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Hille and Nehari-Type Oscillation Criteria for Third-Order Emden–Fowler Neutral Delay Dynamic Equations

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Abstract We establish some oscillation criteria for the third-order Emden–Fowler neutral delay dynamic equations of the form:

$$(a(t)(x(t) + r(t)x(\tau(t)))^{\Delta\Delta})^{\Delta} + p(t)x^{\gamma}(\delta(t)) = 0$$

on a time scale $\mathbb{T}$, where $\gamma > 0$ is a quotient of odd positive integers, and a and p are real-valued positive rd-continuous functions defined on $\mathbb{T}$. Due to the different values of γ, we give not only the oscillation criteria for superlinear neutral delay dynamic equations, but also the oscillation criteria for sublinear neutral delay dynamic equations based on the Hille and Nehari-type oscillation criteria. Our results extend and improve some known results in the literature and are new even for the corresponding third-order differential equations and difference equations as our special cases.

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1 Introduction

In this paper, we study the oscillation for the third-order Emden–Fowler neutral delay dynamic equations of the form:

$$(a(t)(x(t)+r(t)x(\tau(t)))^{\Delta\Delta})^{\Delta}+p(t)x^{\gamma}(\delta(t))=0 \tag{1.1}$$

on a time scale $\mathbb{T}$ with $\sup\mathbb{T}=\infty$, where the following hypotheses hold:

(A_1) $\gamma>0$ is the quotient of odd positive integers;

(A_2) a and p are positive real-valued rd-continuous functions defined on $\mathbb{T}$ such that $\int_{t_0}^{\infty}\Delta t/a(t)=\infty$;

(A_3) r is a real-valued rd-continuous function defined on $\mathbb{T}$ such that either $0\le r(t)<1$ or $-1<r_0\le r(t)<0$;

(A_4) the functions $\tau:\mathbb{T}\to\mathbb{T}$ and $\delta:\mathbb{T}\to\mathbb{T}$ are rd-continuous functions such that $\tau(t)\le t$, $\delta(t)\le t$, $\lim_{t\to\infty}\tau(t)=\lim_{t\to\infty}\delta(t)=\infty$ and $\tau\circ\delta=\delta\circ\tau$.

We note that if $\mathbb{T}=\mathbb{R}$, then $\sigma(t)=t$, $\mu(t)=0$, $x^{\Delta}(t)=x'(t)$. The third-order Emden–Fowler dynamic equation (1.1) becomes third-order nonlinear neutral delay differential equation:

$$(a(t)(x(t)+r(t)x(\tau(t)))'')'+p(t)x^{\gamma}(\delta(t))=0,\quad t\in\mathbb{R}. \tag{1.2}$$

If $\mathbb{T}=\mathbb{Z}$, then $\sigma(t)=t+1$, $\mu(t)=1$, and $x^{\Delta}(t)=\Delta x(t)=x(t+1)-x(t)$, and Eq. (1.1) becomes third-order nonlinear neutral delay difference equation:

$$\Delta(a(t)\Delta\Delta(x(t)+r(t)x(\tau(t))))+p(t)x^{\gamma}(\delta(t))=0,\quad t\in\mathbb{Z}. \tag{1.3}$$

Note that Emden–Fowler dynamic equation with its continuous version, that is, (1.2), has numerous applications in several physical branches, for example, [1,2], and the reference therein. Moreover, when t is a discrete variable, it is (1.3), and it also has many applications to use.

In recent years, there has been much research activity concerning the oscillation and nonoscillation of solutions of various dynamic equations on time scales unbounded above; we refer the reader to the papers [3–10]. And for the oscillation and nonoscillation of the neutral delay dynamic equations, some excellent works have already been established, and we refer the reader to the various articles [11–26].

In [11], Han et al. investigated a third-order neutral Emden–Fowler delay dynamic equation:

$$(r(t)(x(t)-a(t)x(\tau(t)))^{\Delta\Delta})^{\Delta}+p(t)x^{\gamma}(\delta(t))=0,\quad t\in\mathbb{T}, \tag{1.4}$$

where r, a, and p are positive real-valued rd-continuous functions defined on $\mathbb{T}$ with $0<a(t)\le a_0<1$, $\lim_{t\to\infty}a(t)=a<1$. Using Riccati transformation technique and the integral inequality technique, they established some sufficient conditions for the oscillation of (1.4) and one of them is the following: assume $\gamma > 1$ and there exists a positive function $\eta \in C^1_{rd}([t_0, \infty)_{\mathbb{T}}, \mathbb{R})$, such that for some $0 < k < 1$ and for all constants $M > 0$

$$\limsup_{t\to\infty} \int_{t_0}^{t} \left(\eta(s)p(s)\zeta(s) - \frac{r(s)(\eta^{\Delta}(s))^2}{4k\gamma M^{\gamma-1}\eta(s)} \right) \Delta s = \infty, \tag{1.5}$$

where $\zeta(t) := (h_2(\delta(t), t)/t)^{\gamma}$. Then, every solution x of (1.4) oscillates or $\lim_{t\to\infty} x(t) = 0$.

On the foundation of Han's work, Grace [12] studied equation (1.4) again. They established some new criteria for the oscillation of (1.4) and improved the Han's work.

In this paper, we studied the third-order neutral delay dynamic equation (1.1), and note that when $r(t)$ in (1.1) satisfies the case $-1 < r_0 \le r(t) < 0$, this equation is essentially the same as (1.4). Different from the above works, we establish some new oscillation criteria for (1.1) based on the Hille and Nehari-type oscillation criteria. And we have considered both superlinear case with $\gamma > 1$ and sublinear case with $0 < \gamma < 1$. The obtained results are advantageous, since the Hille and Nehari-type oscillation criteria are sharp.

Regarding Hille and Nehari-type oscillation criteria, in 1948, Hille [27] considered the second-order linear differential equation:

$$x''(t) + p(t)x(t) = 0, \tag{1.6}$$

and gave a sufficient condition for the oscillation of (1.6), that is, if the condition

$$\liminf_{t\to\infty} t \int_t^{\infty} p(s)\mathrm{d}s > \frac{1}{4} \tag{1.7}$$

holds, then every solution of (1.6) oscillates. In [28], Nehari by different approach proved that if

$$\liminf_{t\to\infty} \frac{1}{t} \int_t^{\infty} s^2 p(s)\mathrm{d}s > \frac{1}{4}, \tag{1.8}$$

, then Eq. (1.6) is oscillatory. We note that the inequalities (1.7) and (1.8) are sharp and cannot be weakened. Indeed, letting $p(t) = 1/4t^2$ for $t > 1$, we have

$$\liminf_{t\to\infty} t \int_t^{\infty} p(s)\mathrm{d}s = \liminf_{t\to\infty} \frac{1}{t} \int_t^{\infty} s^2 p(s)\mathrm{d}s = \frac{1}{4}, \tag{1.9}$$

and the second-order Euler differential equation

$$x''(t) + \frac{1}{4t^2}x(t) = 0, \; t > 1, \tag{1.10}$$

has a nonoscillatory solution $x(t) = \sqrt{t}$.

Recently, many researchers have used oscillation criteria of this type in many other fields for studying the oscillatory behaviour of solutions. In 2007, Erbe et al. [29] extended Hille and Nehari type oscillatory criteria to dynamic equation on time scales. They studied a third-order dynamic equation

$$x^{\Delta\Delta\Delta}(t) + p(t)x(t) = 0, \tag{1.11}$$

and obtained some sufficient conditions for the oscillation of solutions of the form:

$$\liminf_{t\to\infty} t \int_t^\infty \frac{h_2(s,t_0)}{\sigma(s)} p(s)\Delta s > \frac{1}{4}, \tag{1.12}$$

or

$$\liminf_{t\to\infty} \frac{1}{t} \int_t^\infty \sigma(s) h_2(\sigma(s),t_0) p(s)\Delta s > \frac{l^*}{1+l^*}, \tag{1.13}$$

where $h_2(t,s)$ is the Taylor monomial of degree 2, $l^* := \limsup_{t\to\infty} \sigma(t)/t$. If (1.12) or (1.13) holds, then every solution x of (1.11) is oscillatory or satisfies $\lim_{t\to\infty} x(t) = 0$.

In [30], Agarwal et al. extended Erbe's work to the delay dynamic equations. They investigated the third-order delay dynamic equations

$$(a(rx^{\Delta})^{\Delta})^{\Delta}(t) + p(t)x(\tau(t)) = 0, \tag{1.14}$$

and established some Hille and Nehari oscillatory criteria for the equations which include retarded term $\tau(t)$. They give the results that if the condition

$$\liminf_{t\to\infty} \int_{t_0}^t \frac{\Delta s}{a(s)} \int_t^\infty \frac{\int_{t_2}^{\tau(s)} \frac{\int_{t_1}^{v}(\Delta u/a(u))}{r(v)}}{\int_{t_1}^{\sigma(s)}(\Delta u/a(u))} p(s)\Delta s > \frac{1}{4}, \tag{1.15}$$

or

$$\liminf_{t\to\infty} \frac{1}{\int_{t_0}^t \frac{\Delta s}{a(s)}} \int_{t_3}^t \frac{\left(\int_{t_0}^{\sigma(s)}(\Delta u/a(u))\right)^2 \int_{t_2}^{\tau(s)} \frac{\int_{t_1}^{v}(\Delta u/a(u))}{r(v)}\Delta v}{\int_{t_1}^{\sigma(s)}(\Delta u/a(u))} p(s)\Delta s > \frac{l^*}{1+l^*} \tag{1.16}$$

holds, then every solution x of (1.14) is oscillatory or satisfies $\lim_{t\to\infty} x(t) = 0$.

To the best of our knowledge, there are no results regarding the Hille and Nehari-type oscillation criteria for the third-order neutral delay dynamic equations on time scales up to now, not even for the superlinear or sublinear dynamic equations. The natural question now is: Can one find the Hille and Nehari-type oscillation criteria for third-order neutral delay nonlinear dynamic equations on time scales? The purpose of this paper is to give an affirmative answer to this question. We establish some Hille and Nehari-type oscillation criteria for the oscillation for (1.1) based on Erbe and Agarwal's work. And our results also improve and extend their results for both superlinear and sublinear neutral delay dynamic equations.

2 Preliminary and Lemmas

For completeness, we recall the following concepts related to the notion of time scales. A time scale $\mathbb{T}$ is an arbitrary nonempty closed subset of the real numbers $\mathbb{R}$. Since we are interested in asymptotic behaviour, we suppose that the time scale under consideration is not bounded above and is a time scale interval of the form: $[t_0, \infty)_{\mathbb{T}} := [t_0, \infty) \cap \mathbb{T}$. On any time scale, we defined the forward and backward jump operators by $\sigma(t) := \inf\{s \in \mathbb{T} : s > t\}$ and $\rho(t) := \inf\{s \in \mathbb{T} : s < t\}$, where $\inf \emptyset := \sup \mathbb{T}$ and $\sup \emptyset := \inf \mathbb{T}$, $\emptyset$ denotes the empty set. A point $t \in \mathbb{T}$ is said to be left-dense if $\rho(t) = t$ and $t > \inf \mathbb{T}$, right-dense if $\sigma(t) = t$ and $t < \sup \mathbb{T}$, left-scattered if $\rho(t) < t$ and right-scattered if $\sigma(t) > t$. The graininess μ of the time scale is defined by $\mu(t) := \sigma(t) - t$, and for any function $f : \mathbb{T} \to \mathbb{R}$, we denote $f^{\sigma}(t) := f(\sigma(t))$. A function $g : \mathbb{T} \to \mathbb{R}$ is said to be right-dense continuous (rd-continuous) provided g is continuous at right-dense points and at left-dense points in $\mathbb{T}$, left-hand limits exist and are finite. The set of all such rd-continuous functions is denoted by $C_{rd}(\mathbb{T})$.

For a function $f : \mathbb{T} \to \mathbb{R}$, if f is continuous at t and t is right-scattered, the (delta) derivative is defined by

$$f^{\Delta}(t) = \frac{f(\sigma(t)) - f(t)}{\sigma(t) - t}.$$

If t is right-dense, then the derivative is defined by

$$f^{\Delta}(t) = \lim_{s \to t^{+}} \frac{f(t) - f(s)}{t - s}.$$

Note that if $\mathbb{T} = \mathbb{R}$, then the delta derivative is just the standard derivative, and when $\mathbb{T} = \mathbb{Z}$ the delta derivative is just the forward difference operator. For more details about the time scales, see [31,32].

Next, for the convenience we define that

$$z(t) = x(t) + r(t)x(\tau(t)). \tag{2.1}$$

Note that if $0 \le r(t) < 1$ and $x(\tau(t)) > 0$, we have $z(t) \ge x(t)$. If $-1 < r_0 \le r(t) < 0$ and $x(\tau(t)) > 0$, we have $z(t) \le x(t)$.

Lemma 2.1 *Assume that* $0 \le r(t) < 1$, *then an eventually positive solution* x *of* (1.1) *only satisfies the following two cases for* $t > t_1$ *sufficiently large:*

$$(i)\ z(t) > 0,\ z^{\Delta}(t) > 0,\ z^{\Delta\Delta}(t) > 0, \quad (a(t)z^{\Delta\Delta}(t))^{\Delta} < 0,$$
$$(ii)\ z(t) > 0,\ z^{\Delta}(t) < 0,\ z^{\Delta\Delta}(t) > 0, \quad (a(t)z^{\Delta\Delta}(t))^{\Delta} < 0.$$

Proof Suppose that x is an eventually positive solution of (1.1), there exists $t_1 \in [t_0, \infty)_{\mathbb{T}}$ such that $x(t) > 0$, $x(\tau(t)) > 0$ and $x(\delta(t)) > 0$ on $[t_1, \infty)_{\mathbb{T}}$, which implies that $z(t) > 0$ on $[t_1, \infty)_{\mathbb{T}}$ by $0 \le r(t) < 1$. From (1.1), we have

$$(a(t)z^{\Delta\Delta}(t))^{\Delta} = -p(t)x^{\gamma}(\delta(t)) < 0, \quad t \geq t_1. \tag{2.2}$$

Hence $a(t)z^{\Delta\Delta}(t)$ is strictly decreasing on $[t_1, \infty)_{\mathbb{T}}$ and has one sign eventually. It means $z^{\Delta\Delta}(t)$ has one sign eventually. We claim that $z^{\Delta\Delta}(t) > 0$ eventually. Assume not, there exists $t_2 \geq t_1$ such that

$$z^{\Delta\Delta}(t) < 0, \quad t \geq t_2. \tag{2.3}$$

Then we can choose a negative constant c and $t_3 \geq t_2$ such that

$$a(t)z^{\Delta\Delta}(t) \leq c < 0, \quad t \geq t_3. \tag{2.4}$$

Dividing (2.4) by $a(t)$ and integrating from t_3 to t, we have

$$z^{\Delta}(t) \leq z^{\Delta}(t_3) + c\int_{t_3}^{t} \frac{\Delta s}{a(s)}. \tag{2.5}$$

Let $t \to \infty$. By the hypothesis A_2: $\int_{t_0}^{\infty} \Delta t/a(t) = \infty$, we have that $z^{\Delta}(t) \to -\infty$ as $t \to \infty$. Thus, there is a $t_4 \geq t_3$ such that for $t \geq t_4$,

$$z^{\Delta}(t) \leq z^{\Delta}(t_4) < 0. \tag{2.6}$$

Integrating the previous inequality from t_4 to t, we obtain

$$z(t) - z(t_4) \leq z^{\Delta}(t_4)(t - t_4). \tag{2.7}$$

Letting $t \to \infty$, we get that $z(t) \to -\infty$, which is a contradiction for $z(t) > 0$ eventually. So $z^{\Delta\Delta}(t) > 0$ eventually and the proof is complete. □

Lemma 2.2 ([14, Lemma 2.1 and Lemma 2.2]) *Assume that* $-1 < r_0 \leq r(t) < 0$, *then an eventually positive solution* x *of* (1.1) *only satisfies the following three cases for* $t > t_1$ *sufficiently large:*

$$\begin{aligned}
&(i)\ z(t) > 0,\ z^{\Delta}(t) > 0,\ z^{\Delta\Delta}(t) > 0,\quad (a(t)z^{\Delta\Delta}(t))^{\Delta} < 0,\\
&(ii)\ z(t) > 0,\quad z^{\Delta}(t) < 0,\quad z^{\Delta\Delta}(t) > 0,\quad (a(t)z^{\Delta\Delta}(t))^{\Delta} < 0,\\
&(iii)\ z(t) < 0,\quad z^{\Delta}(t) < 0,\quad z^{\Delta\Delta}(t) > 0,\quad (a(t)z^{\Delta\Delta}(t))^{\Delta} < 0 \ \text{ and } \lim_{t\to\infty} x(t) = 0.
\end{aligned}$$

Lemma 2.3 *Assume that* $0 \leq r(t) < 1$, *and* x *is an eventually positive solution of* (1.1) *and satisfies case* (i) *of Lemma 2.1. Then we get that*

$$z(t) \geq \left(\frac{\int_{t_1}^{t}\int_{t_1}^{s} \frac{1}{a(u)}\Delta u \Delta s}{\int_{t_1}^{t} \frac{1}{a(u)}\Delta u} \right) z^{\Delta}(t) \tag{2.8}$$

holds for $t \in (t_1, \infty)_{\mathbb{T}}$ *and* $z^{\Delta}(t)/\left(\int_{t_1}^{t} \Delta s/a(s)\right)$ *is nonincreasing for* $t \in (t_1, \infty)_{\mathbb{T}}$.

Proof Assume that x is an eventually positive solution and satisfies case (i) of Lemma 2.1. Then from Lemma 2.1 we get when $t \in (t_1, \infty)_{\mathbb{T}}$

$$z^{\Delta}(t) = z^{\Delta}(t_1) + \int_{t_1}^{t} z^{\Delta\Delta}(s)\Delta s = z^{\Delta}(t_1) + \int_{t_1}^{t} \frac{a(s)z^{\Delta\Delta}(s)}{a(s)}\Delta s$$
$$\geq z^{\Delta}(t_1) + a(t)z^{\Delta\Delta}(t)\int_{t_1}^{t} \frac{\Delta s}{a(s)}. \tag{2.9}$$

So we have that

$$\left(\frac{z^{\Delta}(t)}{\int_{t_1}^{t} \frac{\Delta s}{a(s)}}\right)^{\Delta} = \frac{z^{\Delta\Delta}(t)\int_{t_1}^{t} \frac{\Delta s}{a(s)} - \frac{z^{\Delta}(t)}{a(t)}}{\int_{t_1}^{t} \frac{\Delta s}{a(s)} \int_{t_1}^{\sigma(t)} \frac{\Delta s}{a(s)}} \leq 0 \tag{2.10}$$

for all $t > t_1$, which implies $z^{\Delta}(t)/\big(\int_{t_1}^{t} \Delta s/a(s)\big)$ is nonincreasing for $t \in (t_1, \infty)_{\mathbb{T}}$. Using this we easily get

$$z(t) = z(t_1) + \int_{t_1}^{t} z^{\Delta}(s)\Delta s$$
$$= z(t_1) + \int_{t_1}^{t} \frac{z^{\Delta}(s)\int_{t_1}^{s} \frac{\Delta u}{a(u)}}{\int_{t_1}^{s} \frac{\Delta u}{a(u)}}\Delta s$$
$$\geq z(t_1) + \frac{z^{\Delta}(t)}{\int_{t_1}^{t} \frac{\Delta u}{a(u)}} \int_{t_1}^{t}\int_{t_1}^{s} \frac{\Delta u}{a(u)}\Delta s \tag{2.11}$$

for $t \in (t_1, \infty)_{\mathbb{T}}$. The proof is complete. □

Lemma 2.4 *Assume that* $-1 < r_0 \leq r(t) < 0$, *and* x *is an eventually positive solution of* (1.1) *and satisfies case* (i) *of Lemma* 2.2. *Then we get*

$$z(t) \geq \left(\frac{\int_{t_1}^{t}\int_{t_1}^{s} \frac{1}{a(u)}\Delta u\Delta s}{\int_{t_1}^{t} \frac{1}{a(u)}\Delta u}\right) z^{\Delta}(t) \tag{2.12}$$

for $t \in (t_1, \infty)_{\mathbb{T}}$ *and* $z^{\Delta}(t)/\big(\int_{t_1}^{t} \Delta s/a(s)\big)$ *is nonincreasing for* $t \in (t_1, \infty)_{\mathbb{T}}$.

Proof The proof is similar to that of the proof of Lemma 2.3, so we omit the details. □

Lemma 2.5 *Assume that* $0 \leq r(t) < 1$ *and* x *is an eventually positive solution of* (1.1) *which satisfies case (ii) of Lemma* 2.1. *If there is a constant* $\lambda > 0$ *such that*

$$\int_{t_0}^{\infty} p(s)\Big(1 - (1+\lambda)r(\delta(s))\Big)^{\gamma} R^{\sigma}(s)\Delta s = \infty, \tag{2.13}$$

where $R(t) := \int_{t_0}^{t}\Big(\big(\sigma(u) - t_0\big)/a(u)\Big)\Delta u$ *for* $t \in [t_0, \infty)_{\mathbb{T}}$, *then* $\lim_{t\to\infty} x(t) = 0$.

Proof Let x be an eventually positive solution of (1.1) such that case (ii) of Lemma 2.1 holds for $t \geq t_1$. Then $z(t) > 0$ is strictly decreasing eventually and has finite limit. Now we claim that $\lim_{t\to\infty} z(t) = 0$. Otherwise, $\lim_{t\to\infty} z(t) = l > 0$, By the properties of limit, for $\lambda > 0$ there is a $t_2 \in [t_1, \infty)_{\mathbb{T}}$ such that $l \leq z(t) < (1+\lambda)l$ for $t \in [t_2, \infty)_{\mathbb{T}}$. There exists $t_3 \in [t_2, \infty)_{\mathbb{T}}$ such that $t_2 \leq \delta(\tau(t))$ and $t_2 \leq \delta(t)$ for $t \in [t_3, \infty)_{\mathbb{T}}$. Hence we have

$$l \leq z(\delta(t)) < (1+\lambda)l,$$
$$l \leq z(\delta(\tau(t))) < (1+\lambda)l, \quad t \in [t_3, \infty)_{\mathbb{T}}.$$

By the definition of z, since $0 \leq r(t) < 1$ and $x(t) > 0$ for $t \in [t_1, \infty)_{\mathbb{T}}$, we have that for $t \in [t_3, \infty)_{\mathbb{T}}$

$$z(t) \geq x(t).$$

So

$$\begin{aligned} x(\delta(t)) &= z(\delta(t)) - r(\delta(t))x(\tau(\delta(t))) = z(\delta(t)) - r(\delta(t))x(\delta(\tau(t))) \\ &\geq z(\delta(t)) - r(\delta(t))z(\delta(\tau(t))) \geq l[1-(1+\lambda)r(\delta(t))]. \end{aligned} \tag{2.14}$$

Then, from (1.1) we have

$$-(a(t)z^{\Delta\Delta}(t))^{\Delta} \geq p(t)(l[1-(1+\lambda)r(\delta(t))])^{\gamma}. \tag{2.15}$$

Integrating both sides of (2.15) from t to v, and letting $v \to \infty$, due to $z^{\Delta\Delta}(t) > 0$ eventually, we get

$$z^{\Delta\Delta}(t) \geq \frac{l^{\gamma}}{a(t)} \int_t^{\infty} p(s)(1-(1+\lambda)r(\delta(s)))^{\gamma} \Delta s. \tag{2.16}$$

Integrating again from t to v, and letting $v \to \infty$, we have

$$-z^{\Delta}(t) \geq l^{\gamma} \int_t^{\infty} \frac{1}{a(u)} \int_u^{\infty} p(s)(1-(1+\lambda)r(\delta(s)))^{\gamma} \Delta s \Delta u. \tag{2.17}$$

Integrating again from t_3 to v, and letting $v \to \infty$, we have

$$z(t_3) \geq l^{\gamma} \int_{t_3}^{\infty} \int_v^{\infty} \frac{1}{a(u)} \int_u^{\infty} p(s)(1-(1+\lambda)r(\delta(s)))^{\gamma} \Delta s \Delta u \Delta v, \tag{2.18}$$

which contradicts condition (2.13). Since by [11, Lemma 2.4], we have that

$$\begin{aligned} &\int_{t_0}^{\infty} \int_v^{\infty} \frac{1}{a(u)} \int_u^{\infty} p(s)(1-(1+\lambda)r(\delta(s)))^{\gamma} \Delta s \Delta u \Delta v \\ &\quad = \int_{t_0}^{\infty} p(s)(1-(1+\lambda)r(\delta(s)))^{\gamma} \int_{t_0}^{\sigma(s)} \frac{1}{a(u)} \int_{t_0}^{\sigma(u)} \Delta v \Delta u \Delta s \end{aligned}$$

$$= \int_{t_0}^{\infty} p(s)(1-(1+\lambda)r(\delta(s)))^{\gamma} \int_{t_0}^{\sigma(s)} \frac{\sigma(u)-t_0}{a(u)} \Delta u \Delta s$$
$$= \int_{t_0}^{\infty} p(s)(1-(1+\lambda)r(\delta(s)))^{\gamma} R^{\sigma}(s)\Delta s. \tag{2.19}$$

So $\lim_{t\to\infty} z(t) = 0$. From $z(t) \geq x(t)$ for $t \in [t_1, \infty)_{\mathbb{T}}$, we finally get $\lim_{t\to\infty} x(t) = 0$ and complete the proof. □

Lemma 2.6 *Assume that* $-1 < r_0 \leq r(t) < 0$ *and* x *is an eventually positive solution of* (1.1) *which satisfies case (ii) of Lemma* 2.2. *If*

$$\int_{t_0}^{\infty} p(s)R^{\sigma}(s)\Delta s = \infty, \tag{2.20}$$

where $R(t) := \int_{t_0}^{t} \Big(\big(\sigma(u)-t_0\big)/a(u)\Big)\Delta u$ *for* $t \in [t_0, \infty)_{\mathbb{T}}$, *then* $\lim_{t\to\infty} x(t) = 0$.

Proof Let x be an eventually positive solution of (1.1) such that case (ii) of Lemma 2.2 holds for $t \geq t_1$. Then $z(t) > 0$ is strictly decreasing eventually and has finite limit, and we claim that only $\lim_{t\to\infty} z(t) = 0$ holds. Assume not, let $\lim_{t\to\infty} z(t) = l > 0$. Since $-1 < r_0 \leq r(t) < 0$, By the definition of z we have that $x(t) \geq z(t)$ on $[t_1, \infty)_{\mathbb{T}}$. So there exists $t_2 \in [t_1, \infty)_{\mathbb{T}}$ such that $x(t) \geq z(t) \geq l$ for $t \in [t_2, \infty)_{\mathbb{T}}$. So $x(\delta(t)) \geq l$ on $[t_3, \infty)_{\mathbb{T}} \subseteq [t_2, \infty)_{\mathbb{T}}$ and makes (1.1) become

$$-(a(t)z^{\Delta\Delta}(t))^{\Delta} \geq p(t)l^{\gamma}. \tag{2.21}$$

Then putting a same operation which used in Lemma 2.5 towards to (2.15), we get

$$z(t_3) \geq l^{\gamma} \int_{t_3}^{\infty} p(s)R^{\sigma}(s)\Delta s,$$

which a contradiction to (2.20). So we have $\lim_{t\to\infty} z(t) = 0$.

Next we prove that $\lim_{t\to\infty} x(t) = 0$, and first we claim that x is bounded on $[t_2, \infty)_{\mathbb{T}}$. If not, there exists a sequence $\{t_m\}_{m\in\mathbb{N}} \in [t_2, \infty)_{\mathbb{T}}$ with $t_m \to \infty$ as $m \to \infty$ such that

$$x(t_m) = \max\{x(s) : t_0 \leq s \leq t_m\} \quad \text{and} \quad \lim_{m\to\infty} x(t_m) = \infty.$$

It follows from $\tau(t) \leq t$ that

$$z(t_m) = x(t_m) + r(t_m)x(\tau(t_m)) \geq (1+r_0)x(t_m),$$

which implies that $\lim_{t\to\infty} z(t_m) = \infty$, this contradicts the fact that $\lim_{t\to\infty} z(t) = 0$. Hence, we know x is bounded and we can assume that

$$\limsup_{t\to\infty} x(t) = x_1, \qquad \liminf_{t\to\infty} x(t) = x_2.$$

By $-1 < r_0 \leq r(t) < 0$, we get

$$x_1 + r_0 x_1 \leq \lim_{t\to\infty} z(t) \leq x_2 + r_0 x_2,$$
$$x_1 + r_0 x_1 \leq 0 \leq x_2 + r_0 x_2,$$

which implies that $x_1 \leq x_2$, so $x_1 = x_2$, hence, $\lim_{t\to\infty} x(t) = 0$. The proof is complete. □

3 Oscillation Criteria by Comparison Theorem

Theorem 3.1 *Assume that there is a constant* $\lambda > 0$ *such that* (2.13) *holds, and* $0 \leq r(t) < 1$. *Then every solution* x *of* (1.1) *oscillates or* $\lim_{t\to\infty} x(t) = 0$ *if the inequality*

$$(a(t)y^{\Delta}(t))^{\Delta} + A(t)y^{\gamma}(\delta(t)) \leq 0, \tag{3.1}$$

with

$$A(t) = p(t)(1 - r(\delta(t)))^{\gamma} \left(\frac{\int_{t_1}^{\delta(t)} \int_{t_1}^{s} \frac{1}{a(u)} \Delta u \Delta s}{\int_{t_1}^{\delta(t)} \frac{1}{a(u)} \Delta u} \right)^{\gamma} \tag{3.2}$$

has no eventually positive solution.

Proof Suppose that (1.1) has a nonoscillatory solution x. We may assume without loss of generality that $x(t) > 0$, $x(\tau(t)) > 0$, and $x(\delta(t)) > 0$ for all $t \in [t_1, \infty)_{\mathbb{T}}$, $t_1 \in [t_0, \infty)_{\mathbb{T}}$. Then by Lemma 2.1, z satisfies two cases. Assume that z satisfies case (i). By the definition of z, we have that

$$x(t) = z(t) - r(t)x(\tau(t)) \geq z(t) - r(t)z(\tau(t)) \geq (1 - r(t))z(t), \quad t \in [t_1, \infty)_{\mathbb{T}}. \tag{3.3}$$

From (1.1), there exists a $t_2 \geq t_1$ such that

$$(a(t)z^{\Delta\Delta}(t))^{\Delta} + p(t)(1 - r(\delta(t)))^{\gamma}(z(\delta(t)))^{\gamma} \leq 0, \quad t \in [t_2, \infty)_{\mathbb{T}}. \tag{3.4}$$

By Lemma 2.3, there exists a $t_3 \geq t_2$ such that

$$z(\delta(t)) \geq \left(\frac{\int_{t_1}^{\delta(t)} \int_{t_1}^{s} \frac{1}{a(u)} \Delta u \Delta s}{\int_{t_1}^{\delta(t)} \frac{1}{a(u)} \Delta u} \right) z^{\Delta}(\delta(t)). \tag{3.5}$$

Substituting this into (3.4) we obtain for $t \in [t_3, \infty)_{\mathbb{T}}$ that

$$(a(t)z^{\Delta\Delta}(t))^{\Delta} + p(t)(1 - r(\delta(t)))^{\gamma} \left(\frac{\int_{t_1}^{\delta(t)} \int_{t_1}^{s} \frac{1}{a(u)} \Delta u \Delta s}{\int_{t_1}^{\delta(t)} \frac{1}{a(u)} \Delta u} \right)^{\gamma} (z^{\Delta}(\delta(t)))^{\gamma} \leq 0. \tag{3.6}$$

Set $y(t) = z^{\Delta}(t)$. Then from (3.6), y is positive and satisfies the inequality (3.1), and this contradicts the assumption of our theorem. If (*ii*) holds, by Lemma 2.5, $x(t)$ only satisfies $\lim_{t\to\infty} x(t) = 0$. The proof is complete. □

Theorem 3.2 *Assume that* (2.20) *holds, and* $-1 < r_0 \le r(t) < 0$. *Then every solution* x *of* (1.1) *oscillates or* $\lim_{t\to\infty} x(t) = 0$ *if the inequality*

$$(a(t)y^{\Delta}(t))^{\Delta} + B(t)y^{\gamma}(\delta(t)) \le 0, \tag{3.7}$$

with

$$B(t) = p(t)\left(\frac{\int_{t_1}^{\delta(t)}\int_{t_1}^{s}\frac{1}{a(u)}\Delta u\Delta s}{\int_{t_1}^{\delta(t)}\frac{1}{a(u)}\Delta u}\right)^{\gamma} \tag{3.8}$$

has no eventually positive solution.

Proof Suppose to the contrary that (1.1) has a nonoscillatory solution x. We may assume without loss of generality that there exists $t_1 \ge t_0$ such that $x(t) > 0, x(\tau(t)) > 0$, and $x(\delta(t)) > 0$ for all $t \in [t_1, \infty)_{\mathbb{T}}$.

From Lemma 2.2, if (i) holds, by the definition of z, we have that

$$x(t) = z(t) - r(t)x(\tau(t)) \ge z(t) - r(t)z(\tau(t)) \ge z(t), \quad t \in [t_1, \infty)_{\mathbb{T}}. \tag{3.9}$$

From (1.1), there exists a $t_2 \ge t_1$ such that

$$(a(t)z^{\Delta\Delta}(t))^{\Delta} + p(t)(z(\delta(t)))^{\gamma} \le 0, \quad t \in [t_2, \infty)_{\mathbb{T}}. \tag{3.10}$$

By Lemma 2.4, there exists a $t_3 \ge t_2$ such that

$$z(\delta(t)) \ge \left(\frac{\int_{t_1}^{\delta(t)}\int_{t_1}^{s}\frac{1}{a(u)}\Delta u\Delta s}{\int_{t_1}^{\delta(t)}\frac{1}{a(u)}\Delta u}\right) z^{\Delta}(\delta(t)). \tag{3.11}$$

Substituting the last inequality in (3.10), we obtain for $t \in [t_3, \infty)_{\mathbb{T}}$ that

$$(a(t)z^{\Delta\Delta}(t))^{\Delta} + p(t)\left(\frac{\int_{t_1}^{\delta(t)}\int_{t_1}^{s}\frac{1}{a(u)}\Delta u\Delta s}{\int_{t_1}^{\delta(t)}\frac{1}{a(u)}\Delta u}\right)^{\gamma} (z^{\Delta}(\delta(t)))^{\gamma} \le 0. \tag{3.12}$$

Set $y(t) = z^{\Delta}(t)$. Then from (3.12), y is positive and satisfies the inequality (3.7), and this contradicts the assumption of our theorem. If (*ii*) holds, by Lemma 2.6, then $\lim_{t\to\infty} x(t) = 0$. The proof is complete. □

4 Oscillation Criteria for the Linear and Superlinear Dynamic Equations with $\gamma \ge 1$

In this section, we establish some Hille and Nehari-type oscillation criteria for (1.1) with $\gamma \ge 1$.

Theorem 4.1 *Assume that* $0 \le r(t) < 1$, $\gamma \ge 1$ *and there is a constant* $\lambda > 0$ *such that* (2.13) *holds. If*

$$p_* := \liminf_{t\to\infty} \int_{t_1}^{t} \frac{\Delta s}{a(s)} \int_t^{\infty} \left(\frac{\int_{t_1}^{\delta(s)} \int_{t_1}^{v} \frac{1}{a(u)} \Delta u \Delta v}{\int_{t_1}^{\sigma(s)} \frac{1}{a(u)} \Delta u} \right) (1 - r(\delta(s)))^{\gamma} p(s) \Delta s > \frac{1}{4}, \tag{4.1}$$

then every solution x *of* (1.1) *oscillates or satisfies* $\lim_{t\to\infty} x(t) = 0$.

Proof Suppose that (1.1) has a nonoscillatory solution x. We may assume without loss of generality that $x(t) > 0$, $x(\tau(t)) > 0$, and $x(\delta(t)) > 0$ for all $t \in [t_0, \infty)_{\mathbb{T}}$. Then by Lemma 2.1, z satisfies two cases. If z satisfies case (i), by Theorem 3.1, we get (3.4) holds. Since $z^{\Delta\Delta}(t) > 0$ and $z^{\Delta}(t) > 0$ imply that $\lim_{t\to\infty} z(t) = \infty$. Thus, there exists $t_1 \ge t_0$ such that $z(\delta(t)) \ge 1$ on $[t_1, \infty)_{\mathbb{T}}$, and from $\gamma \ge 1$, we know that

$$z^{\gamma}(\delta(t)) \ge z(\delta(t))$$

for $t \in [t_1, \infty)_{\mathbb{T}}$. So (3.4) leads to

$$(a(t) z^{\Delta\Delta}(t))^{\Delta} + p(t)(1 - r(\delta(t)))^{\gamma} z(\delta(t)) \le 0, \quad t \in [t_1, \infty)_{\mathbb{T}}. \tag{4.2}$$

First we define the Reccati function

$$w(t) = \frac{a(t) z^{\Delta\Delta}(t)}{z^{\Delta}(t)}, \quad t \in [t_1, \infty)_{\mathbb{T}}. \tag{4.3}$$

It is easy to see that $w(t) > 0$. Taking the derivatives of both sides and using (4.2), we have

$$\begin{aligned} w^{\Delta}(t) &= \frac{(a(t) z^{\Delta\Delta}(t))^{\Delta} z^{\Delta}(t) - a(t)(z^{\Delta\Delta}(t))^2}{z^{\Delta}(t) z^{\Delta}(\sigma(t))} \\ &= \frac{(a(t) z^{\Delta\Delta}(t))^{\Delta}}{z^{\Delta}(\sigma(t))} - \frac{a(t)(z^{\Delta\Delta}(t))^2}{z^{\Delta}(t) z^{\Delta}(\sigma(t))} \\ &\le -p(t)(1 - r(\delta(t)))^{\gamma} \frac{z(\delta(t))}{z^{\Delta}(\sigma(t))} - w(t) \frac{z^{\Delta\Delta}(t)}{z^{\Delta}(\sigma(t))}. \end{aligned} \tag{4.4}$$

By Lemma 2.3, we have

$$\begin{aligned} \frac{z(\delta(t))}{z^{\Delta}(\sigma(t))} = \frac{z(\delta(t))}{z^{\Delta}(\delta(t))} \frac{z^{\Delta}(\delta(t))}{z^{\Delta}(\sigma(t))} &\ge \left(\frac{\int_{t_1}^{\delta(t)} \int_{t_1}^{s} \frac{1}{a(u)} \Delta u \Delta s}{\int_{t_1}^{\delta(t)} \frac{1}{a(u)} \Delta u} \right) \frac{\int_{t_1}^{\delta(t)} \frac{\Delta s}{a(s)}}{\int_{t_1}^{\sigma(t)} \frac{\Delta s}{a(s)}} \\ &= \frac{\int_{t_1}^{\delta(t)} \int_{t_1}^{s} \frac{1}{a(u)} \Delta u \Delta s}{\int_{t_1}^{\sigma(t)} \frac{1}{a(u)} \Delta u}. \end{aligned} \tag{4.5}$$

Since $a(t)z^{\Delta\Delta}(t)$ is decreasing, we have that

$$w(t)\frac{z^{\Delta\Delta}(t)}{z^{\Delta}(\sigma(t))} = w(t)\frac{a(t)z^{\Delta\Delta}(t)}{a(t)z^{\Delta}(\sigma(t))} \geq w(t)\frac{a^{\sigma}(t)z^{\Delta\Delta}(\sigma(t))}{a(t)z^{\Delta}(\sigma(t))} = \frac{w(t)w^{\sigma}(t)}{a(t)}. \quad (4.6)$$

Substituting (4.5) and (4.6) into (4.4), after rearranging we obtain

$$w^{\Delta}(t) + \frac{\int_{t_1}^{\delta(t)}\int_{t_1}^{s}\frac{1}{a(u)}\Delta u\Delta s}{\int_{t_1}^{\sigma(t)}\frac{1}{a(u)}\Delta u}p(t)(1-r(\delta(t)))^{\gamma} + \frac{w(t)w^{\sigma}(t)}{a(t)} \leq 0. \quad (4.7)$$

Next we prove $\lim_{t\to\infty} w(t) = 0$ and $w(t)\int_{t_1}^{t}\Delta s/a(s) \leq 1$ on $[t_1,\infty)_{\mathbb{T}}$. By (4.7) we easily get

$$w^{\Delta}(t) \leq -\frac{w(t)w^{\sigma}(t)}{a(t)}$$

on $[t_1,\infty)_{\mathbb{T}}$, that is

$$\frac{w^{\Delta}(t)}{w(t)w^{\sigma}(t)} = \left(-\frac{1}{w(t)}\right)^{\Delta} \leq -\frac{1}{a(t)}.$$

Integrating both sides from t_1 to t we have

$$\int_{t_1}^{t}\left(-\frac{1}{w(s)}\right)^{\Delta}\Delta s \leq -\int_{t_1}^{t}\frac{\Delta s}{a(s)}.$$

That is

$$-\frac{1}{w(t)} + \frac{1}{w(t_1)} \leq -\int_{t_1}^{t}\frac{\Delta s}{a(s)},$$

Since $w(t) > 0$, we have that

$$\frac{1}{w(t)} \geq \int_{t_1}^{t}\frac{\Delta s}{a(s)}.$$

By the condition $\int_{t_1}^{\infty}\Delta s/a(s) = \infty$, it is easy to see $\lim_{t\to\infty} w(t) = 0$ and $w(t)\int_{t_1}^{t}\Delta s/a(s) \leq 1$ on $[t_1,\infty)_{\mathbb{T}}$.

Due to above result, we can define

$$r_* := \liminf_{t\to\infty} w(t)\int_{t_1}^{t}\frac{\Delta s}{a(s)} \quad (4.8)$$

and note that $0 \leq r_* \leq 1$. Now we claim that

$$r_* \geq p_* + r_*^2,$$

where p_* is defined as in (4.1). Integrating (4.7) from t to ∞, and by the result $\lim_{t\to\infty} w(t) = 0$, we have that

$$w(t) \geq \int_t^\infty \left(\frac{\int_{t_1}^{\delta(s)} \int_{t_1}^{v} \frac{1}{a(u)} \Delta u \Delta v}{\int_{t_1}^{\sigma(s)} \frac{1}{a(u)} \Delta u} \right) (1 - r(\delta(s)))^\gamma p(s) \Delta s + \int_t^\infty \frac{w^\sigma(s) w(s)}{a(s)} \Delta s. \tag{4.9}$$

Multiplying (4.9) by $\int_{t_1}^t \Delta s / a(s)$, we obtain

$$\begin{aligned}
w(t) \int_{t_1}^t \frac{\Delta s}{a(s)} &\\
\geq \int_{t_1}^t \frac{\Delta s}{a(s)} \int_t^\infty &\left(\frac{\int_{t_1}^{\delta(s)} \int_{t_1}^{v} \frac{1}{a(u)} \Delta u \Delta v}{\int_{t_1}^{\sigma(s)} \frac{1}{a(u)} \Delta u} \right) (1 - r(\delta(s)))^\gamma p(s) \Delta s \\
&+ \int_{t_1}^t \frac{\Delta s}{a(s)} \int_t^\infty \frac{w^\sigma(s) w(s)}{a(s)} \Delta s \\
= \int_{t_1}^t \frac{\Delta s}{a(s)} \int_t^\infty &\left(\frac{\int_{t_1}^{\delta(s)} \int_{t_1}^{v} \frac{1}{a(u)} \Delta u \Delta v}{\int_{t_1}^{\sigma(s)} \frac{1}{a(u)} \Delta u} \right) (1 - r(\delta(s)))^\gamma p(s) \Delta s \\
&+ \int_{t_1}^t \frac{\Delta s}{a(s)} \int_t^\infty \frac{w(s) \int_{t_1}^s \frac{\Delta u}{a(u)} w^\sigma(s) \int_{t_1}^{\sigma(s)} \frac{\Delta u}{a(u)}}{a(s) \int_{t_1}^s \frac{\Delta u}{a(u)} \int_{t_1}^{\sigma(s)} \frac{\Delta u}{a(u)}} \Delta s.
\end{aligned} \tag{4.10}$$

Now for any $\varepsilon > 0$, from the definition of r_*, there exists $t_2 \in [t_1, \infty)_{\mathbb{T}}$ such that for all $t \in [t_2, \infty)_{\mathbb{T}}$

$$w(t) \int_{t_1}^t \frac{\Delta s}{a(s)} \geq r_* - \varepsilon.$$

Taking this into (4.10) we get

$$\begin{aligned}
w(t) \int_{t_1}^t \frac{\Delta s}{a(s)} &\\
\geq \int_{t_1}^t \frac{\Delta s}{a(s)} \int_t^\infty &\left(\frac{\int_{t_1}^{\delta(s)} \int_{t_1}^{v} \frac{1}{a(u)} \Delta u \Delta v}{\int_{t_1}^{\sigma(s)} \frac{1}{a(u)} \Delta u} \right) (1 - r(\delta(s)))^\gamma p(s) \Delta s \\
&+ (r_* - \varepsilon)^2 \int_{t_1}^t \frac{\Delta s}{a(s)} \int_t^\infty \frac{1}{a(s) \int_{t_1}^s \frac{\Delta u}{a(u)} \int_{t_1}^{\sigma(s)} \frac{\Delta u}{a(u)}} \Delta s \\
= \int_{t_1}^t \frac{\Delta s}{a(s)} \int_t^\infty &\left(\frac{\int_{t_1}^{\delta(s)} \int_{t_1}^{v} \frac{1}{a(u)} \Delta u \Delta v}{\int_{t_1}^{\sigma(s)} \frac{1}{a(u)} \Delta u} \right) (1 - r(\delta(s)))^\gamma p(s) \Delta s
\end{aligned}$$

$$
\begin{aligned}
&+ (r_* - \varepsilon)^2 \int_{t_1}^{t} \frac{\Delta s}{a(s)} \int_t^\infty -\left(\frac{1}{\int_{t_1}^{s} \frac{\Delta u}{a(u)}} \right)^{\Delta} \Delta s \\
&= \int_{t_1}^{t} \frac{\Delta s}{a(s)} \int_t^\infty \left(\frac{\int_{t_1}^{\delta(s)} \int_{t_1}^{v} \frac{1}{a(u)} \Delta u \Delta v}{\int_{t_1}^{\sigma(s)} \frac{1}{a(u)} \Delta u} \right) (1 - r(\delta(s)))^{\gamma} p(s) \Delta s + (r_* - \varepsilon)^2,
\end{aligned} \tag{4.11}
$$

for $t \in [t_2, \infty)_{\mathbb{T}}$. Therefore, taking the inferior limits of both sides of (4.11) gives

$$
r_* \geq p_* + (r_* - \varepsilon)^2.
$$

Since $\varepsilon > 0$ is arbitrary, we have

$$
r_* \geq p_* + r_*^2.
$$

It means

$$
p_* \leq r_* - r_*^2 = \frac{1}{4} - \left(r_* - \frac{1}{2} \right)^2 \leq \frac{1}{4},
$$

which contradicts (4.1).

If (ii) holds, by Lemma 2.5, we have $\lim_{t \to \infty} x(t) = 0$. This completes the proof. □

Theorem 4.2 *Assume that* $\gamma \geq 1$, $-1 < r_0 \leq r(t) < 0$ *and* (2.20) *holds. If*

$$
p'_* := \liminf_{t \to \infty} \int_{t_1}^{t} \frac{\Delta s}{a(s)} \int_t^\infty \left(\frac{\int_{t_1}^{\delta(s)} \int_{t_1}^{v} \frac{1}{a(u)} \Delta u \Delta v}{\int_{t_1}^{\sigma(s)} \frac{1}{a(u)} \Delta u} \right) p(s) \Delta s > \frac{1}{4}, \tag{4.12}
$$

then every solution x *of* (1.1) *oscillates or* $\lim_{t \to \infty} x(t) = 0$.

Proof Using Lemmas 2.2, 2.4 and 2.6, the proof is similar to the proof of Theorem 4.1, so we omit the details. □

Remark 4.1 If $r(t) \equiv 0$ and $\gamma = 1$, these results become Theorem 2.8 in [30]. If $r(t) \equiv 0$, $a(t) \equiv 1$, $\delta(t) \equiv t$ and $\gamma = 1$, these results are Theorem 2 in [29]. So our researches extend Erbe [29] and Agarwal [30]'s work.

Theorem 4.3 *Assume that* $\gamma \geq 1$, $0 \leq r(t) < 1$ *and there is a constant* $\lambda > 0$ *such that* (2.13) *holds. Define* $w(t)$ *as in the proof of Theorem 4.1, and*

$$
R_* := \limsup_{t \to \infty} w(t) \int_{t_1}^{t} \frac{\Delta s}{a(s)}, \quad l^* := \limsup_{t \to \infty} \frac{\int_{t_1}^{\sigma(t)} \frac{\Delta s}{a(s)}}{\int_{t_1}^{t} \frac{\Delta s}{a(s)}}.
$$

If

$$q_* := \liminf_{t\to\infty} \frac{\int_{t_2}^{t} \int_{t_1}^{\sigma(s)} \frac{\Delta u}{a(u)} \left(\int_{t_1}^{\delta(s)} \int_{t_1}^{v} \frac{1}{a(u)} \Delta u \Delta v\right) (1 - r(\delta(s)))^{\gamma} p(s)\Delta s}{\int_{t_1}^{t} \frac{\Delta s}{a(s)}} > \frac{l^*}{1 + l^*}, \tag{4.13}$$

then every solution x *of* (1.1) *oscillates or satisfies* $\lim_{t\to\infty} x(t) = 0$.

Proof On the contrary, suppose that (1.1) has a nonoscillatory solution x. We may assume without loss of generality that $x(t) > 0$, $x(\tau(t)) > 0$, and $x(\delta(t)) > 0$ for all $t \in [t_0, \infty)_{\mathbb{T}}$. Then Lemma 2.1 holds. If z satisfies case (i), by proceeding as in the proof of Theorem 4.1, we get (4.4). Since

$$\begin{aligned} w(t)\frac{z^{\Delta\Delta}(t)}{z^{\Delta}(\sigma(t))} &= \frac{w^2(t)}{a(t)} \frac{z^{\Delta}(t)}{z^{\Delta}(\sigma(t))} = \frac{w^2(t)}{a(t)} \frac{z^{\Delta}(t)}{z^{\Delta}(t) + \mu(t)z^{\Delta\Delta}(t)} \\ &= \frac{w^2(t)}{a(t)} \frac{1}{1 + \mu(t)\frac{w(t)}{a(t)}} = \frac{w^2(t)}{a(t) + \mu(t)w(t)}, \end{aligned} \tag{4.14}$$

and

$$w(t)\frac{z^{\Delta\Delta}(t)}{z^{\Delta}(\sigma(t))} = \frac{w^2(t)}{a(t)} \frac{z^{\Delta}(t)}{z^{\Delta}(\sigma(t))} \geq \frac{w^2(t)}{a(t)} \frac{\int_{t_1}^{t} \frac{\Delta s}{a(s)}}{\int_{t_1}^{\sigma(t)} \frac{\Delta s}{a(s)}}. \tag{4.15}$$

Substituting (4.14) and (4.15) into (4.4), we get another two Ricatti inequalities:

$$w^{\Delta}(t) + \frac{\int_{t_1}^{\delta(t)} \int_{t_1}^{s} \frac{1}{a(u)} \Delta u \Delta s}{\int_{t_1}^{\sigma(t)} \frac{1}{a(u)} \Delta u} (1 - r(\delta(t)))^{\gamma} p(t) + \frac{w^2(t)}{a(t) + \mu(t)w(t)} \leq 0, \tag{4.16}$$

$$w^{\Delta}(t) + \frac{\int_{t_1}^{\delta(t)} \int_{t_1}^{s} \frac{1}{a(u)} \Delta u \Delta s}{\int_{t_1}^{\sigma(t)} \frac{1}{a(u)} \Delta u} (1 - r(\delta(t)))^{\gamma} p(t) + \frac{w^2(t)}{a(t)} \frac{\int_{t_1}^{t} \frac{\Delta s}{a(s)}}{\int_{t_1}^{\sigma(t)} \frac{\Delta s}{a(s)}} \leq 0. \tag{4.17}$$

Multiplying (4.16) by $\left(\int_{t_1}^{\sigma(t)} \Delta s/a(s)\right)^2$ and integrating from $t_2 \in [t_1, \infty)_{\mathbb{T}}$ to t, we get

$$\begin{aligned} &\int_{t_2}^{t} \left(\int_{t_1}^{\sigma(s)} \frac{\Delta u}{a(u)}\right)^2 w^{\Delta}(s)\Delta s \\ &\quad + \int_{t_2}^{t} \int_{t_1}^{\sigma(s)} \frac{\Delta u}{a(u)} \left(\int_{t_1}^{\delta(s)} \int_{t_1}^{v} \frac{1}{a(u)} \Delta u \Delta v\right) (1 - r(\delta(s)))^{\gamma} p(s)\Delta s \\ &\quad + \int_{t_2}^{t} \left(\int_{t_1}^{\sigma(s)} \frac{\Delta u}{a(u)}\right)^2 \frac{w^2(s)}{a(s) + \mu(s)w(s)} \Delta s \leq 0. \end{aligned} \tag{4.18}$$

The first term of the above inequality can be expanded as

$$
\begin{aligned}
&\int_{t_2}^{t}\left(\int_{t_1}^{\sigma(s)}\frac{\Delta u}{a(u)}\right)^2 w^{\Delta}(s)\Delta s\\
&=\left(\int_{t_1}^{t}\frac{\Delta s}{a(s)}\right)^2 w(t)-\left(\int_{t_1}^{t_2}\frac{\Delta s}{a(s)}\right)^2 w(t_2)\\
&\quad-\int_{t_2}^{t}\left(\left(\int_{t_1}^{s}\frac{\Delta u}{a(u)}\right)^2\right)^{\Delta} w(s)\Delta s\\
&=\left(\int_{t_1}^{t}\frac{\Delta s}{a(s)}\right)^2 w(t)-\left(\int_{t_1}^{t_2}\frac{\Delta s}{a(s)}\right)^2 w(t_2)\\
&\quad-\int_{t_2}^{t}\frac{1}{a(s)}\left[\int_{t_1}^{s}\frac{\Delta u}{a(u)}+\int_{t_1}^{\sigma(s)}\frac{\Delta u}{a(u)}\right]w(s)\Delta s\\
&=\left(\int_{t_1}^{t}\frac{\Delta s}{a(s)}\right)^2 w(t)-\left(\int_{t_1}^{t_2}\frac{\Delta s}{a(s)}\right)^2 w(t_2)\\
&\quad-\int_{t_2}^{t}\frac{1}{a(s)}\left[2\int_{t_1}^{\sigma(s)}\frac{\Delta u}{a(u)}-\frac{\mu(s)}{a(s)}\right]w(s)\Delta s. \qquad (4.19)
\end{aligned}
$$

Substituting this into (4.18), after rearranging we get

$$
\begin{aligned}
&\left(\int_{t_1}^{t}\frac{\Delta s}{a(s)}\right)^2 w(t)\le\left(\int_{t_1}^{t_2}\frac{\Delta s}{a(s)}\right)^2 w(t_2)\\
&-\int_{t_2}^{t}\int_{t_1}^{\sigma(s)}\frac{\Delta u}{a(u)}\left(\int_{t_1}^{\delta(s)}\int_{t_1}^{v}\frac{1}{a(u)}\Delta u\Delta v\right)(1-r(\delta(s)))^{\gamma}p(s)\Delta s\\
&+\int_{t_2}^{t}H(s,w(s))\Delta s, \qquad (4.20)
\end{aligned}
$$

where

$$
\begin{aligned}
H(s,w(s)) :=& \frac{1}{a(s)}\left[2\int_{t_1}^{\sigma(s)}\frac{\Delta u}{a(u)}-\frac{\mu(s)}{a(s)}\right]w(s)\\
&-\left(\int_{t_1}^{\sigma(s)}\frac{\Delta u}{a(u)}\right)^2\frac{w^2(s)}{a(s)+\mu(s)w(s)}.
\end{aligned}
$$

Noting that from [30, Lemma 2.7], when $w(t)>0$, we have $H(s,w(s))\le 1/a(s)$ for $t\in[t_1,\infty)_{\mathbb{T}}$, and we do not repeat the proof here. So

$$
\int_{t_2}^{t}H(s,w(s))\Delta s\le\int_{t_2}^{t}\frac{\Delta s}{a(s)}.
$$

Dividing (4.20) by $\int_{t_1}^{t}\Delta s/a(s)$, we have

$$w(t)\int_{t_1}^{t}\frac{\Delta s}{a(s)}\le\frac{\left(\int_{t_1}^{t_2}\frac{\Delta s}{a(s)}\right)^2 w(t_2)}{\int_{t_1}^{t}\frac{\Delta s}{a(s)}}+\frac{\int_{t_2}^{t}\frac{\Delta s}{a(s)}}{\int_{t_1}^{t}\frac{\Delta s}{a(s)}}$$
$$-\frac{\int_{t_2}^{t}\int_{t_1}^{\sigma(s)}\frac{\Delta u}{a(u)}\left(\int_{t_1}^{\delta(s)}\int_{t_1}^{v}\frac{1}{a(u)}\Delta u\Delta v\right)(1-r(\delta(s)))^{\gamma}p(s)\Delta s}{\int_{t_1}^{t}\frac{\Delta s}{a(s)}}. \tag{4.21}$$

Now taking the superior limits of both sides of (4.21), we get

$$R_*\le 1-q_*. \tag{4.22}$$

Next, for any $\varepsilon>0$, note that there exists $t_2\in(t_1,\infty)_{\mathbb{T}}$ such that

$$r_*-\varepsilon\le w(t)\int_{t_1}^{t}\frac{\Delta s}{a(s)}\le R_*+\varepsilon\quad\text{for}\quad t\in[t_2,\infty)_{\mathbb{T}},$$

where r_* is defined as in Theorem 4.1. And

$$\frac{\int_{t_1}^{\sigma(t)}\frac{\Delta s}{a(s)}}{\int_{t_1}^{t}\frac{\Delta s}{a(s)}}\le l^*+\varepsilon\quad\text{for}\quad t\in[t_2,\infty)_{\mathbb{T}}.$$

Making a same operation again on (4.17) which we have used on (4.16), we have

$$\left(\int_{t_1}^{t}\frac{\Delta s}{a(s)}\right)^2 w(t)\le\left(\int_{t_1}^{t_2}\frac{\Delta s}{a(s)}\right)^2 w(t_2)+\int_{t_2}^{t}\frac{\left[\int_{t_1}^{s}\frac{\Delta u}{a(u)}+\int_{t_1}^{\sigma(s)}\frac{\Delta u}{a(u)}\right]w(s)}{a(s)}\Delta s$$
$$-\int_{t_2}^{t}\int_{t_1}^{\sigma(s)}\frac{\Delta u}{a(u)}\left(\int_{t_1}^{\delta(s)}\int_{t_1}^{v}\frac{1}{a(u)}\Delta u\Delta v\right)(1-r(\delta(s)))^{\gamma}p(s)\Delta s$$
$$-\int_{t_2}^{t}\left(\int_{t_1}^{\sigma(s)}\frac{\Delta u}{a(u)}\right)^2\frac{w^2(s)}{a(s)}\frac{\int_{t_1}^{s}\frac{\Delta u}{a(u)}}{\int_{t_1}^{\sigma(s)}\frac{\Delta u}{a(u)}}\Delta s,$$

Then

$$w(t)\int_{t_1}^{t}\frac{\Delta s}{a(s)}\le\frac{\left(\int_{t_1}^{t_2}\frac{\Delta s}{a(s)}\right)^2 w(t_2)}{\int_{t_1}^{t}\frac{\Delta s}{a(s)}}+\frac{\int_{t_2}^{t}\frac{1}{a(s)}\left[\int_{t_1}^{s}\frac{\Delta u}{a(u)}+\int_{t_1}^{\sigma(s)}\frac{\Delta u}{a(u)}\right]w(s)\Delta s}{\int_{t_1}^{t}\frac{\Delta s}{a(s)}}$$
$$-\frac{\int_{t_2}^{t}\int_{t_1}^{\sigma(s)}\frac{\Delta u}{a(u)}\left(\int_{t_1}^{\delta(s)}\int_{t_1}^{v}\frac{1}{a(u)}\Delta u\Delta v\right)(1-r(\delta(s)))^{\gamma}p(s)\Delta s}{\int_{t_1}^{t}\frac{\Delta s}{a(s)}}$$
$$-\frac{\int_{t_2}^{t}\left(\int_{t_1}^{s}\frac{\Delta u}{a(u)}\right)^2\frac{w^2(s)}{a(s)}\Delta s}{\int_{t_1}^{t}\frac{\Delta s}{a(s)}}.$$

That is

$$
\begin{aligned}
w(t)\int_{t_1}^{t}\frac{\Delta s}{a(s)} \le\ & \frac{\left(\int_{t_1}^{t_2}\frac{\Delta s}{a(s)}\right)^2 w(t_2)}{\int_{t_1}^{t}\frac{\Delta s}{a(s)}} + \frac{\int_{t_2}^{t}\frac{\frac{1}{a(s)}\left[\int_{t_1}^{s}\frac{\Delta u}{a(u)}+\int_{t_1}^{\sigma(s)}\frac{\Delta u}{a(u)}\right]\left[w(s)\int_{t_1}^{s}\frac{\Delta s}{a(s)}\right]}{\int_{t_1}^{s}\frac{\Delta s}{a(s)}}\Delta s}{\int_{t_1}^{t}\frac{\Delta s}{a(s)}}\\
& -\frac{\int_{t_2}^{t}\int_{t_1}^{\sigma(s)}\frac{\Delta u}{a(u)}\left(\int_{t_1}^{\delta(s)}\int_{t_1}^{v}\frac{1}{a(u)}\Delta u\Delta v\right)(1-r(\delta(s)))^{\gamma}p(s)\Delta s}{\int_{t_1}^{t}\frac{\Delta s}{a(s)}}\\
& -\frac{\int_{t_2}^{t}\left(\int_{t_1}^{s}\frac{\Delta u}{a(u)}\right)^2\frac{w^2(s)}{a(s)}\Delta s}{\int_{t_1}^{t}\frac{\Delta s}{a(s)}}\\
\le\ & \frac{\left(\int_{t_1}^{t_2}\frac{\Delta s}{a(s)}\right)^2 w(t_2)}{\int_{t_1}^{t}\frac{\Delta s}{a(s)}} + (R_*+\varepsilon)(1+l^*+\varepsilon)\frac{\int_{t_2}^{t}\frac{\Delta s}{a(s)}}{\int_{t_1}^{t}\frac{\Delta s}{a(s)}}\\
& -q_* - (r_*-\varepsilon)^2\frac{\int_{t_2}^{t}\frac{\Delta s}{a(s)}}{\int_{t_1}^{t}\frac{\Delta s}{a(s)}}.
\end{aligned}
$$

Taking the superior limits of both sides of this inequality, since $\varepsilon > 0$ is arbitrary, we get

$$R_* \le R_*(1+l^*) - r_*^2 - q_*.$$

After rearranging, we get

$$q_* \le R_* l^* - r_*^2. \tag{4.23}$$

Now combining (4.22) and (4.23), we have that

$$
\begin{aligned}
q_* &\le l^* - l^* q_*,\\
q_* &\le \frac{l^*}{1+l^*},
\end{aligned}
$$

which contradicts condition (4.13).

If (*ii*) holds, by Lemma 2.5, we have $\lim_{t\to\infty} x(t) = 0$. This completes the proof. □

Theorem 4.4 *Assume that* $\gamma \ge 1$, $-1 < r_0 \le r(t) < 0$ *and* (2.20) *holds. If*

$$q'_* := \liminf_{t\to\infty}\frac{\int_{t_2}^{t}\int_{t_1}^{\sigma(s)}\frac{\Delta u}{a(u)}\left(\int_{t_1}^{\delta(s)}\int_{t_1}^{v}\frac{1}{a(u)}\Delta u\Delta v\right)p(s)\Delta s}{\int_{t_1}^{t}\frac{\Delta s}{a(s)}} > \frac{l^*}{1+l^*}, \tag{4.24}$$

where $l^* := \limsup_{t\to\infty}\left(\int_{t_1}^{\sigma(t)}\Delta s/a(s)\right)/\left(\int_{t_1}^{t}\Delta s/a(s)\right)$, *then every solution* x *of* (1.1) *oscillates or* $\lim_{t\to\infty} x(t) = 0$.

Proof By using Lemmas 2.2, 2.4 and 2.6, the proof is similar to that of Theorem 4.3, so we omit the details. □

Remark 4.2 When $a(t) \equiv 1, r(t) \equiv 0, \delta(t) \equiv t$ and $\gamma = 1$, these results are Theorem 3 in [29]. So these results also extend Erbe [29]'s work.

In the following parts, we give some new oscillation criteria also based on the Hille and Nehari-type oscillation criteria. The above ratiocination $z^\gamma(\delta(t)) \geq z(\delta(t))$ is not used.

Theorem 4.5 *Assume that* $\gamma \geq 1$, $0 \leq r(t) < 1$ *and there is a constant* $\lambda > 0$ *such that* (2.13) *holds. If*

$$m_* := \liminf_{t\to\infty} \int_{t_1}^{t} \frac{\Delta s}{a(s)} \int_t^\infty \left(\frac{\int_{t_1}^{\delta(s)} \int_{t_1}^{v} \frac{1}{a(u)} \Delta u \Delta v}{\int_{t_1}^{\sigma(s)} \frac{1}{a(u)} \Delta u} \right)^\gamma (1 - r(\delta(s)))^\gamma p(s) \Delta s = \infty, \tag{4.25}$$

then every solution x *of* (1.1) *oscillates or satisfies* $\lim_{t\to\infty} x(t) = 0$.

Proof Suppose that (1.1) has a nonoscillatory solution x. We may assume without loss of generality that $x(t) > 0$, $x(\tau(t)) > 0$, and $x(\delta(t)) > 0$ for all $t \in [t_0, \infty)_{\mathbb{T}}$. Then by Lemma 2.1, z satisfies two cases. Assume that z satisfies case (i). By proceeding as in the proof of Theorem 3.1, we get (3.4). Define the new Riccati-type function as

$$h(t) = \frac{a(t) z^{\Delta\Delta}(t)}{(z^\Delta(t))^\gamma}, \quad t \in [t_1, \infty)_{\mathbb{T}}. \tag{4.26}$$

Taking the derivative of $h(t)$ and using (3.4), we get that

$$\begin{aligned} h^\Delta(t) &= \frac{(a(t) z^{\Delta\Delta}(t))^\Delta (z^\Delta(t))^\gamma - a(t) z^{\Delta\Delta}(t) ((z^\Delta(t))^\gamma)^\Delta}{(z^\Delta(t))^\gamma (z^\Delta(\sigma(t)))^\gamma} \\ &\leq -p(t)(1 - r(\delta(t)))^\gamma \left(\frac{z(\delta(t))}{z^\Delta(\sigma(t))} \right)^\gamma - h(t) \frac{((z^\Delta(t))^\gamma)^\Delta}{(z^\Delta(\sigma(t)))^\gamma} \\ &\leq -p(t)(1 - r(\delta(t)))^\gamma \left(\frac{\int_{t_1}^{\delta(t)} \int_{t_1}^{s} \frac{1}{a(u)} \Delta u \Delta s}{\int_{t_1}^{\sigma(t)} \frac{1}{a(u)} \Delta u} \right)^\gamma - h(t) \frac{((z^\Delta(t))^\gamma)^\Delta}{(z^\Delta(\sigma(t)))^\gamma}. \end{aligned} \tag{4.27}$$

By Keller's chain rule [31, Theorem 1.90], we obtain

$$\begin{aligned} ((z^\Delta(t))^\gamma)^\Delta &= \gamma \int_0^1 [s z^\Delta(\sigma(t)) + (1 - s) z^\Delta(t)]^{\gamma-1} z^{\Delta\Delta}(t) \mathrm{d}s \\ &\geq \gamma (z^\Delta(t))^{\gamma-1} z^{\Delta\Delta}(t) \geq \gamma (z^\Delta(t_1))^{\gamma-1} z^{\Delta\Delta}(t). \end{aligned} \tag{4.28}$$

Setting $K = \gamma(z^{\Delta}(t_1))^{\gamma-1}$, so (4.27) becomes

$$\begin{aligned}
h^{\Delta}(t) &\le -p(t)(1-r(\delta(t)))^{\gamma}\left(\frac{\int_{t_1}^{\delta(t)}\int_{t_1}^{s}\frac{1}{a(u)}\Delta u\Delta s}{\int_{t_1}^{\sigma(t)}\frac{1}{a(u)}\Delta u}\right)^{\gamma} - h(t)\frac{Kz^{\Delta\Delta}(t)}{(z^{\Delta}(\sigma(t)))^{\gamma}}\\
&= -p(t)(1-r(\delta(t)))^{\gamma}\left(\frac{\int_{t_1}^{\delta(t)}\int_{t_1}^{s}\frac{1}{a(u)}\Delta u\Delta s}{\int_{t_1}^{\sigma(t)}\frac{1}{a(u)}\Delta u}\right)^{\gamma} - h(t)\frac{Ka(t)z^{\Delta\Delta}(t)}{a(t)(z^{\Delta}(\sigma(t)))^{\gamma}}\\
&\le -p(t)(1-r(\delta(t)))^{\gamma}\left(\frac{\int_{t_1}^{\delta(t)}\int_{t_1}^{s}\frac{1}{a(u)}\Delta u\Delta s}{\int_{t_1}^{\sigma(t)}\frac{1}{a(u)}\Delta u}\right)^{\gamma} - h(t)\frac{Ka(\sigma(t))z^{\Delta\Delta}(\sigma(t))}{a(t)(z^{\Delta}(\sigma(t)))^{\gamma}}\\
&\le -p(t)(1-r(\delta(t)))^{\gamma}\left(\frac{\int_{t_1}^{\delta(t)}\int_{t_1}^{s}\frac{1}{a(u)}\Delta u\Delta s}{\int_{t_1}^{\sigma(t)}\frac{1}{a(u)}\Delta u}\right)^{\gamma} - \frac{Kh(t)h^{\sigma}(t)}{a(t)}. \qquad (4.29)
\end{aligned}$$

Now we prove that $0 < h(t)\int_{t_1}^{t}\Delta s/a(s) \le 1/K$ and $\lim_{t\to\infty} h(t) = 0$ on $[t_1,\infty)_{\mathbb{T}}$. By (4.29) we easily get that

$$h^{\Delta}(t) \le -\frac{Kh(t)h^{\sigma}(t)}{a(t)}$$

on $[t_1,\infty)_{\mathbb{T}}$, and so

$$\frac{h^{\Delta}(t)}{h(t)h^{\sigma}(t)} = \left(-\frac{1}{h(t)}\right)^{\Delta} \le -\frac{K}{a(t)}.$$

Integrating both sides from t_1 to t we have

$$\int_{t_1}^{t}\left(-\frac{1}{h(s)}\right)^{\Delta}\Delta s \le -K\int_{t_1}^{t}\frac{\Delta s}{a(s)}.$$

That is

$$-\frac{1}{h(t)} + \frac{1}{h(t_1)} \le -K\int_{t_1}^{t}\frac{\Delta s}{a(s)}.$$

Since $h(t) > 0$, we have that

$$\frac{1}{h(t)} \ge K\int_{t_1}^{t}\frac{\Delta s}{a(s)}.$$

By the condition $\int_{t_1}^{\infty}\Delta s/a(s) = \infty$, it is easy to see $\lim_{t\to\infty} h(t) = 0$ and $h(t)\int_{t_1}^{t}\Delta s/a(s) \le 1/K$ on $[t_1,\infty)_{\mathbb{T}}$. Then we define f_* by

$$f_* := \liminf_{t\to\infty} h(t)\int_{t_1}^{t}\frac{\Delta s}{a(s)}. \tag{4.30}$$

It is clear to see $0 \le f_* \le 1/K$. For any $\varepsilon \ge 0$, from the definition of f_*, there exists $t_2 \in [t_1,\infty)_{\mathbb{T}}$ such that for all $t \in [t_2,\infty)_{\mathbb{T}}$

$$h(t)\int_{t_1}^{t}\frac{\Delta s}{a(s)} \ge f_* - \varepsilon.$$

Now integrating (4.29) from t to ∞, and multiplying by $\int_{t_1}^{t}\Delta s/a(s)$, by $\lim_{t\to\infty} h(t) = 0$, we obtain

$$\begin{aligned}
&h(t)\int_{t_1}^{t}\frac{\Delta s}{a(s)}\\
&\ge \int_{t_1}^{t}\frac{\Delta s}{a(s)}\int_{t}^{\infty}\left(\frac{\int_{t_1}^{\delta(s)}\int_{t_1}^{v}\frac{1}{a(u)}\Delta u\Delta v}{\int_{t_1}^{\sigma(s)}\frac{1}{a(u)}\Delta u}\right)^{\gamma}(1-r(\delta(s)))^{\gamma}p(s)\Delta s\\
&\quad + K\int_{t_1}^{t}\frac{\Delta s}{a(s)}\int_{t}^{\infty}\frac{h^{\sigma}(s)h(s)}{a(s)}\Delta s\\
&= \int_{t_1}^{t}\frac{\Delta s}{a(s)}\int_{t}^{\infty}\left(\frac{\int_{t_1}^{\delta(s)}\int_{t_1}^{v}\frac{1}{a(u)}\Delta u\Delta v}{\int_{t_1}^{\sigma(s)}\frac{1}{a(u)}\Delta u}\right)^{\gamma}(1-r(\delta(s)))^{\gamma}p(s)\Delta s\\
&\quad + K\int_{t_1}^{t}\frac{\Delta s}{a(s)}\int_{t}^{\infty}\frac{h(s)\int_{t_1}^{s}\frac{\Delta u}{a(u)}h^{\sigma}(s)\int_{t_1}^{\sigma(s)}\frac{\Delta u}{a(u)}}{a(s)\int_{t_1}^{s}\frac{\Delta u}{a(u)}\int_{t_1}^{\sigma(s)}\frac{\Delta u}{a(u)}}\Delta s\\
&\ge \int_{t_1}^{t}\frac{\Delta s}{a(s)}\int_{t}^{\infty}\left(\frac{\int_{t_1}^{\delta(s)}\int_{t_1}^{v}\frac{1}{a(u)}\Delta u\Delta v}{\int_{t_1}^{\sigma(s)}\frac{1}{a(u)}\Delta u}\right)^{\gamma}(1-r(\delta(s)))^{\gamma}p(s)\Delta s\\
&\quad + K(f_*-\varepsilon)^2\int_{t_1}^{t}\frac{\Delta s}{a(s)}\int_{t}^{\infty}-\left(\frac{1}{\int_{t_1}^{s}\frac{\Delta u}{a(u)}}\right)^{\Delta}\Delta s\\
&= \int_{t_1}^{t}\frac{\Delta s}{a(s)}\int_{t}^{\infty}\left(\frac{\int_{t_1}^{\delta(s)}\int_{t_1}^{v}\frac{1}{a(u)}\Delta u\Delta v}{\int_{t_1}^{\sigma(s)}\frac{1}{a(u)}\Delta u}\right)^{\gamma}(1-r(\delta(s)))^{\gamma}p(s)\Delta s + K(f_*-\varepsilon)^2
\end{aligned} \tag{4.31}$$

for $t \in [t_2,\infty)_{\mathbb{T}}$. Therefore, taking the inferior limits of both sides of (4.11) gives

$$f_* \ge m_* + K(f_*-\varepsilon)^2.$$

Since $\varepsilon > 0$ is arbitrary, we have

$$f_* \ge m_* + Kr_*^2.$$

It means

$$m_* \leq f_* - K f_*^2 = \frac{1}{4K} - (f_* - \frac{1}{2\sqrt{K}})^2 \leq \frac{1}{4K}.$$

Since $K = \gamma(z^{\Delta}(t_1))^{\gamma-1}$ is a constant, we get a contradiction to (4.25).

If (ii) holds, by Lemma 2.5, we have $\lim_{t\to\infty} x(t) = 0$. This completes the proof. □

Theorem 4.6 *Assume that* $\gamma \geq 1$, $-1 < r_0 \leq r(t) < 0$ *and* (2.20) *holds. If*

$$m'_* := \liminf_{t\to\infty} \int_{t_1}^{t} \frac{\Delta s}{a(s)} \int_{t}^{\infty} \left(\frac{\int_{t_1}^{\delta(s)} \int_{t_1}^{v} \frac{1}{a(u)} \Delta u \Delta v}{\int_{t_1}^{\sigma(s)} \frac{1}{a(u)} \Delta u} \right)^{\gamma} p(s)\Delta s = \infty, \tag{4.32}$$

then every solution x of (1.1) *oscillates or* $\lim_{t\to\infty} x(t) = 0$.

Proof By using Lemmas 2.2, 2.4 and 2.6, the proof is similar to that of Theorem 4.5, so we omit the details. □

5 Oscillation Criteria for the Sublinear Dynamic Equations with $0 < \gamma < 1$

In this section, we present some oscillation criteria for (1.1) with $0 < \gamma < 1$.

Theorem 5.1 *Assume that* $0 < \gamma < 1$, $0 \leq r(t) < 1$ *and there is a constant* $\lambda > 0$ *such that* (2.13) *holds. Further assume* $\mu(t)/a(t)$ *is bounded on* $[t_0, \infty)_{\mathbb{T}}$. *If*

$$n_* := \liminf_{t\to\infty} \left(\int_{t_1}^{t} \frac{\Delta s}{a(s)} \right)^{\gamma} \int_{t}^{\infty} \left(\frac{\int_{t_1}^{\delta(s)} \int_{t_1}^{v} \frac{1}{a(u)} \Delta u \Delta v}{\int_{t_1}^{\sigma(s)} \frac{1}{a(u)} \Delta u} \right)^{\gamma} (1 - r(\delta(s)))^{\gamma} p(s)\Delta s = \infty, \tag{5.1}$$

then every solution x of (1.1) *oscillates or satisfies* $\lim_{t\to\infty} x(t) = 0$.

Proof On the contrary, suppose that (1.1) has a nonoscillatory solution x. We may assume without loss of generality that $x(t) > 0$, $x(\tau(t)) > 0$, and $x(\delta(t)) > 0$ for all $t \in [t_0, \infty)_{\mathbb{T}}$. Then by Lemma 2.1, z satisfies two cases. Assume that z satisfies case (i). By proceeding as in the proof of Theorem 3.1, we get that (3.4) still holds. Define the function $h(t)$ as in Theorem 4.5, using (3.4), we get that

$$\begin{aligned} h^{\Delta}(t) &= \frac{(a(t)z^{\Delta\Delta}(t))^{\Delta}(z^{\Delta}(t))^{\gamma} - a(t)z^{\Delta\Delta}(t)((z^{\Delta}(t))^{\gamma})^{\Delta}}{(z^{\Delta}(t))^{\gamma}(z^{\Delta}(\sigma(t)))^{\gamma}} \\ &\leq -p(t)(1 - r(\delta(t)))^{\gamma} \left(\frac{z(\delta(t))}{z^{\Delta}(\sigma(t))} \right)^{\gamma} - h(t)\frac{((z^{\Delta}(t))^{\gamma})^{\Delta}}{(z^{\Delta}(\sigma(t)))^{\gamma}} \\ &\leq -p(t)(1 - r(\delta(t)))^{\gamma} \left(\frac{\int_{t_1}^{\delta(t)} \int_{t_1}^{s} \frac{1}{a(u)} \Delta u \Delta s}{\int_{t_1}^{\sigma(t)} \frac{1}{a(u)} \Delta u} \right)^{\gamma} - h(t)\frac{((z^{\Delta}(t))^{\gamma})^{\Delta}}{(z^{\Delta}(\sigma(t)))^{\gamma}}. \end{aligned} \tag{5.2}$$

By Keller's chain rule, we obtain

$$((z^{\Delta}(t))^{\gamma})^{\Delta} = \gamma \int_0^1 [sz^{\Delta}(\sigma(t)) + (1-s)z^{\Delta}(t)]^{\gamma-1} z^{\Delta\Delta}(t)\mathrm{d}s$$
$$\geq \gamma (z^{\Delta}(\sigma(t)))^{\gamma-1} z^{\Delta\Delta}(t). \tag{5.3}$$

By Lemma 2.3, note that $z^{\Delta}(t)/\big(\int_{t_1}^{t} \Delta s/a(s)\big)$ is nonincreasing eventually. So for $t \in [t_2, \infty)_{\mathbb{T}}$, $t_2 \in [t_1, \infty)_{\mathbb{T}}$

$$L = \frac{z^{\Delta}(t_2)}{\int_{t_1}^{t_2} \frac{\Delta s}{a(s)}} \geq \frac{z^{\Delta}(t)}{\int_{t_1}^{t} \frac{\Delta s}{a(s)}}. \tag{5.4}$$

Moreover, by the assumption, $\mu(t)/a(t)$ is bounded on $[t_0, \infty)_{\mathbb{T}}$, we can suppose that $\mu(t)/a(t) \leq B$ for all $t \in [t_0, \infty)_{\mathbb{T}}$, thus for $t \in [t_2, \infty)_{\mathbb{T}}$

$$(z^{\Delta}(\sigma(t)))^{\gamma-1} \geq \left(L \int_{t_1}^{\sigma(t)} \frac{\Delta s}{a(s)} \right)^{\gamma-1}$$
$$= L^{\gamma-1} \left(\int_{t_1}^{t} \frac{\Delta s}{a(s)} + \frac{\mu(t)}{a(t)} \right)^{\gamma-1} \geq L^{\gamma-1} \left(\int_{t_1}^{t} \frac{\Delta s}{a(s)} + B \right)^{\gamma-1}. \tag{5.5}$$

Since $\int_{t_1}^{t} \Delta s/a(s) = \infty$, we have

$$\lim_{t\to\infty} \frac{\left(\int_{t_1}^{t} \frac{\Delta s}{a(s)}\right)^{\gamma-1}}{\left(\int_{t_1}^{t} \frac{\Delta s}{a(s)} + B\right)^{\gamma-1}} = 1.$$

So there exists $t_3 \geq t_2$ such that

$$\frac{\left(\int_{t_1}^{t} \frac{\Delta s}{a(s)}\right)^{\gamma-1}}{\left(\int_{t_1}^{t} \frac{\Delta s}{a(s)} + B\right)^{\gamma-1}} \leq \frac{3}{2},$$

for $t \in [t_3, \infty)_{\mathbb{T}}$. That is

$$(z^{\Delta}(\sigma(t)))^{\gamma-1} \geq \frac{2L^{\gamma-1}}{3} \left(\int_{t_1}^{t} \frac{\Delta s}{a(s)} \right)^{\gamma-1}. \tag{5.6}$$

Substituting (5.3) and (5.6) into (5.2), we obtain for $t \in [t_3, \infty)_{\mathbb{T}}$

$$h^{\Delta}(t) \leq -p(t)(1 - r(\delta(t)))^{\gamma} \left(\frac{\int_{t_1}^{\delta(t)} \int_{t_1}^{s} \frac{1}{a(u)} \Delta u \Delta s}{\int_{t_1}^{\sigma(t)} \frac{1}{a(u)} \Delta u} \right)^{\gamma}$$
$$- L_1 \left(\int_{t_1}^{t} \frac{\Delta s}{a(s)} \right)^{\gamma-1} h(t) \frac{z^{\Delta\Delta}(t)}{(z^{\Delta}(\sigma(t)))^{\gamma}}$$

$$\leq -p(t)(1-r(\delta(t)))^{\gamma}\left(\frac{\int_{t_1}^{\delta(t)}\int_{t_1}^{s}\frac{1}{a(u)}\Delta u\Delta s}{\int_{t_1}^{\sigma(t)}\frac{1}{a(u)}\Delta u}\right)^{\gamma}$$

$$-L_1\left(\int_{t_1}^{t}\frac{\Delta s}{a(s)}\right)^{\gamma-1}\frac{h(t)h^{\sigma}(t)}{a(t)}, \tag{5.7}$$

where $L_1=(2\gamma L^{\gamma-1})/3$.

Next we claim that the function $h(t)\left(\int_{t_1}^{t}\Delta s/a(s)\right)^{\gamma}$ is bounded, especially $0<h(t)\left(\int_{t_1}^{t}\Delta s/a(s)\right)^{\gamma}\leq 2/L_1$ for $t\in[t_4,\infty)_{\mathbb{T}}\subset[t_3,\infty)_{\mathbb{T}}$, and $\lim_{t\to\infty}h(t)=0$. From (5.7), we easily get

$$h^{\Delta}(t)\leq -L_1\left(\int_{t_1}^{t}\frac{\Delta s}{a(s)}\right)^{\gamma-1}\frac{h(t)h^{\sigma}(t)}{a(t)}.$$

So

$$\frac{h^{\Delta}(t)}{h(t)h^{\sigma}(t)}=\left(-\frac{1}{h(t)}\right)^{\Delta}\leq -L_1\left(\int_{t_1}^{t}\frac{\Delta s}{a(s)}\right)^{\gamma-1}\frac{1}{a(t)}.$$

Integrating both sides from t_3 to t we have

$$\frac{1}{h(t)}\geq\int_{t_3}^{t}L_1\left(\int_{t_1}^{s}\frac{\Delta u}{a(u)}\right)^{\gamma-1}\frac{1}{a(s)}\Delta s\geq L_1\left(\int_{t_1}^{t}\frac{\Delta s}{a(s)}\right)^{\gamma-1}\int_{t_3}^{t}\frac{\Delta s}{a(s)}$$

$$=L_1\left(\int_{t_1}^{t}\frac{\Delta s}{a(s)}\right)^{\gamma-1}\left(\int_{t_1}^{t}\frac{\Delta s}{a(s)}-\int_{t_1}^{t_3}\frac{\Delta s}{a(s)}\right).$$

Since $\int_{t_1}^{\infty}\Delta s/a(s)=\infty$, we can find a $t_4\geq t_3$ such that for $t\in[t_4,\infty)_{\mathbb{T}}$

$$L_1\left(\int_{t_1}^{t}\frac{\Delta s}{a(s)}-\int_{t_1}^{t_3}\frac{\Delta s}{a(s)}\right)\geq\frac{L_1}{2}\int_{t_1}^{t}\frac{\Delta s}{a(s)}.$$

So for $t\in[t_4,\infty)_{\mathbb{T}}$, we obtain

$$\frac{1}{h(t)}\geq\frac{L_1}{2}\left(\int_{t_1}^{t}\frac{\Delta s}{a(s)}\right)^{\gamma-1}\int_{t_1}^{t}\frac{\Delta s}{a(s)}=\frac{L_1}{2}\left(\int_{t_1}^{t}\frac{\Delta s}{a(s)}\right)^{\gamma}.$$

So $0<h(t)\left(\int_{t_1}^{t}\Delta s/a(s)\right)^{\gamma}\leq 2/L_1$ for $t\in[t_4,\infty)_{\mathbb{T}}$. By the condition $\int_{t_1}^{\infty}\Delta s/a(s)=\infty$, it is easy to see $\lim_{t\to\infty}h(t)=0$.

Then we define h_* by

$$h_*:=\liminf_{t\to\infty}h(t)\left(\int_{t_1}^{t}\frac{\Delta s}{a(s)}\right)^{\gamma}. \tag{5.8}$$

It is clear to see $0 \le h_* \le \frac{2}{L_1}$. For any $\varepsilon \ge 0$, from the definition of h_*, there exists $t_5 \in [t_4, \infty)_{\mathbb{T}}$ such that for all $t \in [t_5, \infty)_{\mathbb{T}}$ s

$$h(t)\left(\int_{t_1}^{t} \frac{\Delta s}{a(s)}\right)^{\gamma} \ge h_* - \varepsilon.$$

Now integrating (5.7) from t to ∞, and multiplying by $\left(\int_{t_1}^{t} \Delta s/a(s)\right)^{\gamma}$, we obtain

$$\begin{aligned} h(t)\left(\int_{t_1}^{t} \frac{\Delta s}{a(s)}\right)^{\gamma} \ge{}& \left(\int_{t_1}^{t} \frac{\Delta s}{a(s)}\right)^{\gamma} \\ &\times \int_t^{\infty} \left(\frac{\int_{t_1}^{\delta(s)} \int_{t_1}^{v} \frac{1}{a(u)} \Delta u \Delta v}{\int_{t_1}^{\sigma(s)} \frac{1}{a(u)} \Delta u}\right)^{\gamma} (1 - r(\delta(s)))^{\gamma} p(s) \Delta s \\ &+ L_1 \left(\int_{t_1}^{t} \frac{\Delta s}{a(s)}\right)^{\gamma} \int_t^{\infty} \left(\int_{t_1}^{s} \frac{\Delta u}{a(u)}\right)^{\gamma-1} \frac{h(s) h^{\sigma}(s)}{a(s)} \Delta s. \end{aligned} \tag{5.9}$$

Note that

$$\begin{aligned} &L_1 \left(\int_{t_1}^{t} \frac{\Delta s}{a(s)}\right)^{\gamma} \int_t^{\infty} \left(\int_{t_1}^{s} \frac{\Delta u}{a(u)}\right)^{\gamma-1} \frac{h(s) h^{\sigma}(s)}{a(s)} \Delta s \\ &= L_1 \left(\int_{t_1}^{t} \frac{\Delta s}{a(s)}\right)^{\gamma} \int_t^{\infty} h(s) \left(\int_{t_1}^{s} \frac{\Delta u}{a(u)}\right)^{\gamma} h^{\sigma}(s) \left(\int_{t_1}^{\sigma(s)} \frac{\Delta u}{a(u)}\right)^{\gamma} \\ &\quad \times \frac{\left(\int_{t_1}^{s} \frac{\Delta u}{a(u)}\right)^{-1} \left(\int_{t_1}^{\sigma(s)} \frac{\Delta u}{a(u)}\right)^{-\gamma}}{a(s)} \Delta s \\ &\ge L_1 (h_* - \varepsilon)^2 \left(\int_{t_1}^{t} \frac{\Delta s}{a(s)}\right)^{\gamma} \int_t^{\infty} \frac{\left(\int_{t_1}^{s} \frac{\Delta u}{a(u)}\right)^{-1} \left(\frac{2}{3}\right)^{\frac{\gamma}{1-\gamma}} \left(\int_{t_1}^{s} \frac{\Delta u}{a(u)}\right)^{-\gamma}}{a(s)} \Delta s \\ &= L_1 \left(\frac{2}{3}\right)^{\frac{\gamma}{1-\gamma}} (h_* - \varepsilon)^2 \left(\int_{t_1}^{t} \frac{\Delta s}{a(s)}\right)^{\gamma} \int_t^{\infty} \frac{\left(\int_{t_1}^{s} \frac{\Delta u}{a(u)}\right)^{-1-\gamma}}{a(s)} \Delta s \\ &\ge L_1 \left(\frac{2}{3}\right)^{\frac{\gamma}{1-\gamma}} (h_* - \varepsilon)^2 \left(\int_{t_1}^{t} \frac{\Delta s}{a(s)}\right)^{\gamma} \\ &\quad \times \int_t^{\infty} \frac{\left[\int_0^1 v \left(\int_{t_1}^{s} \frac{\Delta u}{a(u)}\right)^{-1-\gamma} + (1 - v) \left(\int_{t_1}^{\sigma(s)} \frac{\Delta u}{a(u)}\right)^{-1-\gamma} \mathrm{d}v\right]}{a(s)} \Delta s \\ &= L_1 \left(\frac{2}{3}\right)^{\frac{\gamma}{1-\gamma}} (h_* - \varepsilon)^2 \left(\int_{t_1}^{t} \frac{\Delta s}{a(s)}\right)^{\gamma} \int_t^{\infty} -\frac{1}{\gamma} \left(\left(\int_{t_1}^{s} \frac{\Delta u}{a(u)}\right)^{-\gamma}\right)^{\Delta} \Delta s \\ &= L_1 \left(\frac{2}{3}\right)^{\frac{\gamma}{1-\gamma}} \frac{1}{\gamma} (h_* - \varepsilon)^2. \end{aligned} \tag{5.10}$$

Substituting this into (5.9), we get for $t \in [t_5, \infty)_{\mathbb{T}}$

$$h(t)\left(\int_{t_1}^{t}\frac{\Delta s}{a(s)}\right)^{\gamma} \geq \left(\int_{t_1}^{t}\frac{\Delta s}{a(s)}\right)^{\gamma}$$
$$\times \int_{t}^{\infty}\left(\frac{\int_{t_1}^{\delta(s)}\int_{t_1}^{v}\frac{1}{a(u)}\Delta u\Delta v}{\int_{t_1}^{\sigma(s)}\frac{1}{a(u)}\Delta u}\right)^{\gamma}(1-r(\delta(s)))^{\gamma}p(s)\Delta s + L_2(h_* - \varepsilon)^2, \quad (5.11)$$

where $L_2 = L_1\,(2/3)^{\frac{\gamma}{1-\gamma}}\,(1/\gamma)$. Therefore, taking the inferior limits of both sides of (5.11) gives

$$h_* \geq n_* + L_2(h_* - \varepsilon)^2.$$

Since $\varepsilon > 0$ is arbitrary, we have

$$h_* \geq n_* + L_2 h_*^2.$$

That is

$$n_* \leq h_* - L_2 h_*^2 \leq \frac{1}{4L_2}.$$

Since L_2 is a constant, we get a contradiction to (5.1).

If (*ii*) holds, by Lemma 2.5, we have $\lim_{t\to\infty} x(t) = 0$. This completes the proof. □

Remark 5.1 It is easy to see that the condition "$\mu(t)/a(t)$ is bounded on $[t_0, \infty)_{\mathbb{T}}$" in Theorem 5.1 can be removed for the continuous case (1.2). And for the discrete case (1.3), by the hypothesis $\int_{t_0}^{\infty}\Delta s/a(s) = \infty$, we have $\mu(t)/a(t)$ is still bounded on $[t_0, \infty)_{\mathbb{T}}$. So it also can be removed, and we get the following two corollaries.

Corollary 5.1 *If* $\mathbb{T} = \mathbb{R}$, $0 < \gamma < 1$ *and* $0 \leq r(t) < 1$. *Assume that there is a constant* $\lambda > 0$ *such that* (2.13) *holds. Then every solution* x *of* (1.1) *oscillates or* $\lim_{t\to\infty} x(t) = 0$ *if*

$$\liminf_{t\to\infty}\left(\int_{t_1}^{t}\frac{ds}{a(s)}\right)^{\gamma}\int_{t}^{\infty}\left(\frac{\int_{t_1}^{\delta(s)}\int_{t_1}^{v}\frac{1}{a(u)}du dv}{\int_{t_1}^{s}\frac{1}{a(u)}du}\right)^{\gamma}(1-r(\delta(s)))^{\gamma}p(s)ds = \infty. \quad (5.12)$$

Corollary 5.2 *If* $\mathbb{T} = \mathbb{Z}$, $0 < \gamma < 1$ *and* $0 \leq r(t) < 1$. *Assume that there is a constant* $\lambda > 0$ *such that* (2.13) *holds and* $\liminf_{t\to\infty} a(t) \neq 0$. *Then every solution* x *of* (1.1) *oscillates or* $\lim_{t\to\infty} x(t) = 0$ *if*

$$\liminf_{t\to\infty}\left(\sum_{t=n_0}^{\infty}\frac{1}{a(t)}\right)^{\gamma}\sum_{t}^{\infty}\left(\frac{\sum_{s=n_0}^{\delta(t)}\sum_{u=n_0}^{s}\frac{1}{a(u)}}{\sum_{s=n_0}^{t+1}\frac{1}{a(s)}}\right)^{\gamma}(1-r(\delta(t)))^{\gamma}p(t) = \infty. \quad (5.13)$$

Theorem 5.2 *Assume that* $0 < \gamma < 1$, $-1 < r_0 \le r(t) < 0$ *and* (2.20) *holds. Further assume* $\mu(t)/a(t)$ *is bounded on* $[t_0, \infty)_{\mathbb{T}}$. *If*

$$n'_* := \liminf_{t\to\infty} \left(\int_{t_1}^{t} \frac{\Delta s}{a(s)}\right)^{\gamma} \int_t^{\infty} \left(\frac{\int_{t_1}^{\delta(s)} \int_{t_1}^{v} \frac{1}{a(u)}\Delta u \Delta v}{\int_{t_1}^{\sigma(s)} \frac{1}{a(u)}\Delta u}\right)^{\gamma} p(s)\Delta s = \infty, \quad (5.14)$$

then every solution x *of* (1.1) *oscillates or satisfies* $\lim_{t\to\infty} x(t) = 0$.

Proof By using Lemmas 2.2, 2.4 and 2.6, the proof is similar to that of Theorem 5.1, so we omit the details. □

Corollary 5.3 *If* $\mathbb{T} = \mathbb{R}$, $0 < \gamma < 1$ *and* $-1 < r_0 \le r(t) < 0$. *Assume that* (2.20) *holds. Then every solution* x *of* (1.1) *oscillates or* $\lim_{t\to\infty} x(t) = 0$ *if*

$$\liminf_{t\to\infty} \left(\int_{t_1}^{t} \frac{\mathrm{d}s}{a(s)}\right)^{\gamma} \int_t^{\infty} \left(\frac{\int_{t_1}^{\delta(s)} \int_{t_1}^{v} \frac{1}{a(u)}\mathrm{d}u\mathrm{d}v}{\int_{t_1}^{s} \frac{1}{a(u)}\mathrm{d}u}\right)^{\gamma} p(s)\mathrm{d}s = \infty. \quad (5.15)$$

Corollary 5.4 *If* $\mathbb{T} = \mathbb{Z}$, $0 < \gamma < 1$ *and* $-1 < r_0 \le r(t) < 0$. *Assume that* (2.20) *holds and* $\liminf_{t\to\infty} a(t) \ne 0$. *Then every solution* x *of* (1.1) *oscillates or* $\lim_{t\to\infty} x(t) = 0$ *if*

$$\liminf_{t\to\infty} \left(\sum_{t=n_0}^{\infty} \frac{1}{a(t)}\right)^{\gamma} \sum_{t}^{\infty} \left(\frac{\sum_{s=n_0}^{\delta(t)} \sum_{u=n_0}^{s} \frac{1}{a(u)}}{\sum_{s=n_0}^{t+1} \frac{1}{a(s)}}\right)^{\gamma} p(t) = \infty. \quad (5.16)$$

6 Examples

In this section, we give the following examples to illustrate our main results.

Example 6.1 Consider the third-order neutral delay dynamic equations on time-scales

$$\left(x(t) + \frac{1}{2}x(\tau(t))\right)^{\Delta\Delta\Delta} + \frac{\beta}{t}\left(\frac{t}{h_2(\delta(t), t_0)}\right)^{\gamma} x^{\gamma}(\delta(t)) = 0, \quad t \in [t_0, \infty)_{\mathbb{T}}, \quad (6.1)$$

where $\delta(t) = 1/t$, $\beta > 0$, $1 < \gamma < 2$ is a quotient of odd positive integers, $h_2(\delta(t), t_0) < t^2$.

Let $a(t) = 1$, $r(t) = 1/2$, $p(t) = (\beta/t)(t/h_2(\delta(t), t_0))^{\gamma}$. First choosing a constant λ with $0 < \lambda < 2t_0 - 1$, we get that

$$\int_{t_0}^{\infty} p(s)\Big(1-(1+\lambda)r(\delta(s))\Big)^{\gamma} R^{\sigma}(s)\Delta s > \int_{t_0}^{\infty} p(s)\left(1-\frac{(1+\lambda)}{2s}\right)R^{\sigma}(s)\Delta s = \infty.$$

So that (2.13) holds. Also

$$p_* := \liminf_{t\to\infty} \int_{t_1}^{t} \frac{\Delta s}{a(s)} \int_t^\infty \left(\frac{\int_{t_1}^{\delta(s)} \int_{t_1}^{v} \frac{1}{a(u)} \Delta u \Delta v}{\int_{t_1}^{\sigma(s)} \frac{1}{a(u)} \Delta u} \right) (1 - r(\delta(s)))^\gamma p(s) \Delta s$$
$$> \liminf_{t\to\infty} (t - t_1) \int_t^\infty \left(\frac{h_2(\delta(t), t_0)}{\sigma(s) - t_1} \right) (1 - \frac{1}{2s})^\gamma p(s) \Delta s$$
$$= \infty > \frac{1}{4}.$$

Hence by Theorem 4.1, every solution x of (6.1) is either oscillatory or $\lim_{t\to\infty} x(t) = 0$.

Example 6.2 Consider the third-order neutral delay differential equation:

$$\left(x(t) - \frac{1}{5} x(t-3) \right)''' + \left(1 - \frac{e^3}{5} \right) e^{-\frac{t}{2}} x^{\frac{1}{2}}(t) = 0, \quad t \in [t_0, \infty). \tag{6.2}$$

Let $\gamma = 1/2$, $a(t) = 1$, $r(t) = -1/5$, $p(t) = (1 - e^2/10)e^{2t}$. It is easy to see that condition (2.20) holds, and of Corollary 5.3 hold. Then by Corollary 5.3, every solution x of (6.2) is either oscillatory or satisfies $\lim_{t\to\infty} x(t) = 0$. In fact, e^{-t} is a solution of (6.2).

Example 6.3 Consider the third-order delay q-difference equation:

$$\left(x(t) - \frac{1}{2} x(\tau(t)) \right)^{\Delta\Delta\Delta} + \frac{\beta t^{\gamma-1}}{\delta^2(t)} x^\gamma(\delta(t)) = 0, \quad t \in [1, \infty)_{\mathbb{T}}, \tag{6.3}$$

where $\mathbb{T} = q^{\mathbb{N}_0}$, $\beta > 0$, $\gamma > 1$ is a quotient of odd positive integers.

For $\mathbb{T} = q^{\mathbb{N}_0}$, we have $h_2(\delta(t), t_0) = h_2(\delta(t), 1) = (\delta(t) - 1)(\delta(t) - q)/(1 + q)$, $\sigma(t) = qt$. Let $r(t) = -1/2$, $p(t) = \beta t^{\gamma-1}/\delta^2(t)$. It is easy to see that (2.20) holds, and

$$m'_* := \liminf_{t\to\infty} \int_{t_1}^{t} \frac{\Delta s}{a(s)} \int_t^\infty \left(\frac{\int_{t_1}^{\delta(s)} \int_{t_1}^{v} \frac{1}{a(u)} \Delta u \Delta v}{\int_{t_1}^{\sigma(s)} \frac{1}{a(u)} \Delta u} \right)^\gamma p(s) \Delta s$$
$$> \liminf_{t\to\infty} (t - t_0) \int_t^\infty \left(\frac{h_2(\delta(s), t_1)}{\sigma(s)} \right)^\gamma p(s) \Delta s$$
$$= \liminf_{t\to\infty} \beta q^{-\gamma} (t - t_0) \int_t^\infty \frac{1}{s\delta^2(s)} \left(\frac{(\delta(s) - 1)(\delta(s) - q)}{1 + q} \right)^\gamma \Delta s$$
$$= \infty,$$

so (4.32) holds. By Theorem 4.6, every solution x of (6.3) is either oscillatory or satisfies $\lim_{t\to\infty} x(t) = 0$.

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