge f_{j_l}(z) = 0$ *on* $\cup_{j=1}^{q}\{z : (f_1, a_j)(z) = 0\}$.

If $q > \frac{3n(n+1)\lambda - 2(n-1)(\lambda-1)}{2(\lambda-l+1)}$, *then* $f_1, \ldots, f_\lambda$ *are algebraically dependent over* $\mathscr{R}$, *i.e.* $f_1 \wedge \cdots \wedge f_\lambda \equiv 0$ *on* $\mathscr{R}$.

For the case $\lambda = 3$ and $l = 2$ in Theorem 1, we have the following corollary, which is better than Theorem B(a).

Corollary 1 *Assume* $f_1, f_2, f_3 \in \mathscr{G}(f, \{a_j\}_{j=1}^{q}, 1)$ *and* $n \ge 2$. *If* $q \ge \frac{9n^2+5n+4}{4}$, *then* $f_1 \wedge f_2 \wedge f_3 \equiv 0$.

On the other hand, the uniqueness problem with truncated multiplicities for meromorphic mappings from $\mathbb{C}^m$ into $P^n(\mathbb{C})$ sharing a finite set of fixed (or moving) targets in $P^n(\mathbb{C})$ has been studied very intensively by many authors in the last few decades, and they related to many problems in Nevanlinna theory and hyperbolic complex analysis (see ref. [2], [4], [13]). In [3], Chen-Li-Yan studied the uniqueness problem without the assumption of the linearly non-degeneracy for meromorphic mappings.

Theorem C (Chen-Li-Yan [3]) *If* $q = 4n^2 + 2n$, $n \ge 2$, *then* $\sharp\mathscr{G}(f, \{a_j\}_{j=1}^{q}, 1) = 1$.

Later, Thoan et al. [8] *showed the result still valid if* $q = 4n^2 + 2$.

Set $rank_{\mathscr{R}}(f) := rank\{f_0, \ldots, f_n\}$ *over* $\mathscr{R}$. *It is easy to see that the definition of* $rank_{\mathscr{R}}(f)$ *does not depend on the choice of the reduced representation of* f. *In 2013, Quang and An* [9] *got the following uniqueness theorem with fewer moving hyperplanes.*

Theorem D (Quang-An [9]) *If* $q > 4nk + 2$, $n \ge 2$, *where* $k + 1 = rank_{\mathscr{R}}(f)$, *then* $\sharp\mathscr{G}(f, \{a_j\}_{j=1}^{q}, 1) = 1$.

Quang [11] *established some new second main theorem for meromorphic mappings intersecting moving hyperplanes. For their application, Quang showed the following result.*

Theorem E (Quang [11])

(a) *If* $q > \frac{9n^2+9n+4}{4}$, $n \ge 2$, *then* $\sharp\mathscr{G}(f, \{a_j\}_{j=1}^{q}, 1) \le 2$.
(b) *If* $q > 3n^2 + n + 2$ *and* $n \ge 2$, *then* $\sharp\mathscr{G}(f, \{a_j\}_{j=1}^{q}, 1) = 1$.

The following question arises naturally: are there uniqueness theorem with few moving hyperplanes? Based on the propagation of dependence in Theorem 1, *the second purpose of this paper is to give an answer to the above question. Namely, we will prove the following result.*

Theorem 2 *Let* $f : \mathbb{C}^m \to P^n(\mathbb{C})$ *be a meromorphic mapping and let* $\{a_j\}_{j=1}^{q}$ *be "small" (with respect to* f*) meromorphic mappings of* $\mathbb{C}^m$ *into* $P^n(\mathbb{C})$ *in general position such that*

$$\dim\{z : (f, a_i)(z) = (f, a_j)(z) = 0\} \le m - 2 \ (1 \le i < j \le q).$$

Set $k + 1 = rank_{\mathscr{R}}(f)$, *then the following assertions hold:*

(i) If $k \geq \frac{n+1}{2}$ *and* $q > 2(2n+1)k - 2k^2 + 2$ $(n \geq 2)$*, then* $\sharp\mathscr{G}(f, \{a_j\}_{j=1}^q, 1) = 1$.

(ii) If $k < \frac{n+1}{2}$ *and* $q > \frac{2k^2+2(n-1)k+(n+3)+\sqrt{(2k^2+2(n-1)k+n-1)^2+8(3n+1)k}}{2}$ $(n \geq 2)$*, then* $\sharp\mathscr{G}(f, \{a_j\}_{j=1}^q, 1) = 1$.

Put $Q = \frac{2k^2+2(n-1)k+n+3+\sqrt{(2k^2+2(n-1)k+n-1)^2+8(3n+1)k}}{2}$. For $1 \leq k < \frac{n+1}{2}$, we have

$$2(2n+1)k - 2k^2 + 2 - Q$$
$$= \frac{8k(k - \frac{n+1}{2})(2k^2 - (4n+3)k + 2n)}{-6k^2 + 6(n+1)k - (n-1) + \sqrt{(2k^2 + 2(n-1)k + n - 1)^2 + 8(3n+1)k}} > 0.$$

By Theorem 2, we get the following corollary.

Corollary 2 *Let* $f : \mathbb{C}^m \to P^n(\mathbb{C})$ *be a meromorphic mapping and let* $\{a_j\}_{j=1}^q$ *be "small" (with respect to* f*) meromorphic mappings of* $\mathbb{C}^m$ *into* $P^n(\mathbb{C})$ *in general position such that*

$$\dim\{z : (f, a_i)(z) = (f, a_j)(z) = 0\} \leq m - 2 \ (1 \leq i < j \leq q).$$

If $q > 2(2n+1)k - 2k^2 + 2$, $n \geq 2$*, where* $k+1 = rank_{\mathscr{R}}(f)$*, then* $\sharp\mathscr{G}(f, \{a_j\}_{j=1}^q, 1) = 1$.

Since $1 \leq k \leq n$, Corollary 2 implies that

Corollary 3 *If* $q > 2n^2 + 2n + 2$ *and* $n \geq 2$*, then* $\sharp\mathscr{G}(f, \{a_j\}_{j=1}^q, 1) = 1$.

We would like to notice that Theorem 2 is an improvement of not only the above mentioned theorems, but also of many uniqueness theorem of meromorphic mappings for moving targets without counting multiplicity.

2 Basic Notions and Preliminaries in Nevanlinna Theory

We set $\|z\| = (|z_1|^2 + \cdots + |z_m|^2)^{\frac{1}{2}}$ for $z = (z_1, \cdots, z_m) \in \mathbb{C}^m$, $B(r) := \{z : \|z\| \leq r\}$, $S(r) := \{z : \|z\| = r\}$, $d^c := \frac{\sqrt{-1}}{4\pi}(\bar{\partial} - \partial)$, $\upsilon := (dd^c\|z\|^2)^{m-1}$ and $\sigma := d^c \log \|z\|^2 \wedge (dd^c \log \|z\|^2)^{m-1}$.

Let $F(z)$ be a nonzero holomorphic function on a domain Ω in $\mathbb{C}^m$. For $\alpha = (\alpha_1, \cdots, \alpha_m)$ with $\alpha_i \in \mathbb{Z}_+$ $(1 \leq i \leq m)$, set $|\alpha| = \sum_{i=1}^m \alpha_i$ and $D^\alpha F = \frac{\partial^{|\alpha|} F}{\partial^{(\alpha_1)} z_1 \cdots \partial^{(\alpha_m)} z_m}$, where $\mathbb{Z}_+$ denote the set of nonnegative integers. We define the map $\nu_F : \Omega \to \mathbb{Z}_+$ by $\nu_F(a) := \max\{m : D^\alpha F = 0 \ for \ all \ \alpha \ with \ |\alpha| < m\} (a \in \Omega)$.

We mean by a divisor on a domain Ω in $\mathbb{C}^m$ a map $\nu : \Omega \to \mathbb{Z}$ such that, for each $a \in \Omega$, there are nonzero holomorphic functions F and G on a connected neighborhood $U \subset \Omega$ of a such that $\nu(z) = \nu_F(z) - \nu_G(z)$ for each $z \in U$ outside an analytic set of dimension $\leq m - 2$. Two divisors are regarded as the same if they are identical outside an analytic set of dimension $\leq m - 2$. For a divisor ν on Ω, we set $|\nu| := \overline{\{z : \nu(z) \neq 0\}}$, which is a purely $(m-1)$-dimensional analytic subset of Ω or empty. For a divisor ν on $\mathbb{C}^m$ and a positive integer M or $M = +\infty$, we define the truncated divisor $\nu^{[M]}$ by $\nu^{[M]}(z) := \min\{M, \nu(z)\}$.

Let φ be a nonzero meromorphic function on Ω in $\mathbb{C}^m$. We define the divisor ν_φ as follows: for each $a \in \mathbb{C}^m$, choose nonzero holomorphic functions F and G on a neighborhood U of a such that $\varphi = \frac{F}{G}$ on U and $\dim(F^{-1}(0) \cap G^{-1}(0)) \leq m-2$. Then, put $\nu_\varphi(a) := \nu_F(a)$, $\nu_\varphi(z) := \nu_F(z)$ and $\nu_\varphi^\infty := \nu_G$, which are independent of choices of F and G and so globally well-defined on Ω.

Define the counting function of ν by

$$N(r, \nu) := \int_1^r \frac{n(t)}{t^{2m-1}} dt, \quad (1 < r < +\infty)$$

where

$$n(t) := \begin{matrix} \int_{|\nu| \cap B(r)} \nu(z) \upsilon & \text{if} \quad m \geq 2, \\ \sum_{|z| \leq t} \nu(z) & \text{if} \quad m = 1. \end{matrix}$$

Similarly, we define $n^{[M]}(t)$ and $N(r, \nu^{[M]})$ and denote by $N^{[M]}(r, \nu)$, respectively.

Let $\varphi : \mathbb{C}^m \to \mathbb{C}$ be a meromorphic function. Define

$$N_\varphi(r) := N(r, \nu_\varphi),\ N_\varphi^{[M]}(r) := N^{[M]}(r, \nu_\varphi).$$

For brevity, we will omit the character ^{M} if $M = +\infty$.

Let $f : \mathbb{C}^m \to P^n(\mathbb{C})$ be a meromorphic mapping. We can choose holomorphic functions $f_0, \cdots, f_n$ on $\mathbb{C}^m$ such that $I_f := \{z : f_0(z) = \cdots = f_n(z)\}$ is of dimension at most $m-2$, and $f(z) = (f_0 : \cdots : f_n)$ outside I_f. Usually, $f(z) = (f_0 : \cdots : f_n)$ is called a reduced representation of f.

Set $\|f\| := (|f_0|^2 + \cdots + |f_n|^2)^{\frac{1}{2}}$, the characteristic function of f is defined by

$$T_f(r) := \int_{S(r)} \log \|f\| \sigma - \int_{S(1)} \log \|f\| \sigma, \quad r > 1.$$

Note that $T_f(r)$ is independent on the choice of the reduced representation.

Let a be a meromorphic mapping of $\mathbb{C}^m$ into $P^n(\mathbb{C})$ with reduced representation $a = (a_0 : \cdots : a_n)$. We define

$$m_{(f,a)}(r) := \int_{S(r)} \log \frac{\|f\| \|a\|}{|(f, a)|} \sigma - \int_{S(1)} \log \frac{\|f\| \|a\|}{|(f, a)|} \sigma,$$

where $\|a\| := (|a_0|^2 + \cdots + |a_n|^2)^{\frac{1}{2}}$ and $r > 1$. Throughout this paper, we always assume that the homogeneous coordinates of $P^n(\mathbb{C})$ are chosen so that for a given meromorphic mapping $a = (a_0 : \cdots : a_n)$ of $\mathbb{C}^m$ into $P^n(\mathbb{C})$, $a_0 \not\equiv 0$. Then, we set $\tilde{a} = (\frac{a_0}{a_0} : \cdots : \frac{a_n}{a_0})$.

Let $a_1, \ldots, a_q$ $(q \geq n+1)$ be q meromorphic mappings of $\mathbb{C}^m$ into $P^n(\mathbb{C})$ with reduced representations $a_j = (a_{j0} : \cdots : a_{jn}) (1 \leq j \leq q)$. We say that $a_1, \ldots, a_q$

located in general position if $\det(a_{j_k}) \not\equiv 0$ for any $1 \le j_0 < j_1 < \cdots < j_n \le q$. Denote by $\mathscr{M}$ the field of meromorphic functions on $\mathbb{C}^m$ and denote by $\mathscr{R}$ the smallest subfield of $\mathscr{M}$ which contains $\mathbb{C}$ and all $\frac{a_{j_k}}{a_{j_l}}$ with $a_{j_l} \not\equiv 0$.

Let f and a be meromorphic mappings of $\mathbb{C}^m$ into $P^n(\mathbb{C})$ with reduced representations $f = (f_0 : \cdots : f_n)$ and $a = (a_0 : \cdots : a_n)$. We say that f is linearly non-degenerate over $\mathscr{R}$ if $f_0, \ldots, f_n$ are linearly independent over $\mathscr{R}$. Put $(f, a) := \sum_{i=0}^n a_i f_i$, then we say that a is "small" with respect to f if $T_a(r) = o(T_f(r))$ as $r \to +\infty$. Denote by $\nu_{(f,a)}$ the map of $\mathbb{C}^m$ into $\mathbb{Z}$ whose value $\nu_{(f,a)}(z)$ is the intersection of the images of f and a at $f(z)$.

If $f, a : \mathbb{C}^m \to P^n(\mathbb{C})$ be meromorphic mappings such that $(f, a) \not\equiv 0$, then the First Main Theorem for moving targets in value distribution theory states that

$$T_f(r) + T_a(r) = m_{(f,a)}(r) + N_{(f,a)}(r) + O(1).$$

For a nonzero meromorphic function φ on $\mathbb{C}^m$, the proximity function $m(r, \varphi)$ is defined by

$$m(r, \varphi) = \int_{S(r)} \log^+ |\varphi| \sigma,$$

where $\log^+ |x| = \max\{\log x, 0\}$ for $x \ge 0$. The Nevanlinna's characteristic function is defined by

$$T(r, \varphi) := N(r, \nu_\varphi^\infty) + m(r, \varphi).$$

We regard φ as a meromorphic mapping of $\mathbb{C}^m$ into $P^1(\mathbb{C})$, then

$$T_\varphi(r) = T(r, \varphi) + O(1).$$

As usual, by the notation "$\| P$," we mean the assertion P holds for all $r \in [0, +\infty)$ excluding a Borel subset E of the interval $[0, +\infty)$ with $\int_E dr < +\infty$.

The First Main Theorem for general position (see [15], p. 326) Assume that $1 \le \lambda \le n + 1$, $f_i : \mathbb{C}^m \to P^n(\mathbb{C}) (1 \le i \le \lambda)$ are λ meromorphic mappings located in general position. Then

$$N(r, \mu_{f_1 \wedge \cdots \wedge f_\lambda}) + m(r, f_1 \wedge \cdots \wedge f_\lambda) \le \sum_{1 \le i \le \lambda} T_{f_i}(r) + O(1).$$

Let V be a complex vector space of dimension $n \ge 1$. The vectors $\{v_1, \ldots, v_k\}$ are said to be in general position if for each selection of integers $1 \le i_1 < \cdots < i_p \le k$ with $p \le n$, $v_{i_1} \wedge \cdots \wedge v_{i_p} \ne 0$. The vectors $\{v_1, \ldots, v_k\}$ are said to be in special position if they are not in general position. Take $1 \le p \le k$, then $\{v_1, \ldots, v_k\}$ are said to be in p-special position if for each selection of integers $1 \le i_1 < \cdots < i_p \le k$ the vectors $v_{i_1}, \ldots, v_{i_p}$ are in special position.

The Second Main Theorem for general position (see [15], Theorem 2.1, P.326) Let M be a connected complex manifold of dimension m. Let A be a pure $(m-1)$-dimensional analytic subset of M. Let V be a complex vector space of dimension $n+1>1$. Let p and k be integers with $1 \leq p \leq k \leq n+1$. Let $f_j : M \to P(V)$, $1 \leq j \leq k$ be meromorphic mappings. Assume that $f_1, \dots, f_k$ are in general position. Also assume that $f_1, \dots, f_k$ are in p-special position on A. Then we have

$$\mu_{f_1 \wedge \cdots \wedge f_k} \geq (k-p+1)\nu_A.$$

The following is the "second main theorem type" for meromorphic mappings intersecting moving targets with truncated counting function according to S.D.Quang [11].

Second Main Theorem for moving targets (Quang [11]) Let $f : \mathbb{C}^m \to P^n(\mathbb{C})$ be a meromorphic mapping and $\{a_i\}_{i=1}^q$ ($q \geq 2n-k+2$) be meromorphic mappings of $\mathbb{C}^m$ into $P^n(\mathbb{C})$ located in general position such that $(f, a_i) \not\equiv 0 (1 \leq i \leq q)$, where $k+1 = \operatorname{rank}_{\mathscr{R}}(f)$. Then the following assertions hold:

(a) $\| \frac{q}{2n-k+2} T_f(r) \leq \sum_{i=1}^q N^{[k]}_{(f,a_i)}(r) + o(T_f(r)) + O(\max_{1\leq i\leq q} T_{a_i}(r))$,

(b) $\| \frac{q-n+2k-1}{n+k+1} T_f(r) \leq \sum_{i=1}^q N^{[k]}_{(f,a_i)}(r) + o(T_f(r)) + O(\max_{1\leq i\leq q} T_{a_i}(r))$.

Remark:

(1) If $k \geq \frac{n+1}{2}$ then the above theorem (a) is stronger than (b), otherwise, if $k < \frac{n+1}{2}$ then (b)is stronger than (a).

(2) If $k \geq 1$, we have the following estimates:

$$\min_{1\leq k\leq n} \{\max\{\frac{q}{2n-k+2}, \frac{q-n+2k-1}{n+k+1}\}\} \geq \begin{cases} \frac{2q}{3(n+1)} & \text{if } q \geq 3n+3, \\ \frac{q-n+1}{n+2} & \text{if } q < 3n+3. \end{cases}$$

Therefore, S.D.Quang [11] obtained the following corollary.

Corollary 2.1 *Let $f : \mathbb{C}^m \to P^n(\mathbb{C})$ be a meromorphic mapping and $\{a_i\}_{i=1}^q$ ($q \geq 2n+1$) be meromorphic mappings of $\mathbb{C}^m$ into $P^n(\mathbb{C})$ located in general position such that $(f, a_i) \not\equiv 0 (1 \leq i \leq q)$.*

(a) Then we have

$$\| \frac{2q-n+1}{3(n+1)} T_f(r) \leq \sum_{i=1}^q N^{[n]}_{(f,a_i)}(r) + o(T_f(r)) + O(\max_{1\leq i\leq q} T_{a_i}(r)).$$

(b) If $q \geq 3n+3$, then

$$\| \frac{2q}{3(n+1)} T_f(r) \leq \sum_{i=1}^q N^{[n]}_{(f,a_i)}(r) + o(T_f(r)) + O(\max_{1\leq i\leq q} T_{a_i}(r)).$$

(c) If $q < 3n+3$, then

$$\| \frac{q-n+1}{n+2} T_f(r) \leq \sum_{i=1}^q N^{[n]}_{(f,a_i)}(r) + o(T_f(r)) + O(\max_{1\leq i\leq q} T_{a_i}(r)).$$

3 Proof of Theorem 1

Proof It suffices to prove Theorem 1 in the case of $\lambda \le n+1$. Assume that $f_1 \wedge \cdots \wedge f_\lambda \not\equiv 0$. Considering $\lambda - 1$ arbitrary moving targets $a_{i_1}, \ldots, a_{i_{\lambda-1}}$, then there exists a_{i_0} with $i_0 \notin \{i_1, \ldots, i_{\lambda-1}\}$ such that the matrix $\begin{pmatrix} (f_1, a_{i_0}) & \cdots & (f_\lambda, a_{i_0}) \\ (f_1, a_{i_1}) & \cdots & (f_\lambda, a_{i_1}) \\ \vdots & \vdots & \vdots \\ (f_1, a_{i_{\lambda-1}}) & \cdots & (f_\lambda, a_{i_{\lambda-1}}) \end{pmatrix}$ is nondegenerate.

Indeed, suppose on contrary, the matrix $\begin{pmatrix} (f_1, a_1) & \cdots & (f_\lambda, a_1) \\ (f_1, a_2) & \cdots & (f_\lambda, a_2) \\ \vdots & \vdots & \vdots \\ (f_1, a_{n+1}) & \cdots & (f_\lambda, a_{n+1}) \end{pmatrix}$ is of rank $\le \lambda - 1 (< n+1)$. Let $a_j = (a_{j0} : \cdots : a_{jn}) (1 \le j \le n+1)$ be the reduced representations of $a_1, \ldots, a_{n+1}$ and $f_j = (f_{j0} : \cdots : f_{jn}) (1 \le j \le \lambda)$ be the reduced representations of $f_1, \ldots, f_\lambda$. Then

$$\begin{pmatrix} (f_1, a_1) & \cdots & (f_\lambda, a_1) \\ (f_1, a_2) & \cdots & (f_\lambda, a_2) \\ \vdots & \vdots & \vdots \\ (f_1, a_{n+1}) & \cdots & (f_\lambda, a_{n+1}) \end{pmatrix} = \begin{pmatrix} a_{10} & \cdots & a_{1n} \\ a_{20} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots \\ a_{n+10} & \cdots & a_{n+1n} \end{pmatrix} \cdot \begin{pmatrix} f_{10} & \cdots & f_{\lambda 0} \\ f_{11} & \cdots & f_{\lambda 1} \\ \vdots & \vdots & \vdots \\ f_{1n} & \cdots & f_{\lambda n} \end{pmatrix}.$$

Since $a_1, \cdots, a_{n+1}$ are in general position, we get $f_1 \wedge \cdots \wedge f_\lambda \equiv 0$. This is a contradiction.

We denote by $\mu_{\widetilde{f_1} \wedge \cdots \wedge \widetilde{f_\lambda}}$ the divisor associated with $\widetilde{f_1} \wedge \cdots \wedge \widetilde{f_\lambda}$, $N(r, \mu_{\widetilde{f_1} \wedge \cdots \wedge \widetilde{f_\lambda}})$ the counting function associated with the divisor $\mu_{\widetilde{f_1} \wedge \cdots \wedge \widetilde{f_\lambda}}$, where $\tilde{f_t} := ((f_t, a_{i_0}) : \cdots : (f_t, a_{i_{\lambda-1}}))$ $(1 \le t \le \lambda)$. We will prove the following claim.

Claim 3.1 *For every $z \in \mathbb{C}^m$ outside an analytic set of codimension ≥ 2, we have*

$$\sum_{j=0}^{\lambda-1} \left(\lambda \min_{1 \le t \le \lambda} \{\nu_{(f_t, a_{i_j})}(z)\} + (\lambda - l) \sum_{t=1}^{\lambda} \min\{\nu_{(f_t, a_{i_j})}(z), 1\} \right)$$
$$+ \sum_{j \in \{1,\ldots,q\} \setminus \{i_0, i_1, \ldots, i_{\lambda-1}\}} \sum_{t=1}^{\lambda} (\lambda - l + 1) \min\{\nu_{(f_t, a_j)}(z), 1\} \le \lambda \mu_{\tilde{f_1} \wedge \cdots \wedge \tilde{f_\lambda}}(z).$$

In fact, the Claim is the Claim 3.2 in [8], *we give the proof here for completeness of our proof. Let $I = \{i_0, i_1, \ldots, i_{\lambda-1}\}$, $\overline{I} = \{1, 2, \ldots, q\} \setminus I$, $\mathscr{A} := \cup_{i \in I} (f_1, a_i)^{-1}(0)$, $\overline{\mathscr{A}} := \cup_{i \in \overline{I}} (f_1, a_i)^{-1}(0)$ and $A = \bigcup_{1 \le i < j \le q} ((f_1, a_i)^{-1}(0) \cap (f_1, a_j)^{-1}(0))$.*

Case 1 Let $z_0 \in \mathscr{A} \setminus (A \cup \cup_{i=1}^{\lambda} I(f_i) \cup (a_{i_0} \wedge \cdots \wedge a_{i_{\lambda-1}})^{-1}(0))$ be a regular point of $\mathscr{A}$. Without loss of generality, we assume that z_0 is a zero of (f_1, a_{i_0}) where $i_0 \in I$. Let S be an irreducible analysis subset of $\mathscr{A}$ containing z_0 and U be an open neighborhood of z_0 in $\mathbb{C}^m$, such that $U \cap (\mathscr{A} \setminus S) = \emptyset$. Choosing a holomorphic function h on neighborhood $U' \subset U$ of z_0 such that $\nu_h(z) = \min_{1\le t\le\lambda}\{\nu_{(f_t,a_{i_0})}(z)\}$ if $z \in S$ and $\nu_h(z) = 0$ if $z \notin S$. Then $(f_t, a_{i_0}) = hu_t$ $(1 \le t \le \lambda)$, where u_t is holomorphic function.

Since the matrix $\begin{pmatrix} (f_1, a_{i_1}) & \cdots & (f_\lambda, a_{i_1}) \\ \vdots & \vdots & \vdots \\ (f_1, a_{i_{\lambda-1}}) & \cdots & (f_\lambda, a_{i_{\lambda-1}}) \end{pmatrix}$ is of rank $\le \lambda - 1$. Therefore, there exist not all zero holomorphic function $b_1, \ldots, b_\lambda$ such that

$$\sum_{t=1}^{\lambda} b_t (f_t, a_{i_j}) = 0 \ (1 \le j \le \lambda - 1).$$

It implies that

$$\sum_{t=1}^{\lambda} b_t \tilde{f}_t = \left(\sum_{t=1}^{\lambda} b_t (f_t, a_{i_0}), 0, \ldots, 0 \right).$$

Without loss of generality, we may assume that the set of the common zeros of $\{b_t\}_{t=1}^{\lambda}$ is an analysis subset of codimension ≥ 2. Then there exists an index, say λ, such that $S \not\subset b_\lambda^{-1}(0)$.

Thus, for each $z \in (U' \cap S) \setminus b_\lambda^{-1}(0)$, we have

$$\begin{aligned} \tilde{f}_1(z) \wedge \cdots \wedge \tilde{f}_\lambda(z) =& \tilde{f}_1(z) \wedge \cdots \wedge \widetilde{f_{\lambda-1}}(z) \wedge (\tilde{f}_\lambda(z) + \sum_{t=1}^{\lambda-1} \frac{b_t}{b_\lambda} \tilde{f}_t(z)) \\ =& \tilde{f}_1(z) \wedge \cdots \wedge \widetilde{f_{\lambda-1}}(z) \wedge (V(z)h(z)) \\ =& h(z)\tilde{f}_1(z) \wedge \cdots \wedge \widetilde{f_{\lambda-1}}(z) \wedge V(z), \end{aligned}$$

where $V(z) = (u_\lambda + \sum_{t=1}^{\lambda-1} \frac{b_t}{b\lambda} u_t, 0, \ldots, 0)$. On the other hand,

$$\begin{pmatrix} (f_{j_1}, a_{i_0}) & \cdots & (f_{j_l}, a_{i_0}) \\ (f_{j_1}, a_{i_1}) & \cdots & (f_{j_l}, a_{i_1}) \\ \vdots & \vdots & \vdots \\ (f_{j_1}, a_{i_{\lambda-1}}) & \cdots & (f_{j_1}, a_{i_{\lambda-1}}) \end{pmatrix} = \begin{pmatrix} a_{i_0 0} & \cdots & a_{i_0 n} \\ a_{i_1 0} & \cdots & a_{i_1 n} \\ \vdots & \vdots & \vdots \\ a_{i_{\lambda-1} 0} & \cdots & a_{i_{\lambda-1} n} \end{pmatrix} \cdot \begin{pmatrix} f_{j_1 0} & \cdots & f_{j_l 0} \\ f_{j_1 1} & \cdots & f_{j_l 1} \\ \vdots & \vdots & \vdots \\ f_{j_1 n} & \cdots & f_{j_l n} \end{pmatrix}.$$

Since $a_1, \cdots, a_q$ are in general position, $\mathrm{rank}\{a_{i_0}, \cdots, a_{i_{\lambda-1}}\} = \lambda$. By the assumption, for any increasing sequence $1 \le j_1 < \cdots < j_l \le \lambda - 1$, $f_{j_1}(z) \wedge \cdots \wedge f_{j_l}(z) = 0$ on S. Thus,

$$\begin{aligned} \mathrm{rank}\{\widetilde{f_{j_1}}, \cdots, \widetilde{f_{j_l}}\} &\le \min\{\mathrm{rank}\{f_{j_1}, \cdots, f_{j_l}\}, \mathrm{rank}\{a_{i_0}, \cdots, a_{i_{\lambda-1}}\}\} \\ &= \mathrm{rank}\{f_{j_1}, \cdots, f_{j_l}\}. \end{aligned}$$

Hence $\tilde{f}_{j_1}(z) \wedge \cdots \wedge \tilde{f}_{j_l}(z) = 0$ on S. It yields that the family $\{\tilde{f}_1, \ldots, \tilde{f}_l, V(z)\}$ is in $(l+1)-$ special position on S. By the Second Main Theorem for general position, we have

$$\mu_{\tilde{f}_1 \wedge \cdots \wedge \widetilde{f_{\lambda-1}} \wedge V}(z) \geq \lambda - l, \ z \in S.$$

Hence, $\mu_{\tilde{f}_1 \wedge \cdots \wedge \tilde{f}_\lambda}(z) \geq \nu_h(z) + \lambda - l, \ z \in (U' \cap S) \setminus b_\lambda^{-1}(0)$. In particular,

$$\mu_{\tilde{f}_1 \wedge \cdots \wedge \tilde{f}_\lambda}(z_0) \geq \min_{1 \leq t \leq \lambda} \{\nu_{(f_t, a_{i_0})}(z_0)\} + \lambda - l.$$

Therefore

$$\begin{aligned}
&\sum_{i \in I} (\lambda \min_{1 \leq t \leq \lambda} \{\nu_{(f_t, a_i)}(z_0)\} + (\lambda - l) \sum_{t=1}^{\lambda} \min\{\nu_{(f_t, a_i)}(z_0), 1\}) \\
&\quad + \sum_{i \in \bar{I}} \sum_{t=1}^{\lambda} (\lambda - l + 1) \min\{\nu_{(f_t, a_i)}(z_0), 1\} \\
&= \lambda(\min_{1 \leq t \leq \lambda} \{\nu_{(f_t, a_{i_0})}(z_0)\} + \lambda - l) \leq \lambda \mu_{\tilde{f}_1 \wedge \cdots \wedge \tilde{f}_\lambda}(z_0).
\end{aligned}$$

Case 2 Let $z_0 \in \widetilde{\mathscr{A}} \setminus (A \cup \cup_{i=1}^{\lambda} I(f_i) \cup (a_{i_0} \wedge \cdots \wedge a_{i_{\lambda-1}})^{-1}(0))$ be a regular point of $\widetilde{\mathscr{A}}$. Then z_0 is a zero of one holomorphic mapping (f_1, a_t) where $t \in \bar{I}$. According to the assumption and the proof in Case 1, the family $\tilde{f}_1, \ldots, \tilde{f}_\lambda$ are in l-special position on an irreducible analytic subset of codimension 1 of $\widetilde{\mathscr{A}}$ which contains z_0. By the Second Main Theorem for general position, we have

$$\mu_{\tilde{f}_1 \wedge \cdots \wedge \tilde{f}_\lambda}(z_0) \geq \lambda - l + 1.$$

Hence

$$\begin{aligned}
&\sum_{i \in I} (\lambda \min_{1 \leq t \leq \lambda} \{\nu_{(f_t, a_i)}(z_0)\} + (\lambda - l) \sum_{t=1}^{\lambda} \min\{\nu_{(f_t, a_i)}(z_0), 1\}) \\
&\quad + \sum_{i \in \bar{I}} \sum_{t=1}^{\lambda} (\lambda - l + 1) \min\{\nu_{(f_t, a_i)}(z_0), 1\} \\
&= \lambda(\lambda - l + 1) \leq \lambda \mu_{\tilde{f}_1 \wedge \cdots \wedge \tilde{f}_\lambda}(z_0).
\end{aligned}$$

From the above two cases, Claim 3.1 is proved.

For nonnegative integers $c_1, c_2, \ldots, c_\lambda$, it is easy to check that

$$\min_{1 \leq t \leq \lambda} c_t \geq \sum_{t=1}^{\lambda} \min\{c_t, n\} - (\lambda - 1)n.$$

Since $\min\{\nu_{(f_1,a_{i_0})},1\} = \cdots = \min\{\nu_{(f_\lambda,a_{i_0})},1\}$ and $\nu_{(f_t,a_i)}(z_0) \geq \min\{\nu_{(f_t,a_{i_0})}(z_0),1\}$. Claim 3.1 implies that

$$\sum_{j=1}^{\lambda-1}\sum_{t=1}^{\lambda}\Big(\lambda\min\{\nu_{(f_t,a_{i_j})}(z),n\} - [(\lambda-1)n-(\lambda-l)]\min\{\nu_{(f_t,a_{i_j})}(z),1\}\Big)$$
$$+\sum_{j\in\{1,\ldots,q\}\setminus\{i_1,\ldots,i_{\lambda-1}\}}\sum_{t=1}^{\lambda}(\lambda-l+1)\min\{\nu_{(f_t,a_j)}(z),1\} \leq \lambda\mu_{\tilde{f}_1\wedge\cdots\wedge\tilde{f}_\lambda}(z).$$

Thus,

$$\sum_{j=1}^{\lambda-1}\sum_{t=1}^{\lambda}\Big(\lambda N^{[n]}_{(f_t,a_{i_j})}(r) - [(\lambda-1)n-(\lambda-l)]N^{[1]}_{(f_t,a_{i_j})}(r)\Big)$$
$$+\sum_{j\in\{1,\ldots,q\}\setminus\{i_1,\ldots,i_{\lambda-1}\}}\sum_{t=1}^{\lambda}(\lambda-l+1)N^{[1]}_{(f_t,a_j)}(r)$$
$$\leq \lambda N_{\tilde{f}_1\wedge\cdots\wedge\tilde{f}_\lambda}(r).$$

From $T_{\tilde{f}_t}(r) \leq T_{f_t}(r)+o(\max_{1\leq i\leq\lambda}T_{f_i}(r))$ $(1\leq t\leq\lambda)$ and the First Main Theorem for general position, we get

$$\sum_{j=1}^{\lambda-1}\sum_{t=1}^{\lambda}\Big(\lambda N^{[n]}_{(f_t,a_{i_j})}(r) - [(\lambda-1)n-(\lambda-l)]N^{[1]}_{(f_t,a_{i_j})}(r)\Big)$$
$$+\sum_{j\in\{1,\ldots,q\}\setminus\{i_1,\ldots,i_{\lambda-1}\}}\sum_{t=1}^{\lambda}(\lambda-l+1)N^{[1]}_{(f_t,a_j)}(r) \leq \lambda\sum_{t=1}^{\lambda}T_{f_t}(r)+o(\max_{1\leq t\leq\lambda}T_{f_t}(r)).$$

Summing both sides of the above inequality over all sequences $1\leq i_1<\cdots<i_{\lambda-1}\leq q$, we have

$$\sum_{i=1}^{q}\sum_{t=1}^{\lambda}\Big(\lambda(\lambda-1)N^{[n]}_{(f_t,a_i)}(r) + ((\lambda-l+1)q-(\lambda-1)((\lambda-1)n+1))N^{[1]}_{(f_t,a_i)}(r)\Big)$$
$$\leq q\lambda\sum_{t=1}^{\lambda}T_{f_t}(r)+o(\max_{1\leq t\leq\lambda}T_{f_t}(r)).$$

Since $N^{[1]}_{(f_t,a_i)}(r) \geq \frac{1}{n}N^{[n]}_{(f_t,a_i)}(r)$, then

$$\frac{(\lambda-l+1)q+(\lambda-1)(n-1)}{n}\sum_{i=1}^{q}\sum_{t=1}^{\lambda}N^{[n]}_{(f_t,a_i)}(r) \leq q\lambda\sum_{t=1}^{\lambda}T_{f_t}(r)+o(\max_{1\leq t\leq\lambda}T_{f_t}(r)).$$

For $2 \leq l \leq \lambda \leq n$,

$$q > \frac{3n(n+1)\lambda - 2(n-1)(\lambda-1)}{2(\lambda-l+1)} \geq \frac{3n^2+4n+5}{2} \geq 3n+3$$

Applying the Second Main Theorem for moving targets (Corollary 2.1(b)), we set

$$\| \frac{(\lambda-l+1)q + (\lambda-1)(n-1))}{n} \cdot \frac{2q}{3(n+1)} \sum_{t=1}^{\lambda} T_{f_t}(r)$$
$$\leq q\lambda \sum_{t=1}^{\lambda} T_{f_t}(r) + o(\max_{1 \leq t \leq \lambda} T_{f_t}(r)).$$

Letting $r \to +\infty$, we have

$$\frac{(\lambda-l+1)q + (\lambda-1)(n-1))}{n} \cdot \frac{2q}{3(n+1)} \leq q\lambda.$$

Therefore

$$q \leq \frac{3n(n+1)\lambda - 2(n-1)(\lambda-1)}{2(\lambda-l+1)}.$$

This is a contradiction. Thus, $f_1 \wedge \cdots \wedge f_\lambda \equiv 0$. Theorem 1 is proved completely. □

4 Proof of Theorem 2

Let $f : \mathbb{C}^m \to P^n(\mathbb{C})$ be a meromorphic mapping and $\{a_1, \ldots, a_q\}$ be meromorphic mappings of $\mathbb{C}^m$ into $P^n(\mathbb{C})$. Take a reduced representation $f = (f_1 : \cdots : f_q)$ of f, we set

$$V(f) := \{(c_1, \ldots, c_n) \in \mathscr{R}^{n+1} : \sum_{i=0}^{n} c_i f_i = 0\}.$$

It is easy to see that $V(f)$ does nor depend on the choice of the reduced representation of f and $\operatorname{rank}_{\mathscr{R}}(f) = n + 1 - \dim_{\mathscr{R}} V(f)$.

In order to prove Theorem 2, we need the following lemma.

Lemma 4.1 *Let $f : \mathbb{C}^m \to P^n(\mathbb{C})$ be a meromorphic mapping and d be a positive integer or $+\infty$. Let $\{a_j\}_{j=1}^q$ be "small" (with respect to f) meromorphic mappings of $\mathbb{C}^m$ into $P^n(\mathbb{C})$ in general position such that*

$$\dim\{z : (f, a_i)(z) = (f, a_j) = 0\} \leq m - 2 \ (1 \leq i < j \leq q).$$

Assume that one of the following conditions satisfies:

(i) $k \geq \frac{n+1}{2}$ *and* $q > 2(n+1)k - k^2$, *or*
(ii) $k < \frac{n+1}{2}$ *and* $q > k^2 + (n-1)k + (n+1)$.

Then for every $g \in \mathscr{G}(f, \{a_j\}_{j=1}^q, 1)$, *the following hold:*

(a) $V(f) = V(g)$ *and* $rank_{\mathscr{R}}(f) = rank_{\mathscr{R}}(g)$.
(b) $\| T_f(r) = O(T_g(r))$ *and* $\| T_g(r) = O(T_f(r))$.

Proof Without loss of generality, we may assume that $\mathrm{rank}_{\mathscr{R}}(f) \geq \mathrm{rank}_{\mathscr{R}}(g)$. Taking reduced representations $f = (f_0 : \cdots : f_n)$ and $g = (g_0 : \cdots : g_n)$, respectively.

We will show that $V(g) \subseteq V(f)$.

Indeed, suppose that there exists an element $c = (c_0, \ldots, c_n) \in \mathscr{R}^{n+1}$ such that $\sum_{i=1}^n c_i g_i = 0$, but $\sum_{i=1}^n c_i f_i \neq 0$. Since $f(z) = g(z)$ on $\cup_{j=1}^q \{z : (f, a_j)(z) = 0\}$, we have $\cup_{i=1}^q \{z : (f, a_j)(z) = 0\} \subset \{z : (f, c)(z) = 0\}$. This implies that

$$\sum_{i=1}^q N^{[1]}_{(f,a_i)}(r) \leq N^{[1]}_{(f,c)}(r) + o(T_f(r)).$$

By the Second Main Theorem for moving hyperplanes, we have

– If $k \geq \frac{n+1}{2}$, then

$$\begin{aligned} \| \frac{q}{2n-k+2} T_f(r) &\leq \sum_{i=1}^q N^{[k]}_{(f,a_i)}(r) + o(T_f(r)) \\ &\leq k \sum_{i=1}^q N^{[1]}_{(f,a_i)}(r) + o(T_f(r)) \\ &\leq k N^{[1]}_{(f,c)}(r) + o(T_f(r)) \leq k T_f(r) + o(T_f(r)). \end{aligned}$$

Let $r \to +\infty$, we get

$$q \leq 2(n+1)k - k^2.$$

This is in contradiction with (i).

– If $k < \frac{n+1}{2}$, then

$$\begin{aligned} \| \frac{q-n+2k-1}{n+1+k} T_f(r) &\leq \sum_{i=1}^q N^{[k]}_{(f,a_i)}(r) + o(T_f(r)) \\ &\leq k \sum_{i=1}^q N^{[1]}_{(f,a_i)}(r) + o(T_f(r)) \\ &\leq k N^{[1]}_{(f,c)}(r) + o(T_f(r)) \leq k T_f(r) + o(T_f(r)). \end{aligned}$$

Let $r \to +\infty$, we get

$$q \leq k^2 + (n-1)k + (n+1).$$

This is in contradiction with (ii).

Hence $V(g) \subseteq V(f)$. It implies that

$$\mathrm{rank}_{\mathscr{R}}(f) = n + 1 - \dim_{\mathscr{R}} V(f) \leq n + 1 - \dim_{\mathscr{R}} V(g) = \mathrm{rank}_{\mathscr{R}}(g).$$

Therefore, $\mathrm{rank}_{\mathscr{R}}(f) = \mathrm{rank}_{\mathscr{R}}(g)$, which yields $V(f) = V(g)$.

By the Second Main Theorem for moving hyperplanes, $q \geq 2n - k + 2$, we have

$$\begin{aligned}\| \frac{q}{2n+2-k} T_f(r) &\leq \sum_{i=1}^{q} N^{[k]}_{(f,a_i)}(r) + o(T_f(r)) \\ &\leq k \sum_{i=1}^{q} N^{[1]}_{(g,a_i)}(r) + o(T_f(r)) \leq kq(T_g(r)) + o(T_f(r)).\end{aligned}$$

Thus, $\| T_f(r) = O(T_g(r))$. Since $\mathrm{rank}_{\mathscr{R}}(f) = \mathrm{rank}_{\mathscr{R}}(g)$, similarly we get $\| T_g(r) = O(T_f(r))$. The Lemma is proved. □

Proof of Theorem 2 Suppose that $g \in \mathscr{G}(f, \{a_j\}_{j=1}^{q}, 1)$ and $f \not\equiv g$. By changing indices if necessary, we may assume that

$$\begin{aligned}&\underbrace{\frac{(f,a_1)}{(g,a_1)} \equiv \frac{(f,a_2)}{(g,a_2)} \equiv \cdots \equiv \frac{(f,a_{k_1})}{(g,a_{k_1})}}_{group\ 1} \not\equiv \underbrace{\frac{(f,a_{k_1+1})}{(g,a_{k_1+1})} \equiv \cdots \equiv \frac{(f,a_{k_2})}{(g,a_{k_2})}}_{group\ 2} \\ &\qquad \not\equiv \cdots \not\equiv \underbrace{\frac{(f,a_{k_{s-1}+1})}{(g,a_{k_{s-1}+1})} \equiv \cdots \equiv \frac{(f,a_{k_s})}{(g,a_{k_s})}}_{group\ s},\end{aligned}$$

where $k_s = q$. For each $1 \leq i \leq q$, we set

$$\sigma(i) = \begin{cases} i + k & \text{if} \quad i + k \leq q; \\ i + k - q, & \text{if} \quad i + k > q. \end{cases}$$

and

$$P_i := (f, a_i)(g, a_{\sigma(i)}) - (g, a_i)(f, a_{\sigma(i)}).$$

We claim that the number of elements in each group is at most k.

Indeed, suppose this claim does not hold. Without loss of generality, we may assume that $k_1 \geq k + 1$. Since $\mathrm{rank}_{\mathscr{R}}(f) = \dim(\{(f, \widetilde{a}_i)\}_{i=1}^{k+1})_{\mathscr{R}}$, for every $0 \leq j \leq n$, there exist meromorphic mappings $c_{ji} \in \mathscr{R}$ such that

$$f_j = \sum_{i=1}^{k+1} c_{ji}(f, \widetilde{a}_i).$$

By Lemma 4.1, we have

$$V(f) = V(g).$$

It yields that

$$g_j = \sum_{i=1}^{k+1} c_{ji}(g, \widetilde{a}_i) = \frac{(f, a_1)}{(g, a_1)} f_j$$

for every $0 \le j \le n$. Hence $f = g$. This is a contradiction. Thus, the number of elements of each group is at most k.

Hence $\frac{(f,a_i)}{(g,a_i)}$ and $\frac{(f,a_{\sigma(i)})}{(g,a_{\sigma(i)})}$ belong to distinct groups. This implies that $P_i \not\equiv 0 (1 \le i \le q)$.

By Claim 3.1 for $\lambda = l = 2$ and $i_0 = \sigma(i)$, we have

$$\sum_{j=i,\sigma(i)} 2\min\{\nu_{(f,a_j)}(z), \nu_{(g,a_j)}(z)\} + 2 \sum_{j=1.j\neq i,\sigma(i)}^{q} \min\{\nu_{(f,a_j)}(z), 1\} \le 2\mu_{\tilde{f}\wedge\tilde{g}}(z).$$

for every $z \in \mathbb{C}^m$ outside an analytic set of codimension ≥ 2. Because

$$\begin{aligned}&\min\left\{\nu_{(f,a_j)}(z), \nu_{(g,a_j)}(z)\right\}\\ &\ge \min\left\{\nu_{(f,a_j)}(z), k\} + \min\{\nu_{(g,a_j)}(z), k\right\} - k\min\{\nu_{(f,a_j)}(z), 1\}.\end{aligned}$$

Thus,

$$\begin{aligned}&2\sum_{j=i,\sigma(i)} \left(\min\{\nu_{(f,a_j)}(z), k\} + \min\{\nu_{(g,a_j)}(z), k\} - k\min\{\nu_{(f,a_j)}(z), 1\}\right)\\ &\quad + 2\sum_{j=1.j\neq i,\sigma(i)}^{q} \min\{\nu_{(f,a_j)}(z), 1\} \le 2\mu_{\tilde{f}\wedge\tilde{g}}(z).\end{aligned}$$

for every $z \in \mathbb{C}^m$ outside an analytic set of codimension ≥ 2. It yields that

$$\begin{aligned}&\sum_{j=i,\sigma(i)} \sum_{u=f,g} (2N^{[k]}_{(u,a_j)}(r) - kN^{[1]}_{(u,a_j)}(r))\\ &\quad + \sum_{j=1.j\neq i,\sigma(i)}^{q} \sum_{u=f,g} N^{[1]}_{(u,a_j)}(r) \le 2T(r) + o(T(r)),\end{aligned}$$

where $T(r) := T_f(r) + T_g(r)$. By summing up over all i, we have

$$\sum_{j=1}^{q} \sum_{u=f,g} \left(4N^{[k]}_{(u,a_j)}(r) + (q-2k-2)N^{[1]}_{(u,a_j)}(r)\right) \le 2qT(r) + o(T(r)).$$

Since $N^{[1]}_{(u,a_j)}(r) \geq \frac{1}{k} N^{[k]}_{(u,a_j)}(r)$, we obtain

$$\frac{q+2k-2}{k} \sum_{j=1}^{q} \sum_{u=f,g} N^{[k]}_{(u,a_j)}(r) \leq 2qT(r) + o(T(r)).$$

By the Second Main Theorem for moving targets, we get

- If $k \geq \frac{n+1}{2}$,

$$\frac{q+2k-2}{k} \cdot \frac{q}{2n-k+2} T(r) \leq 2qT(r) + o(T(r)).$$

  Letting $r \to +\infty$, we have

$$q \leq -2k^2 + 2(2n+1)k + 2.$$

  It is a contradiction.
- If $k < \frac{n+1}{2}$,

$$\frac{q+2k-2}{k} \cdot \frac{q-n+2k-1}{n+k+1} T(r) \leq 2qT(r) + o(T(r)).$$

  Letting $r \to +\infty$, we have

$$q^2 - [2k^2 + 2(n-1)k + n + 3]q - 2(n+1-2k)(k-1) \leq 0.$$

  Thus,

$$q \leq \frac{2k^2 + 2(n-1)k + (n+3) + \sqrt{(2k^2 + 2(n-1)k + n - 1)^2 + 8(3n+1)k}}{2}.$$

  This is a contradiction.

Therefore, we have $f = g$. This completes the proof of Theorem 2. □

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