Bull. Malays. Math. Sci. Soc. (2017) 40:1175–1186
DOI 10.1007/s40840-016-0344-0

BULLETIN *of the* MALAYSIAN MATHEMATICAL SCIENCES SOCIETY
http://www.springer.com/mathematics/journal/40840

Maximal Regular Subsemibands of the Finite Order-Preserving Partial Transformation Semigroups $\mathcal{PO}(n, r)$

Ping Zhao[1] · Huabi Hu[2] · Taijie You[1]

Received: 12 February 2012 / Revised: 1 August 2015 / Published online: 30 March 2016

Abstract Let $\mathcal{PO}_n$ be the semigroup of all order-preserving partial transformations on the finite set $X_n = \{1, 2, \ldots, n\}$. For $1 \le r \le n-1$, set $\mathcal{PO}(n, r) = \{\alpha \in \mathcal{PO}_n : |\operatorname{im}(\alpha)| \le r\}$. In this paper, we investigate the maximal regular subsemigroups and the maximal regular subsemibands of the semigroup $\mathcal{PO}(n, r)$. First, we completely describe the maximal regular subsemigroups of the semigroup $\mathcal{PO}(n, r)$, for $1 \le r \le n-1$. Secondly, we show that, for $2 \le r \le n-2$, any maximal regular subsemigroup of the semigroup $\mathcal{PO}(n, r)$ is a semiband and obtain that the maximal regular subsemigroups and the maximal regular subsemibands of the semigroup $\mathcal{PO}(n, r)$ coincide, for $2 \le r \le n-2$. Finally, we obtain the complete classification of maximal regular subsemibands of the semigroup $\mathcal{PO}_n$.

Keywords Transformation semigroup · Maximal regular subsemigroup · Maximal regular subsemiband

Mathematics Subject Classification 20M20

Communicated by Kar Ping Shum.

✉ Ping Zhao
zhaoping731108@hotmail.com

Huabi Hu
huhuabi@hotmail.com

Taijie You
taijieyou@163.com

1 School of Mathematics and Computer Sciences, Guizhou Normal University, Guiyang 550001, Guizhou, China

2 School of Biology & Engineering, Guizhou Medical University, Guiyang 550004, Guizhou, China

1 Introduction

A subset U of a finite semigroup S is a generating set for S if every element of S may be written as a finite product of elements of U; in this case we write $S = \langle U \rangle$. An element ε of S is said to be idempotent if $\varepsilon^2 = \varepsilon$. If S has a generating set consisting of idempotents, then S is said to be *idempotent generated* or *semiband*. The latter term was introduced by Pastijn [1]. In the abstract theory of semigroups, idempotents are extremely important in the structure theory of semigroups, both finite and infinite. They help classify different types of semigroups, identify subgroups, determine left or right ideals, and describe the general structure of a given semigroup. Our interest in idempotents stems from the question whether some semigroups are semibands or not.

A large body of work involving semibands arises in the context of the singular (non-invertible) endomorphisms of a structured set. Erdos [2] proved that the semigroup of singular endomorphisms of a finite dimensional vector space is a semiband. Fountain and Lewin [3] proved that the semigroup of singular order-preserving endomorphisms of an independence algebra of finite rank is a semiband, while Oliveira [4] proved a similar result for order-independence algebras.

Let X_n be a finite chain with n elements, say $X_n = \{1 < 2 < \cdots < n\}$. As usual, we denote by $\mathcal{PT}_n$ the monoid of all partial transformations of X_n (under composition) and by $\mathcal{T}_n$ the submonoid of $\mathcal{PT}_n$ of all full transformations of X_n.

We say that a transformation $\alpha \in \mathcal{PT}_n$ is order-preserving if $x \le y$ implies $x\alpha \le y\alpha$, for all $x, y \in \mathrm{dom}(\alpha)$. We denote by $\mathcal{PO}_n$ the subsemigroup of $\mathcal{PT}_n$ of all order-preserving partial transformations (excluding the identity map) and by $\mathcal{O}_n$ the semigroup $\mathcal{PO}_n \cap \mathcal{T}_n$ of all full transformations that preserve the order. Let $\mathcal{M}_n \in \{\mathcal{O}_n, \mathcal{PO}_n\}$. Set

$$\mathcal{M}(n, r) = \{\alpha \in \mathcal{M}_n : |\operatorname{im}(\alpha)| \le r\}.$$

The sets $\mathcal{M}(n, r)$, with $0 \le r \le n - 1$, together with the empty set (if necessary), are the two-sided ideals of $\mathcal{M}_n$ and each is a regular subsemigroup of $\mathcal{M}_n$. Various properties of $\mathcal{M}(n, r)$ are known. In particular, Howie [5] showed that the semigroup $\mathcal{O}_n = \mathcal{O}(n, n-1)$ is a semiband and later jointly with Gomes [6] proved that $\mathcal{PO}_n = \mathcal{PO}(n, n-1)$ is also a semiband. Garba [7] proved that, for $2 \le r \le n-2$, $\mathcal{M}(n, r)$ is a semiband. We observe that the semigroup $\mathcal{O}_n$ is a semiband but its maximal (regular) subsemigroups in general need not be a semiband (see [8, Theorem 3.1 and Theorem 4.1]). Dimitrova and Koppitz [9] classified completely maximal regular subsemigroups of $\mathcal{O}(n, r)$, for $2 \le r \le n-2$. Zhao [10] showed that, for $2 \le r \le n-2$, any maximal regular subsemigroup of the semigroup $\mathcal{O}(n, r)$ is a semiband and obtained that the maximal regular subsemigroups and the maximal regular subsemibands of $\mathcal{O}(n, r)$ coincide, for $2 \le r \le n-2$ (see [10, Theorem 2.26]). In this paper, we investigate the maximal regular subsemigroups and the maximal regular subsemibands of $\mathcal{PO}(n, r)$. First, we completely describe the maximal regular subsemigroups of the semigroup $\mathcal{PO}(n, r)$, for $1 \le r \le n-1$. Secondly, using a similar approach from Zhao [10], we show that, for $2 \le r \le n-2$, any maximal regular subsemigroup of the semigroup $\mathcal{PO}(n, r)$ is a semiband and obtain that the maximal regular subsemigroups and the maximal regular subsemibands of $\mathcal{PO}(n,r)$ coincide, for $2 \le r \le n-2$. Finally, we obtain the complete classification of maximal regular subsemibands of $\mathcal{PO}_n$.

Let $\alpha \in \mathcal{PO}_n$. As usual, we write $\mathrm{im}(\alpha)$ and $\mathrm{dom}(\alpha)$ for the image of α and domain of α, respectively. The *kernel* of α is the equivalence $\ker(\alpha) = \{(x,y) \in \mathrm{dom}(\alpha) \times \mathrm{dom}(\alpha) : x\alpha = y\alpha\}$. Given a subset U of $\mathcal{PO}_n$, we denote by $E(U)$ its set of idempotents. We denote by $V(\alpha)$ the set of all inverses of α, and by L_α , R_α, H_α, and J_α the $\mathcal{L}$-class, $\mathcal{R}$-class, $\mathcal{H}$-class, and $\mathcal{J}$-class, respectively, of an element $\alpha \in \mathcal{PO}_n$. For general background on Semigroup Theory, we refer the reader to Howie's book [11].

2 Main Result

Recall that the Green's relations on $\mathcal{PO}_n$ can be characterized as $\alpha\mathcal{L}\beta$ if and only if $\mathrm{im}(\alpha) = \mathrm{im}(\beta)$, $\alpha\mathcal{R}\beta$ if and only if $\ker(\alpha) = \ker(\beta)$, and $\alpha\mathcal{J}\beta$ if and only if $|\mathrm{im}(\alpha)| = |\mathrm{im}(\beta)|$, for every transformations α and β. The semigroup $\mathcal{PO}_n$ has trivial $\mathcal{H}$-classes and has n $\mathcal{J}$-classes: $J_0, J_1, \ldots, J_{n-1}$, where J_0 consists of the empty mapping and $J_r = \{\alpha \in \mathcal{PO}_n : |\mathrm{im}(\alpha)| = r\}$. Clearly, $\mathcal{PO}(n,r) = J_0 \cup J_1 \cup \cdots \cup J_r$.

It is well known that $\mathcal{PO}_n$ is a regular semiband and $\mathcal{PO}(n,r)$ $(2 \le r \le n-1)$ is generated by idempotents of rank r (see [7, Lemma 3.3 and Proposition of Page 195] and [6, Lemma 3.14]). Notice that $\mathcal{PO}(n,r)$ is an ideal of $\mathcal{PO}_n$. From the result [[12], Corollary 1.4] that every ideal of a regular semiband S is also a regular semiband, we immediately deduce:

Lemma 1 *Let $2 \le r \le n-1$. Then $\mathcal{PO}(n,r)$ is a regular subsemiband of $\mathcal{PO}_n$ and $\mathcal{PO}(n,r) = \langle E(J_r)\rangle$.*

The following lemma is from Clifford and Preston [13, Theorems 2.17 and 2.18].

Lemma 2 *(1) For any two elements a, b in a semigroup S, $ab \in R_a \cap L_b$ if and only if $E(R_b \cap L_a) \ne \emptyset$. (2) Let a be a regular element of a semigroup S. (i) Every inverse of a lies in D_a. (ii) An $\mathcal{H}$-class H_b contains an inverse of a if and only if both of the $\mathcal{H}$-classes $R_a \cap L_b$ and $R_b \cap L_a$ contain idempotents.*

Let $Q_1, Q_2, \ldots, Q_m$ be all subsets of X_n with cardinality r, where $m = \binom{n}{r}$, and let

$$R(Q_i) = \{\alpha \in J_r : \mathrm{dom}(\alpha) = Q_i\},$$

where $1 \le i \le m$. Then $R(Q_1), R(Q_2), \ldots, R(Q_m)$ are some of the $\mathcal{R}$-classes of J_r. Clearly, $|E(R(Q_j))| = 1$ for all $1 \le j \le m$. We denote the unique idempotent of $R(Q_j)$ by ζ_j. Then ζ_j is the identity mapping on Q_j.

Lemma 3 *Let $2 \le r \le n-1$. Let S be a regular subsemigroup of $\mathcal{PO}(n,r)$. If $S \cap R_\alpha \ne \emptyset$, for all $\alpha \in J_r$, then $S = \mathcal{PO}(n,r)$.*

Proof We claim that

$$S \cap L_\beta \neq \emptyset, \ \ for \ all \ \beta \in J_r. \tag{2.1}$$

Otherwise, if there exist $\beta \in J_r$ such that $L_\beta \subseteq J_r \backslash S$. Let $\mathrm{im}(\beta) = Q_i$, then $\zeta_i \in E(R(Q_i)) \cap L_\beta$ and so $\zeta_i \notin S$. Notice that $R_{\zeta_i} = R(Q_i)$ and $|E(R(Q_i))| = 1$. This will yield $S \cap R_{\zeta_i} = \emptyset$ (otherwise, since S is regular, we have $\zeta_i \in S$). Notice that S is regular. By condition and (2.1), we have

$$S \cap E(R_\alpha) \neq \emptyset, \quad S \cap E(L_\beta) \neq \emptyset, \text{for all}\, \alpha, \beta \in J_r. \tag{2.2}$$

We show that $E(J_r) \subseteq E(S)$ and so $S = \mathcal{PO}(n,r)$ by Lemma 1. Suppose that $e \in E(J_r) \backslash E(S)$. Notice that $\mathcal{D} = \mathcal{J}$ in every finite semigroup. By (2.2), we can choose $f \in E(S) \cap L_e$ and $g \in E(S) \cap R_e$. Since $e \notin E(S)$, we have $e \neq f$ and $e \neq g$. By Lemma 2, we have $fg \in S \cap R_f \cap L_g$ (since $e \in E(L_f \cap R_g)$). As usual, by $\mathcal{L}^S$, $\mathcal{R}^S$ denote Green relations of the subsemigroup S of $\mathcal{PO}(n,r)$. Since S is regular, we have that $\mathcal{R}^S = \mathcal{R} \cap (S \times S)$ and $\mathcal{L}^S = \mathcal{L} \cap (S \times S)$. Then $f\mathcal{R}^S fg \mathcal{L}^S g$ (since $f \mathcal{R} fg \mathcal{L} g$). Notice that $f, g \in E(S)$ and $fg \in S$. By Lemma 2, there exist $\delta \in V(fg)$ such that $\delta \in L_f^S \cap R_g^S$. Since $f \mathcal{L} e \mathcal{R} g$, we have $\delta \in L_f^S \cap R_g^S \subseteq S \cap L_f \cap R_g = S \cap L_e \cap R_e = S \cap H_e$. Then $e = \delta \in S$ (since every $\mathcal{H}$-class of $\mathcal{PO}(n,r)$ is trivial), which contradicts $e \in E(J_r) \backslash E(S)$. Thus $E(J_r) \subseteq E(S)$. □

Let $2 \leq r \leq n-1$. A proper subsemigroup S of $\mathcal{PO}(n,r)$ is called a *maximal regular subsemigroup* if S is a regular semigroup, and any regular subsemigroup of $\mathcal{PO}(n,r)$ properly containing S must must be $\mathcal{PO}(n,r)$.

Our first main result is:

Theorem 4 *Let $2 \leq r \leq n-1$. Then any maximal regular subsemigroups of the semigroup $\mathcal{PO}(n,r)$ are of the form: $\mathcal{PO}(n,r-1) \cup (J_r \backslash R_\alpha)$, for some $\alpha \in J_r$.*

Proof Let $\alpha \in J_r$, and let $M_\alpha = \mathcal{PO}(n,r-1) \cup (J_r \backslash R_\alpha)$. We shall show that M_α is a maximal regular subsemigroup of $\mathcal{PO}(n,r)$. For any $\beta, \gamma \in J_r$, either $\beta\gamma \mathcal{R} \beta$ or $\beta\gamma \in \mathcal{PO}(n,r-1)$ by Lemma 2, so M_α is a subsemigroup of $\mathcal{PO}(n,r)$. From Lemma 1, we know that $\mathcal{PO}(n,r-1)$ is regular. Let $\beta \in J_r \backslash R_\alpha$. Suppose that $\mathrm{im}(\beta) = \{b_1 < \cdots < b_r\}$. Let e_1 be the identity mapping on $\mathrm{im}(\beta)$, and let

$$e_2 = \begin{pmatrix} B_1 & B_2 & \cdots & B_r \\ b_1 & b_2 & \cdots & b_r \end{pmatrix},$$

where $B_1 = \{1, \ldots, b_1\}$, $B_i = \{b_{i-1}+1, \ldots, b_i\}$, for $2 \leq i \leq r-1$, and $B_r = \{b_{r-1}+1, \ldots, n\}$. Clearly, $e_1, e_2 \in E(L_\beta)$ and $(e_1, e_2) \notin \mathcal{R}$. Since $\mathcal{PO}_n$ is regular, we have $|E(R_\beta)| \geq 1$. Let $f \in E(R_\beta)$. Then $e_i \mathcal{L} \beta \mathcal{R} f$. By Lemma 2, we have that $R_{e_i} \cap L_f$ contains inverse of α. Let $\gamma_i \in R_{e_i} \cap L_f \cap V(\alpha)$, $i = 1, 2$, then $e_i \mathcal{R} \gamma_i \mathcal{L} f$ and so $(\gamma_1, \gamma_2) \notin \mathcal{R}$. It follows that $\gamma_1 \in J_r \backslash R_\alpha \subseteq M_\alpha$ or $\gamma_2 \in J_r \backslash R_\alpha \subseteq M_\alpha$. Then β is regular and so M_α is a regular subsemigroup of $\mathcal{PO}(n,r)$.

Suppose T is a regular subsemigroup of $\mathcal{PO}(n,r)$ properly containing M_α. Then $T \cap R_\beta \neq \emptyset$, for all $\beta \in J_r$. Thus, by Lemma 3, $T = \mathcal{PO}(n,r)$ and so M_α is a maximal regular subsemigroup of $\mathcal{PO}(n,r)$.

Conversely, suppose M is a maximal regular subsemigroup of $\mathcal{PO}(n,r)$. Then there exist $\alpha \in J_r$ such that $M \cap R_\alpha = \emptyset$ (otherwise, by Lemma 3, we have $M = \mathcal{PO}(n,r)$) and so $M \subseteq \mathcal{PO}(n,r-1) \cup (J_r \backslash R_\alpha) = M_\alpha$. Thus, by the maximality of M, $M = M_\alpha$. $\square$

Let A be a subset of X_n. We say that a subset C of A is convex on A if

$$x, y \in C, z \in A \textit{ and } x \leq z \leq y \Rightarrow z \in C.$$

We shall refer to an equivalence π on the subset A of X_n as convex if its classes are convex subsets of A, and we shall say that π is of weight r on A if $|\pi/A| = r$. Now, we consider the top class J_r of $\mathcal{PO}(n,r)$. We denote by Λ_r the collection of all subsets of X_n of cardinality r. Let $A \in \Lambda_r$. Then a typical $\mathcal{L}$-class in J_r may be denoted by $L_A = \{\alpha \in \mathcal{PO}(n,r) : \mathrm{im}(\alpha) = A\}$. The number of $\mathcal{L}$-classes within J_r is the number of image sets in X_n of cardinality r, namely, $\binom{n}{r}$. We denote by Ω_r the collection of all convex equivalences of weight r on all subsets of X_n. Let $\pi \in \Omega_r$. Then a typical $\mathcal{R}$-class in J_r may be denoted by $R_\pi = \{\alpha \in \mathcal{PO}(n,r) : \ker(\alpha) = \pi\}$. Thus, J_r has $\Sigma_{k=r}^{n}\binom{n}{k}\binom{k-1}{r-1})$ $\mathcal{R}$-classes corresponding to the $\Sigma_{k=r}^{n}\binom{n}{k}\binom{k-1}{r-1})$ convex equivalences of weight r on all subsets of X_n, and $\binom{n}{r}$ $\mathcal{L}$-classes corresponding to the $\binom{n}{r}$ subsets of X_n of cardinality r. It follows that J_r has $\binom{n}{r}(\Sigma_{k=r}^{n}\binom{n}{k}\binom{k-1}{r-1})$ $\mathcal{H}$-classes. Also we may write $H_{(\pi,A)}$ for the $\mathcal{H}$-class, which is the intersection of R_π and L_A. A is said to be a transversal of π if $|A \cap \tilde{x}| = 1$ for every equivalence class $\tilde{x}$ of π. Then $H_{(\pi,A)}$ is a group $\mathcal{H}$-class if and only if A is a transversal of π. Notice that every $\mathcal{H}$-class of $\mathcal{PO}(n,r)$ is trivial. Thus every group $\mathcal{H}$-class consists of an idempotent.

The following lemma is from G.U. Garba [7, Lemma 3.3 and Proposition of Page 195]:

Lemma 5 *There is a way of listing the subsets of X_n of cardinality r as $A_1, A_2, \ldots, A_m$ (where $m = \binom{n}{r}$ and $r \geq 2$) so that there exist distinct convex equivalences $\pi_1, \pi_2, \ldots, \pi_m$ of weight r with the property that A_{i-1}, A_i are both transversals of π_i ($i = 2, \ldots, m$) and A_m, A_1 are traversals of π_1. Each $\mathcal{H}$-class $H_{(\pi_i,A_i)}$ consists of an idempotent ε_i ($i = 1, \ldots, m$), $\mathcal{H}$-class $H_{(\pi_i,A_{i-1})}$ consists of an idempotent η_i ($i = 2, \ldots, m$) and $\mathcal{H}$-class $H_{(\pi_1,A_m)}$ consists of an idempotent η_1, and there exist idempotents $\varepsilon_{m+1}, \ldots, \varepsilon_p$ (where $p = \Sigma_{k=r}^{n}\binom{n}{k}\binom{k-1}{r-1})$) such that $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_p$ cover all the $\mathcal{R}$-classes in J_r and $\{\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_p\}$ is a set of generators for $\mathcal{PO}(n,r)$.*

Before presenting our next lemma, we introduce the following notation.

Let A_i, π_i, ε_i, and η_i be as defined in Lemma 5, and let $\pi_i = \ker(\varepsilon_i)$, for $i = m+1, \ldots, p$. Then $\Omega_r = \{\pi_1, \pi_2, \ldots, \pi_p\}$ ($p = \Sigma_{k=r}^{n}\binom{n}{k}\binom{k-1}{r-1})$) and $\Lambda_r = \{A_1, A_2, \ldots, A_m\}$ ($m = \binom{n}{r}$). Let $\Sigma = \cup_{i,j=1}^{m} H_{(\pi_i,A_j)}$, $G^+(\Sigma) = \{\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_m\}$, $G^-(\Sigma) = \{\eta_1, \eta_2, \ldots, \eta_m\}$, and $G(\Sigma) = G^+(\Sigma) \cup G^-(\Sigma)$.

As in [14], let $k \in \{0, 1, 2, \ldots, m\}$ and define a total order $\leq_k$ on the set $\{1, \ldots, m\}$ by

$$k+1 \leq_k k+2 \leq_k \cdots \leq_k m \leq_k 1 \leq_k \cdots \leq_k k,$$

where $i <_k j$ if $i \leq_k j$ and $i \neq j$. In the following lemma, it will always be clear from context when additions are modular m.

Lemma 6 $\Sigma \subseteq \langle G(\Sigma) \rangle$.

Proof Let

$$\alpha_{ij} = \eta_i \eta_{i-1} \cdots \eta_{j+1}, 1 \leq j < i \leq m,$$
$$\beta_{ij} = \varepsilon_i \varepsilon_{i+1} \cdots \varepsilon_j, 1 \leq i \leq j \leq m.$$

Then, by Lemmas 2 and 5, $\alpha_{ij} \in \langle G^-(\Sigma) \rangle \cap H_{(\pi_i, A_j)}$ and $\beta_{ij} \in \langle G^+(\Sigma) \rangle \cap H_{(\pi_i, A_j)}$. Since $\mathcal{PO}_n$ is $\mathcal{H}$-trivial, we have

$$H_{(\pi_i, A_j)} = \{\alpha_{ij}\} \subseteq \langle G^-(\Sigma) \rangle, 1 \leq j < i \leq m,$$
$$H_{(\pi_i, A_j)} = \{\beta_{ij}\} \subseteq \langle G^+(\Sigma) \rangle, 1 \leq i \leq j \leq m.$$

Notice that $\Sigma = \cup_{i,j=1}^m H_{(\pi_i, A_j)}$. It follows immediately that $\Sigma \subseteq \langle G^+(\Sigma) \cup G^-(\Sigma) \rangle = \langle G(\Sigma) \rangle$. □

Lemma 7 $\Sigma \backslash R_{\pi_k} \subseteq \langle E(J_r \backslash R_{\pi_k}) \rangle$, *for all* $1 \leq k \leq m$.

Proof Let π_i, A_i, ε_i, and η_i be defined as before. Let

$$\alpha_{i,j}^{[k]} = \eta_i \eta_{i-1} \cdots \eta_j, \quad j \leq_k i <_k k,$$
$$\beta_{i,j}^{[k]} = \varepsilon_i \varepsilon_{i+1} \cdots \varepsilon_j, \quad i \leq_k j <_k k.$$

Notice that $G(\Sigma) \backslash \{\varepsilon_k, \eta_k\} \subseteq E(J_r \backslash R_{\pi_k})$. Then, by Lemmas 2 and 5,

$$\alpha_{i,j}^{[k]} \in \langle G^-(\Sigma) \backslash \{\eta_k\} \rangle \cap H_{(\pi_i, A_{j-1})} \subseteq \langle E(J_r \backslash R_{\pi_k}) \rangle \cap H_{(\pi_i, A_{j-1})},$$
$$\beta_{i,j}^{[k]} \in \langle G^+(\Sigma) \backslash \{\varepsilon_k\} \rangle \cap H_{(\pi_i, A_j)} \subseteq \langle E(J_r \backslash R_{\pi_k}) \rangle \cap H_{(\pi_i, A_j)}.$$

Since $\mathcal{PO}_n$ is $\mathcal{H}$-trivial, we have

$$H_{(\pi_i, A_{j-1})} = \{\alpha_{i,j}^{[k]}\} \subseteq \langle E(J_r \backslash R_{\pi_k}) \rangle, j \leq_k i <_k k,$$
$$H_{(\pi_i, A_j)} = \{\beta_{i,j}^{[k]}\} \subseteq \langle E(J_r \backslash R_{\pi_k}) \rangle, i \leq_k j <_k k.$$

It follows immediately that $\Sigma \backslash R_{\pi_k} = \cup_{\substack{i,j=1 \\ i \neq k}}^{m} H_{(\pi_i, A_j)} = [(\cup_{j \leq_k i <_k k} H_{(\pi_i, A_{j-1})}) \cup (\cup_{i \leq_k j <_k k} H_{(\pi_i, A_j)})] \subseteq \langle E(J_r \backslash R_{\pi_k}) \rangle$. □

Lemma 8 *Let* $2 \leq r \leq n-2$. *Then* $J_r \backslash R_\alpha = \langle E(J_r \backslash R_\alpha) \rangle \cap J_r$, *for all* $\alpha \in J_r$.

Proof Let π_i, A_i, ε_i, and η_i be defined as before. Notice that $\Omega_r = \{\pi_1, \pi_2, \ldots, \pi_p\}$ ($p = \Sigma_{k=r}^n \binom{n}{k}\binom{k-1}{r-1}$) and $\Lambda_r = \{A_1, A_2, \ldots, A_m\}$ ($m = \binom{n}{r}$). Let $\widetilde{\Sigma} = J_r \backslash \Sigma$. Then $\widetilde{\Sigma} = \cup_{i=m+1}^p \cup_{j=1}^m H_{(\pi_i, A_j)}$. Let $\alpha \in J_r$ and $\ker(\alpha) = \pi$, then there exist

$k \in \{1, 2, \ldots, p\}$ such that $\pi = \pi_k$. Clearly, $R_\alpha = R_{\pi_k}$ and $J_r \backslash R_\alpha = J_r \backslash R_{\pi_k}$. We first prove that $J_r \backslash R_\alpha \subseteq \langle E(J_r \backslash R_\alpha)\rangle \cap J_r$, i.e., $J_r \backslash R_{\pi_k} \subseteq \langle E(J_r \backslash R_{\pi_k})\rangle \cap J_r$. We distinguish two cases:

Case 1 $1 \leq k \leq m$. Clearly $J_r \backslash R_{\pi_k} = (\Sigma \backslash R_{\pi_k}) \cup \widetilde{\Sigma}$. We shall show that $\widetilde{\Sigma} \subseteq \langle E(J_r \backslash R_{\pi_k})\rangle$. Let $\alpha \in \widetilde{\Sigma}$. Then $\alpha \in H_{(\pi_i, A_j)}$ for some $i \in \{m+1, \ldots, p\}$, $j \in \{1, \ldots, m\}$. Notice that $\alpha \mathcal{R} \varepsilon_i$. Suppose that $\varepsilon_i \in H_{(\pi_i, A_s)}$ for some $s \in \{1, \ldots, m\}$. Let

$$\sigma_s = \begin{cases} \eta_{s+1} & s = k \\ \varepsilon_s & s \neq k. \end{cases}$$

Then $\varepsilon_i \mathcal{L} \sigma_s$ and $\sigma_s \in \Sigma \backslash R_{\pi_k}$. Let $\beta \in R_{\sigma_s} \cap L_\alpha$. Clearly $\sigma_s \in E(L_{\varepsilon_i} \cap R_\beta)$ and $\beta \in \Sigma \backslash R_{\pi_k}$. Notice that $\mathcal{PO}_n$ is $\mathcal{H}$-trivial and $\varepsilon_i \in E(\widetilde{\Sigma}) \subseteq E(J_r \backslash R_{\pi_k})$. Then, by Lemmas 2 and 7,

$$\alpha = \varepsilon_i \beta \in E(J_r \backslash R_{\pi_k}) \cdot \Sigma \backslash R_{\pi_k} \subseteq E(J_r \backslash R_{\pi_k}) \cdot \langle E(J_r \backslash R_{\pi_k})\rangle \subseteq \langle E(J_r \backslash R_{\pi_k})\rangle$$

and so $\widetilde{\Sigma} \subseteq \langle E(J_r \backslash R_{\pi_k})\rangle$. Thus, by Lemma 7, $J_r \backslash R_{\pi_k} = (\Sigma \backslash R_{\pi_k}) \cup \widetilde{\Sigma} \subseteq \langle E(J_r \backslash R_{\pi_k})\rangle$ and so $J_r \backslash R_{\pi_k} = (J_r \backslash R_{\pi_k}) \cap J_r \subseteq \langle E(J_r \backslash R_{\pi_k})\rangle \cap J_r$.

Case 2 $m+1 \leq k \leq p$. Clearly $J_r \backslash R_{\pi_k} = \Sigma \cup (\widetilde{\Sigma} \backslash R_{\pi_k})$. We shall show that $\widetilde{\Sigma} \backslash R_{\pi_k} \subseteq \langle E(J_r \backslash R_{\pi_k})\rangle$. Let $\alpha \in \widetilde{\Sigma} \backslash R_{\pi_k}$. Then $\alpha \in H_{(\pi_i, A_j)}$ for some $i \in \{m+1, \ldots, p\} \backslash \{k\}$, $j \in \{1, \ldots, m\}$. Notice that $\alpha \mathcal{R} \varepsilon_i$. Suppose that $\varepsilon_i \in H_{(\pi_i, A_s)}$ for some $s \in \{1, \ldots, m\}$. Then $\varepsilon_i \mathcal{L} \varepsilon_s$ and $\varepsilon_s \in \Sigma$. Let $\beta \in R_{\varepsilon_s} \cap L_\alpha$. Clearly $\varepsilon_s \in E(L_{\varepsilon_i} \cap R_\beta)$ and $\beta \in \Sigma$. Notice that $\varepsilon_i \in E(\widetilde{\Sigma} \backslash R_{\pi_k}) \subseteq E(J_r \backslash R_{\pi_k})$, $G(\Sigma) \subseteq E(J_r \backslash R_{\pi_k})$, and $\mathcal{PO}_n$ is $\mathcal{H}$-trivial. Then, by Lemmas 2 and 7,

$$\alpha = \varepsilon_i \beta \in E(J_r \backslash R_{\pi_k}) \cdot \Sigma \subseteq E(J_r \backslash R_{\pi_k}) \cdot \langle G(\Sigma)\rangle \subseteq \langle E(J_r \backslash R_{\pi_k})\rangle$$

and so $\widetilde{\Sigma} \backslash R_{\pi_k} \subseteq \langle E(J_r \backslash R_{\pi_k})\rangle$. Thus, by Lemma 6, $J_r \backslash R_{\pi_k} = \Sigma \cup (\widetilde{\Sigma} \backslash R_{\pi_k}) \subseteq \langle G(\Sigma)\rangle \cup \langle E(J_r \backslash R_{\pi_k})\rangle \subseteq \langle E(J_r \backslash R_{\pi_k})\rangle$ and so $J_r \backslash R_{\pi_k} = (J_r \backslash R_{\pi_k}) \cap J_r \subseteq \langle E(J_r \backslash R_{\pi_k})\rangle \cap J_r$.

It remains to prove that $\langle E(J_r \backslash R_\alpha)\rangle \cap J_r \subseteq J_r \backslash R_\alpha$. Let $S = \mathcal{PO}(n, r-1) \cup (J_r \backslash R_\alpha)$. From Theorem 4, we know that S is a subsemigroup of $\mathcal{PO}(n, r)$. It is obvious that $E(J_r \backslash R_\alpha) \subseteq S$. Then $\langle E(J_r \backslash R_\alpha)\rangle \subseteq S$ and so $\langle E(J_r \backslash R_\alpha)\rangle \cap J_r \subseteq S \cap J_r = J_r \backslash R_\alpha$. □

Our next main result is:

Theorem 9 *Let $2 \leq r \leq n-2$. Then any maximal regular subsemigroup of $\mathcal{PO}(n, r)$ is a semiband.*

Proof From Theorem 4, we know that each maximal regular subsemigroup of $\mathcal{PO}(n, r)$ must be in the following form: $\mathcal{PO}(n, r-1) \cup (J_r \backslash R_\alpha)$, for some $\alpha \in J_r$. By Lemmas 1 and 8, we have

$$\mathcal{PO}(n, r-1) \cup (J_r \backslash R_\alpha) = \langle E(J_{r-1})\rangle \cup (\langle E(J_r \backslash R_\alpha)\rangle \cap J_r) = \langle E(J_{r-1}) \cup E(J_r \backslash R_\alpha)\rangle.$$

Then any maximal regular subsemigroup of $\mathcal{PO}(n, r)$ is a semiband. □

As in [7], we shall refer to an element α in $\mathcal{PO}_n$ as being of type (k, r) or belonging to the set $[k, r]$ if $|\operatorname{dom}(\alpha)| = k$, $|\operatorname{im}(\alpha)| = r$. Clearly $J_r = \cup_{i=r}^{n}[i, r]$ and $J_{n-1} = [n, n-1] \cup [n-1, n-1]$. We draw attention to the $\mathcal{J}$-class J_{n-1}. Gomes and Howie [6] used the notation $[i \to i+1]$ for the increasing idempotent ε defined by $i\varepsilon = i+1$, $x\varepsilon = x$ $(x \neq i)$ and the notation $[i \to i-1]$ for the decreasing idempotent η defined by $i\eta = i-1$, $x\eta = x$ $(x \neq i)$. They also used the notation δ_k for the identity mapping on $X_n \backslash \{k\}$. Let $E_{n-1}^{+} = \{[i \to i+1] : 1 \leq i \leq n-1\}$ and $E_{n-1}^{-} = \{[i \to i-1] : 2 \leq i \leq n\}$ be the increasing and decreasing idempotent sets of $[n, n-1]$, respectively. Let $E_{n-1} = E_{n-1}^{+} \cup E_{n-1}^{-}$, and let $F_{n-1} = \{\delta_1, \ldots, \delta_n\}$. Then $E([n, n-1]) = E_{n-1}$ and $E([n-1, n-1]) = F_{n-1}$. For convenience, we use $[n \to n+1]$ or $[1 \to 0]$ to denote θ (the empty mapping).

For any $i, j \in X_n$, let

$$M_{i,j} = \{\alpha \in \mathcal{O}_n : (\forall x, y \in X_n)\ x \leq i \Longrightarrow x\alpha \leq i,\ \ y \geq j \Longrightarrow y\alpha \geq j\}, \tag{2.3}$$

$$PM_{i,j} = \{\alpha \in \mathcal{PO}_n : (\forall x, y \in \operatorname{dom}(\alpha))\ x \leq i \Longrightarrow x\alpha \leq i,\ \ y \geq j \Longrightarrow y\alpha \geq j\} \cup J_0. \tag{2.4}$$

Let $\alpha \in \mathcal{O}_n$ and fix some $i \in X_n$. It is easy to prove that

$$i\alpha \leq i \Leftrightarrow (x \in X_n) x \leq i \Rightarrow x\alpha \leq i,$$
$$i\alpha \geq i \Leftrightarrow (y \in X_n) y \geq i \Rightarrow y\alpha \geq i.$$

Notice that $[n, n-1] = \mathcal{O}_n \cap J_{n-1}$. From the above fact and [15, Lemma 2.5], we easily deduce the following:

Lemma 10 *Let $n \geq 3$. Then $M_{i,j} = \langle E([n, n-1]) \backslash \{[i \to i+1], [j \to j-1]\} \rangle$, for all $i, j \in X_n$.*

Let 1_{X_n} be the identity mapping on X_n, and let $\mathcal{O}_n^1 = \mathcal{O}_n \cup \{1_{X_n}\}$. For a strictly partial transformation $\beta = \begin{pmatrix} A \\ a \end{pmatrix} \in \mathcal{PO}_n$, let $c = min(A \cup \{a\})$ and $d = max(A \cup \{a\})$. Now, we define the completion β^* of β in $\mathcal{O}_n^1$ as follows:

$$x\beta^* = \begin{cases} a, & c \leq x \leq d, \\ x, & otherwise. \end{cases}$$

Clearly $\beta^* = 1_{X_n}$ if and only if $A = \{a\}$.

Lemma 11 *Let $n \geq 3$. Then $PM_{i,j} = \langle E(J_{n-1}) \backslash \{[i \to i+1], [j \to j-1]\} \rangle$, for all $i, j \in X_n$.*

Proof Let $\tilde{E} = E(J_{n-1}) \backslash \{[i \to i+1], [j \to j-1]\}$. We first show that $PM_{i,j} \subseteq \langle \tilde{E} \rangle$. It is obvious that $\theta = \delta_1 \delta_2 \cdots \delta_n$. Notice that $F_{n-1} \subseteq \tilde{E}$. Thus $J_0 = \{\theta\} \subseteq \langle \tilde{E} \rangle$. Let $\alpha \in PM_{i,j} \backslash J_0$. We distinguish three cases.

Case 1 $\alpha \in \mathcal{O}_n$. Clearly $\alpha \in M_{i,j}$. By Lemma 10, we have

$$\alpha \in M_{i,j} = \langle E([n, n-1])\backslash\{[i \to i+1], [j \to j-1]\}\rangle \subseteq \langle \tilde{E} \rangle.$$

Case 2 α is an identity mapping in $[r, r]$, where $1 \le r \le n-1$. Now, put

$$X_n \backslash \operatorname{dom}(\alpha) = \{x_1, x_2, \ldots, x_{n-r}\},$$

and define e_k to be the identity mapping on $X_n \backslash \{x_k\}$, then $e_k \in F_{n-1} \subseteq \tilde{E}$, for all $k \in \{1, \ldots, n-r\}$. It is obvious that $\alpha = e_1 e_2 \cdots e_{n-r}$. Thus $\alpha \in \langle \tilde{E} \rangle$.

Case 3 α is in $[r, s]$ $(1 \le s \le r \le n-1)$, but α is not an identity mapping. Suppose that

$$\alpha = \begin{pmatrix} A_1 & A_2 & \cdots & A_s \\ a_1 & a_2 & \cdots & a_s \end{pmatrix} \in [r, s].$$

Let α_k^* be a completion of $\begin{pmatrix} A_k \\ a_k \end{pmatrix}$ in $\mathcal{O}_n^1$. We shall prove that

$$\alpha_1^*, \alpha_2^*, \ldots, \alpha_s^* \in M_{i,j} \cup \{1_{X_n}\}.$$

Suppose that $\alpha_k^* \neq 1_{X_n}$. Notice that $(\forall x \in X_n)\ x\alpha_k^* \in \{x, a_k\}$. Let $x \in X_n$ such that $x \ge j$. (i) If $a_k \ge j$, then $x\alpha_k^* \ge j$. (ii) If $a_k < j$, then $x > a_k$. Suppose that $x\alpha_k^* = a_k$. By the definition of α_k^* and $x > a_k$, we have $x \le max A_k$ and so $j \le max A_k$. Since $max A_k \in A_k \subseteq \operatorname{dom}(\alpha)$ and $\alpha \in PM_{i,j} \backslash J_0$, we have $j \le (max A_k)\alpha = a_k$, a contradiction. Then $x\alpha_k^* = x$ and so $x\alpha_k^* \ge j$. Similarly, we can prove that $x \le i \Rightarrow x\alpha_k^* \le i$. Thus $\alpha_k^* \in M_{i,j}$.

Let α_0 be an identity mapping on $\operatorname{dom}(\alpha)$. It is easy to prove that

$$\alpha = \alpha_0 \alpha_1^* \cdots \alpha_s^*.$$

From Case 2 above, we know that $\alpha_0 \in \langle \tilde{E} \rangle$. By Lemma 10, we have

$$\alpha_k^* \in M_{i,j} = \langle E([n, n-1]\}\backslash\{[i \to i+1], [j \to j-1]\}\rangle \subseteq \langle \tilde{E} \rangle.$$

Thus $\alpha \in \langle \tilde{E} \rangle$.

It remains to prove that $\langle \tilde{E} \rangle \subseteq PM_{i,j}$. Notice that $F_{n-1} \subseteq PM_{i,j}$ and $\tilde{E} = (E([n, n-1])\backslash\{[i \to i+1], [j \to j-1]\}) \cup F_{n-1}$. By Lemma 10, we have

$$E([n, n-1])\backslash\{[i \to i+1], [j \to j-1]\} \subseteq M_{i,j} \subseteq PM_{i,j}.$$

Then $\tilde{E} \subseteq PM_{i,j}$. It is easy to prove that $PM_{i,j}$ is a subsemigroup of $\mathcal{PO}_n$. Thus $\langle \tilde{E} \rangle \subseteq PM_{i,j}$. □

Lemma 12 *Let* $1 \le i \le n-1$. *Then* $PM_{i,i+1}$ *is a regular subsemiband of* $\mathcal{PO}_n$.

Proof From Lemma 11, we know that $PM_{i,i+1}$ is a subsemiband of $\mathcal{PO}_n$. Let $\alpha \in PM_{i,i+1}$. If $|\operatorname{im}(\alpha)| = 0$, then clearly $\alpha = \theta$. Then $\alpha = \alpha^2$ and so α is regular. If $|\operatorname{im}(\alpha)| \geq 1$, suppose that

$$\alpha = \begin{pmatrix} A_1 & A_2 & \cdots & A_r \\ a_1 & a_2 & \cdots & a_r \end{pmatrix}.$$

Let $c_k \in A_k$, $1 \leq k \leq r$. Notice that if $a_k \leq i$, then $c_k \leq i$ (otherwise, since $\alpha \in PM_{i,i+1}$, we have $a_k = A_k\alpha = c_k\alpha \geq i+1$, a contradiction); if $a_k \geq i+1$, then $c_k \geq i+1$ (otherwise, since $\alpha \in PM_{i,i+1}$, we have $a_k = A_k\alpha = c_k\alpha \leq i$, a contradiction). Let

$$\beta = \begin{pmatrix} A_1 & A_2 & \cdots & A_r \\ c_1 & c_2 & \cdots & c_r \end{pmatrix}.$$

Then clearly $\beta \in PM_{i,i+1}$ and $\alpha = \alpha\beta\alpha$. Thus α is regular. □

Let $2 \leq r \leq n-1$. A proper subsemigroup S of $\mathcal{PO}(n,r)$ is called a *maximal regular subsemiband* of $\mathcal{PO}(n,r)$ if S is a regular semiband, and any regular subsemiband of $\mathcal{PO}(n,r)$ properly containing S must be $\mathcal{PO}(n,r)$.

Lemma 13 *Let* $1 \leq i \leq n-1$. *Then* $\mathcal{PO}(n,n-2) \cup PM_{i,i+1}$ *is a maximal regular subsemiband of* $\mathcal{PO}_n$.

Proof Let $B_i = \mathcal{PO}(n,n-2) \cup PM_{i,i+1}$. From Lemmas 1 and 12, we easily deduce that B_i is a regular subsemigroup of $\mathcal{PO}_n$. By Lemmas 1 and 10, we have $B_i = \mathcal{PO}(n,n-2) \cup PM_{i,i+1} = \langle E(J_{n-2})\rangle \cup \langle E(J_{n-1})\backslash\{[i \to i+1], [i+1 \to i]\}\rangle = \langle E(J_{n-2}) \cup (E(J_{n-1})\backslash\{[i \to i+1], [i+1 \to i]\})\rangle$. Then B_i is a regular subsemiband of $\mathcal{PO}_n$.

Let T be a regular semiband of $\mathcal{PO}_n$ properly containing B_i. Then $E(B_i) \subset E(T)$. Notice that $\mathcal{PO}(n,n-2) \subseteq B_i \subset T$ and $E(PM_{i,i+1} \cap J_{n-1}) = E(J_{n-1})\backslash\{[i \to i+1], [i+1 \to i]\}$. Thus

$$\begin{aligned} E(J_{n-1})\backslash\{[i \to i+1], [i+1 \to i]\} &= E(PM_{i,i+1} \cap J_{n-1}) \\ &= E(B_i \cap J_{n-1}) \subset E(T \cap J_{n-1}), \end{aligned}$$

whence $E^-_{n-1} \cup F_{n-1} \subseteq E(J_{n-1})\backslash\{[i \to i+1]\} \subseteq T$ or $E^+_{n-1} \cup F_{n-1} \subseteq E(J_{n-1})\backslash\{[i+1 \to i]\} \subseteq T$. Notice that $E^-_{n-1} \cup F_{n-1}$ ($E^+_{n-1} \cup F_{n-1}$) contains exactly one (idempotent) element from each $\mathcal{R}$-class of $\mathcal{PO}_n$ of rank $n-1$. It follows that $T \cap E(R_\alpha) \neq \emptyset$, for all $\alpha \in J_{n-1}$. Thus, by Lemma 3, $T = \mathcal{PO}_n$. □

Lemma 14 *Let* $1 \leq i \leq n-1$. *Then* $\mathcal{PO}(n,n-2) \cup \langle E(J_{n-1}\backslash\{\delta_i\}\rangle = \mathcal{PO}(n,n-2) \cup \{\alpha \in \mathcal{PO}_n : i \in \operatorname{dom}(\alpha)\}$.

Proof Let $S_i = \{\alpha \in \mathcal{PO}_n : i \in \operatorname{dom}(\alpha)\}$, and let $W_i = \mathcal{PO}(n,n-2) \cup S_i$. It is obvious that $W_i = \mathcal{PO}(n,n-2) \cup (S_i \cap J_{n-1})$. For any $\alpha, \beta \in W_i$, either $\alpha\beta \in \mathcal{PO}(n,n-2)$ or $\alpha\beta \in J_{n-1}$. If $\alpha\beta \in J_{n-1}$, then $\alpha, \beta \in W_i \cap J_{n-1} = S_i \cap J_{n-1} \subseteq S_i$

and $\alpha\mathcal{R}\alpha\beta$. Thus $\ker(\alpha) = \ker(\alpha\beta)$ and so $\alpha\beta \in S_i$. Hence W_i is a subsemigroup of $\mathcal{PO}_n$. It is obvious that $E(J_{n-1})\backslash\{\delta_i\} \subseteq S_i \subseteq W_i$. Then $\langle E(J_{n-1})\backslash\{\delta_i\}\rangle \subseteq W_i$ and so $\mathcal{PO}(n, n-2) \cup \langle E(J_{n-1})\backslash\{\delta_i\}\rangle \subseteq W_i$. From [16, Lemma 3.6], we know that $\langle S_i\rangle = \langle E(J_{n-1})\backslash\{\delta_i\}\rangle$. It follows that $W_i \subseteq \mathcal{PO}(n, n-2) \cup \langle S_i\rangle = \mathcal{PO}(n, n-2) \cup \langle E(J_{n-1}\backslash\{\delta_i\}\rangle$. □

Our final main result is:

Theorem 15 *Let $n \geq 3$. Then each maximal regular subsemiband of $\mathcal{PO}_n$ must be in one of the following forms:*
(A) $A_i = \mathcal{PO}(n, n-2) \cup \{\alpha \in \mathcal{PO}_n : i \in \text{dom}(\alpha)\}$, $1 \leq i \leq n$.
(B) $B_i = \mathcal{PO}(n, n-2) \cup PM_{i,i+1}$, $1 \leq i \leq n-1$.

Proof Let $S_i = \{\alpha \in \mathcal{PO}_n : i \in \text{dom}(\alpha)\}$. From Lemma 13, we know that B_i is a maximal regular subsemiband of $\mathcal{PO}_n$. By the definition of S_i, we easily deduce that $S_i \cap J_{n-1} = J_{n-1}\backslash R_{\delta_i}$ and so $A_i = \mathcal{PO}(n, n-2) \cup (J_{n-1}\backslash R_{\delta_i})$. Then, by Theorem 4, A_i is a maximal regular subsemigroup of $\mathcal{PO}_n$. To show that A_i is a maximal regular subsemiband of $\mathcal{PO}_n$, it is enough to verify that A_i is a semiband. By Lemmas 1 and 14, we have

$$A_i = \mathcal{PO}(n, n-2) \cup S_i = \langle E(J_{n-2})\rangle \cup \langle E(J_{n-1})\backslash\{\delta_i\}\rangle = \langle E(J_{n-2}) \cup (E(J_{n-1})\backslash\{\delta_i\})\rangle.$$

Conversely, let S be an arbitrary maximal regular subsemiband of $\mathcal{PO}_n$ not of the form A_i or B_i. Notice that

$$E(A_i \cap J_{n-1}) = E(S_i \cap J_{n-1}) = E(J_{n-1})\backslash\{\delta_i\},$$
$$E(B_i \cap J_{n-1}) = E(PM_{i,i+1} \cap J_{n-1}) = E(J_{n-1})\backslash\{[i \to i+1], [i+1 \to i]\}.$$

We claim that S satisfies

$$S \cap E(R_\alpha) \neq \emptyset, \ \ for\ all\ \alpha \in J_{n-1}. \tag{2.5}$$

Otherwise, there exists A_i or B_i for some $i \in X_n$ such that $E(S \cap J_{n-1}) \subseteq E(A_i \cap J_{n-1})$ or $E(S \cap J_{n-1}) \subseteq E(B_i \cap J_{n-1})$. Notice that $\mathcal{PO}(n, n-2) \subseteq A_i, B_i$. Then $E(S) \subseteq E(A_i)$ or $E(S) \subseteq E(B_i)$. It follows that $S \subseteq A_i$ or $S \subseteq B_i$, since S, A_i, and B_i are semibands. Thus, by the maximality of S, $S = A_i$ or $S = B_i$

By Lemma 3 and (2.5), We have $S = \mathcal{PO}_n$ and this is a contradiction. □

From Theorem 4 and Theorem 15, we know that the maximal regular and the maximal regular subsemibands of $\mathcal{PO}_n$ do not coincide. As an immediate consequence of Theorem 4 and Theorem 9, we have the following:

Corollary 16 *Let $2 \leq r \leq n-2$. Then the maximal regular subsemigroups and maximal regular subsemibands of $\mathcal{PO}(n, r)$ coincide.*

Acknowledgements The authors would like to thank the referee for his/her valuable suggestions and comments which helped improve the presentation of this paper. This work is supported by the National Natural Science Foundation of China (No. 11461014) and the Natural Science Fund of Guizhou (No. [2013]2225).

References

1. Pastijin, F.: Embedding semigroups in semibands. Semigr. Forum **14**(1), 247–263 (1977)
2. Erdos, J.A.: On products of idempotent matrices. Glasg. Math. J. **8**(2), 118–122 (1967)
3. Fountain, J.B., Lewin, A.: Products of idempotent endomorphisms of an independence algebra of finite rank. Proc. Edinb. Math. Soc. **35**(2), 493–500 (1992)
4. Oliveira, A.: Order-independence algebras. Algebra Univers. **39**(3–4), 171–196 (1998)
5. Howie, J.M.: Products of idempotents in certain semigroups of transformations. Proc. Edinb. Math. Soc. **17**(2), 223–236 (1971)
6. Gomes, G.M.S., Howie, J.M.: On the ranks of certain semigroups of order-preserving transformations. Semigr. Forum **45**(3), 272–282 (1992)
7. Garba, G.U.: On the idempotent ranks of certain semigroups of order-preserving transformations. Port. Math. **51**(2), 185–204 (1994)
8. Yang, X.L., Lu, C.H.: Maximal properties of some subsemigroup in finite order-preserving transformation semigroups. Commun. Algebra **28**(7), 3125–3135 (2000)
9. Dimitrova, I., Koppitz, J.: On the maximal regular subsemigroups of ideals of order-preserving or order-reversing transformations. Semigr. Forum **82**(1), 172–180 (2011)
10. Zhao, P.: Maximal regular subsemibands of finite order-preserving transformation semigroups K(n, r). Semigr. Forum **84**(1), 97–115 (2012)
11. Howie, J.M.: Fundamentals of Semigroup Theory, London Mathematical Society Monographs. New Series, 12. Oxford University Press, New York (1995)
12. Eberhart, C., Williams, W., Kinch, L.: Idempotent-generated regular semigroups. J. Aust. Math. Soc. **15**(1), 27–34 (1973)
13. Clifford, H., Preston, G.B.: The Algebraic Theory of Semigroups, vol. 1. American Mathematical Society, Providence (1961)
14. Catarino, P.M., Higgins, P.M.: The monoid of orientation-preserving mappings on a chain. Semigr. Forum **58**(2), 190–206 (1999)
15. Zhao, P., Yang, M.: Maximal properties of some subsemigroups of order-preserving full transformation. Bull. Korean Math. Soc. **50**(2), 627–637 (2013)
16. Zhao, P., Yang, M.: Locally maximal idempotent-generated subsemigroups of finite orientation-preserving singular partial transformation semigroups. Algebra Colloq. **20**(3), 435–442 (2013)