Bull. Malays. Math. Sci. Soc. (2018) 41:709–721
https://doi.org/10.1007/s40840-016-0342-2

BULLETIN of the
MALAYSIAN MATHEMATICAL
SCIENCES SOCIETY
http://www.springer.com/mathematics/journal/40840

CrossMark

Note on the Distribution Composition $(x_+^\mu)^\lambda$

Limonka Lazarova[1] · Biljana Jolevska-Tuneska[2] · İnci Aktürk[3] · Emin Özçağ[3]

Received: 2 September 2015 / Revised: 8 March 2016 / Published online: 28 March 2016

Abstract The composition of the distributions x^λ and x_+^μ is evaluated for $\lambda = -1, -2, \ldots \quad \mu > 0$ and $\lambda\mu \in \mathbb{Z}^-$. Further results are deduced.

Keywords Distribution · Composition of distributions · Neutrix calculus · Neutrix limit · Neutrix composition · Hadamard's finite part

Mathematics Subject Classification Primary 46F10 · Secondary 46F30

1 Introduction

In the following, we let $\mathcal{D}$ be the space of infinitely differentiable functions with compact support, let $\mathcal{D}[a, b]$ be the space of infinitely differentiable functions with

---

Communicated by See Keong Lee.

✉ Biljana Jolevska-Tuneska
biljanaj@feit.ukim.edu.mk

Limonka Lazarova
limonka.lazarova@ugd.edu.mk

Emin Özçağ
ozcag1@hacettepe.edu.tr

1 Department of Mathematics, Faculty of Computer Science, University of Goce Delcev Stip, Štip, Republic of Macedonia

2 Faculty of Electrical Engineering and Informational Technologies, St. Cyrill and Methodius University in Skopje, Karpos II bb, Skopje, Republic of Macedonia

3 Department of Mathematics, Hacettepe University, Beytepe, Ankara, Turkey

support contained in the interval $[a, b]$, and let $\mathcal{D}'$ be the space of distributions defined on $\mathcal{D}$.

we define the locally summable function x_+^λ, for $\lambda > -1$, by

$$x_+^\lambda = \begin{cases} x^\lambda, & x > 0, \\ 0, & x < 0. \end{cases}$$

The distribution x_+^λ is then defined inductively for $\lambda < -1$ and $\lambda \neq -2, -3, \dots$ by $(x_+^\lambda)' = \lambda x_+^{\lambda-1}$. It follows that if s is a positive integer and $-s-1 < \lambda < -s$, then

$$\langle x_+^\lambda, \varphi(x)\rangle = \int_0^\infty x^\lambda \left[\varphi(x) - \sum_{k=0}^{s-1} \frac{\varphi^{(k)}(0)}{k!} x^k\right] dx$$

for arbitrary φ in $\mathcal{D}$. In particular, if φ has its support contained in the interval $[-1, 1]$, then

$$\langle x_+^\lambda, \varphi(x)\rangle = \int_0^1 x^\lambda \left[\varphi(x) - \sum_{k=0}^{s-1} \frac{\varphi^{(k)}(0)}{k!} x^k\right] dx + \sum_{k=0}^{s-1} \frac{\varphi^{(k)}(0)}{k!(\lambda+k+1)}.$$

The distribution x_+^{-s} is defined by

$$x_+^{-s} = \frac{(-1)^{s-1}(\ln x_+)^{(s)}}{(s-1)!}$$

for $s = 1, 2, \dots$ and not as in Gelfand and Shilov [5].

It is easily shown that if φ is an arbitrary function in $\mathcal{D}[-1, 1]$, then

$$\langle x_+^{-s}, \varphi(x)\rangle = \int_0^1 x^{-s} \left[\varphi(x) - \sum\nolimits_{k=0}^{s-1} \frac{\varphi^{(k)}(0)}{k!} x^k\right] dx$$
$$- \sum\nolimits_{k=0}^{s-2} \frac{\varphi^{(k)}(0)}{(s-k-1)k!} - \frac{\phi(s-1)}{(s-1)!}\varphi^{(s-1)}(0), \quad (1)$$

for $s = 1, 2, \dots$, where

$$\phi(s) = \begin{cases} \sum_{k=1}^{s} k^{-1}, & s \geq 1, \\ 0, & s = 0. \end{cases}$$

The composition of a distribution and an infinitely differentiable function is extended to distributions by continuity provided that the derivative of the infinitely differentiable function is different from zero, see [1]. Fisher defined the composition of a distribution F and a summable function f which has a single simple root in the open interval (a, b) and it was recently generalized in [6] by allowing f to be a distribution.

Now let $\rho(x)$ be a function in $\mathcal{D}$ having the following properties:

$$\text{(i)}\ \rho(x) = 0 \text{ for } |x| \geq 1, \qquad \text{(ii)}\ \rho(x) \geq 0,$$
$$\text{(iii)}\ \rho(x) = \rho(-x), \qquad \text{(iv)}\ \int_{-1}^{1} \rho(x)\,dx = 1.$$

Putting $\delta_n(x) = n\rho(nx)$ for $n = 1, 2, \ldots$, it follows that $\{\delta_n(x)\}$ is a regular sequence of infinitely differentiable functions converging to the Dirac delta function $\delta(x)$.

Further, if F is a distribution in $\mathcal{D}'$ and $F_n(x) = \langle F(x-t), \delta_n(x)\rangle$, then $\{F_n(x)\}$ is a regular sequence of infinitely differentiable functions converging to $F(x)$.

The following definition for the neutrix composition of distributions was given in [3] and originally called the composition of distributions.

Definition 1.1 Let F be a distribution in $\mathcal{D}'$ and let f be a locally summable function. We say that the neutrix composition $F(f(x))$ exists and is equal to h on the open interval (a, b) if

$$\mathrm{N}-\lim_{n\to\infty} \int_{-\infty}^{\infty} F_n(f(x))\varphi(x)dx = \langle h(x), \varphi(x)\rangle$$

for all φ in $\mathcal{D}[a, b]$, where $F_n(x) = F(x)*\delta_n(x)$ for $n = 1, 2, \ldots$ and N is the neutrix, see [2], having domain N' the positive integers and range N'' the real numbers, with negligible functions which are finite linear sums of the functions

$$n^\lambda \ln^{r-1} n,\ \ln^r n : \qquad \lambda > 0,\ r = 1, 2, \ldots$$

and all functions which converge to zero in the usual sense as n tends to infinity.

In particular, we say that the composition $F(f(x))$ exists and is equal to h on the open interval (a, b) if

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} F_n(f(x))\varphi(x)dx = \langle h(x), \varphi(x)\rangle$$

for all φ in $\mathcal{D}[a, b]$.

Note that if a function $f(n)$ tends to α in the usual sense as n tends to infinity, it converges to α in the neutrix sense. The reader may find the general definition of the neutrix limit with some examples in [2].

2 Results

Nicholas and Fisher defined the composition $(x_+^r)^{-s}$ as the neutrix limit of regular sequence $[(x_+^r)^{-s}]_n$ for $r, s = 1, 2, \ldots$, see [7]. Further Özçağ et. al. consider the case $r = 0$, in other words the s-th power of the Heaviside function $H(x)$ defined by $[H(x)]^{-s} = H(x)$, see [9]. Recently, some compositions such as $(|x|^{r-1/2})^{-4s}$ and $(|x|^\mu)^{-s}$ were defined in [4,8] respectively.

We first of all need the following Lemma which can be easily proved by induction.

Lemma 2.1

$$\int_{-1}^{1} v^i \rho^{(r)}(v)\,dv = \begin{cases} 0, & 0 \le i < r, \\ (-1)^r r!, & i = r \end{cases} \tag{2}$$

and because $v^r \rho^{(r)}$ *is an even function, we have*

$$\int_{-1}^{0} v^r \rho^{(r)}(v)\,dv = \int_{0}^{1} v^r \rho^{(r)}(v)\,dv = \frac{1}{2}(-1)^r r! \tag{3}$$

and

$$\begin{aligned} \int_{-1}^{0} v^r \ln|v|\rho^{(r)}(v)\,dv &= \int_{0}^{1} v^r \ln|v|\rho^{(r)}(v)\,dv \\ &= \frac{1}{2}(-1)^r r!\phi(r) + (-1)^r r! c(\rho) \end{aligned} \tag{4}$$

for $r = 0, 1, 2, \ldots$ *where* $c(\rho) = \int_0^1 \ln t \rho(t)\,dt$.

We now prove the following theorem.

Theorem 2.2 *The distribution* $(x_+^\mu)^{-m}$ *exists and*

$$(x_+^\mu)^{-m} = x_+^{-s} - (-1)^s \frac{(-1)^m m!\,[2c(\rho) + \phi(m-1)] + s\phi(s-1)}{s!}\delta^{(s-1)}(x) \tag{5}$$

for $\mu > 0, \quad m = 1, 2, \ldots$ *and* $\mu m = s (s \in \mathbb{Z}^+)$.
In particular, we have

$$\left(x_+^{\frac{1}{r}}\right)^{-r} = x_+^{-1} - (-1)^r r!\,[2c(\rho) + \phi(r-1)]\,\delta(x) \tag{6}$$

for $r = 1, 2, \ldots$.

Proof We first put

$$(-1)^{m-1}(m-1)![(x_+^\mu)^{-m}]_n = \begin{cases} \int_{-1/n}^{1/n} \ln|x^\mu - t|\delta_n^{(m)}(t)dt, & if \quad x \ge 0 \\ \int_{-1/n}^{1/n} \ln|t|\delta_n^{(m)}(t)dt, & if \quad x < 0. \end{cases} \tag{7}$$

Then

$$
\begin{aligned}
(-1)^{m-1}(m-1)! \int_{-1}^{1} x^k[(x_+^\mu)^{-m}]_n\,dx &= \int_{-1}^{1} x^k \int_{-1/n}^{1/n} \ln|x_+^\mu - t|\delta_n^{(m)}(t)dtdx \\
&= \int_{-1/n}^{1/n} \delta_n^{(m)}(t) \int_0^{n^{-1/\mu}} x^k \ln|x^\mu - t|\,dx\,dt \\
&\quad + \int_{-1/n}^{1/n} \delta_n^{(m)}(t) \int_{n^{-1/\mu}}^{1} x^k \ln|x^\mu - t|\,dx\,dt + \int_{-1}^{0} x^k \int_{-1/n}^{1/n} \ln|t|\delta_n^{(m)}(t)\,dx\,dt \\
&= \frac{n^{m-(k+1)/\mu}}{\mu} \int_{-1}^{1} \rho^{(m)}(v) \int_0^1 y^{-1+(k+1)/\mu} \ln|y-v|\,dy\,dv \\
&\quad + \frac{n^{m-(k+1)/\mu}}{\mu} \int_{-1}^{1} \rho^{(m)}(v) \int_1^n y^{-1+(k+1)/\mu} \ln|y-v|\,dy\,dv \\
&\quad + \frac{n^{m-(k+1)/\mu}}{\mu} \ln n \int_{-1}^{1} \rho^{(m)}(v)\,dv \int_0^n y^{-1+(k+1)/\mu}\,dy \\
&\quad + \frac{(-1)^{k+1}n^m}{k+1} \int_{-1}^{1} \ln|v/n|\rho^{(m)}(v)\,dv \\
&= I_1 + I_2 + I_3 + I_4,
\end{aligned} \tag{8}
$$

on using the substitutions $y = nx^\mu$ and $v = nt$.

It follows immediately that

$$
\mathrm{N} - \lim_{n\to\infty} I_3 = 0 \qquad \textit{and} \qquad \mathrm{N} - \lim_{n\to\infty} I_4 = 0 \tag{9}
$$

for $k = 0, 1, \ldots$ and

$$
\mathrm{N} - \lim_{n\to\infty} I_1 = 0 \tag{10}
$$

for all $k = 0, 1, 2, \ldots, s-2$.

Further

$$
\begin{aligned}
\int_1^n y^{-1+(k+1)/\mu} \ln|y-v|\,dy &= \int_1^n y^{-1+(k+1)/\mu} \ln y\,dy \\
&\quad + \int_1^n y^{-1+(k+1)/\mu} \ln|1 - v/y|\,dy \\
&= I_2' + I_2'',
\end{aligned} \tag{11}
$$

where

$$
I_2' = \frac{\mu n^{(k+1)/\mu} \ln n}{(k+1)} + \frac{\mu^2[1 - n^{(k+1)/\mu}]}{(k+1)^2} \tag{12}
$$

and

$$I_2'' = -\sum_{i=1}^{\infty} \frac{v^i}{i} \int_1^n y^{-1-i+(k+1)\mu}\, dy$$
$$= -\sum_{i=1}^{\infty} \frac{v^i \mu[n^{-i+(k+1)/\mu} - 1]}{i(k+1-\mu i)} \tag{13}$$

$k = 0, 1, 2, \ldots, s-2$.

It follows from Lemma 2.1 and Eqs. (11), (12), and (13) that

$$\mathrm{N} - \lim_{n\to\infty} I_2 = \frac{(-1)^m (m-1)!}{s-k-1} \tag{14}$$

for $k = 0, 1, \ldots, s-2$, and it then follows from Eqs. (8), (9), (10), and (14) that

$$\mathrm{N} - \lim_{n\to\infty} \int_{-1}^{1} x^k \left[(x_+^{\mu})^{-m}\right]_n dx = -\frac{1}{s-k-1} \tag{15}$$

$k = 0, 1, 2, \ldots, s-2$.

When $k = s-1$, then we have

$$\begin{aligned} I_1 &= \frac{1}{\mu} \int_{-1}^{1} \rho^{(m)}(v) \int_0^1 y^{m-1} \ln|y-v|\, dy\, dv \\ &= \frac{1}{\mu} \int_0^1 \rho^{(m)}(v) \left[\int_0^v + \int_v^1 y^{m-1} \ln|y-v|\, dy\right] dv \\ &\quad + \frac{1}{\mu} \int_{-1}^{0} \rho^{(m)}(v) \left[\int_0^{-v} + \int_{-v}^{1} y^{m-1} \ln|y-v|\, dy\right] dv \\ &= J_1 + J_2 + J_3 + J_4. \end{aligned} \tag{16}$$

Now making the substitution $y = uv$

$$J_1 = \frac{1}{\mu} \int_0^1 v^m \rho^{(m)}(v) \int_0^1 u^{m-1} [\ln v + \ln(1-u)]\, du\, dv, \tag{17}$$

and using Lemma 2.1, we have

$$\int_0^1 v^m \rho^{(m)}(v) \ln v \int_0^1 u^{m-1}\, du\, dv = (-1)^m (m-1)! \left[c(\rho) + \frac{1}{2}\phi(m)\right] \tag{18}$$

and

$$\begin{aligned}
&\int_0^1 v^m \rho^{(m)}(v) \int_0^1 u^{m-1} \ln(1-u)\, du\, dv \\
&= \frac{1}{2}(-1)^m (m-1)! \int_0^1 \ln(1-u) d(u^m - 1) \\
&= \frac{1}{2}(-1)^m (m-1)! \int_0^1 \frac{u^m - 1}{1-u}\, du \\
&= \frac{1}{2}(-1)^{m-1} (m-1)! \phi(m),
\end{aligned} \tag{19}$$

and it follows from Eqs. (17), (18), and (19) that

$$\mathrm{N} - \lim_{n\to\infty} J_1 = \frac{(-1)^m (m-1)! c(\rho)}{\mu}. \tag{20}$$

Similarly, using the substitution $y = uv$ again we have

$$J_3 = -\frac{1}{\mu} \int_{-1}^0 v^m \rho^{(m)}(v) \int_{-1}^0 u^{m-1} \left[\ln|v| + \ln(1-u)\right] du\, dv. \tag{21}$$

Thus

$$\begin{aligned}
\int_{-1}^0 v^m \rho^{(m)}(v) \ln|v| \int_{-1}^0 u^{m-1}\, du\, dv &= \frac{(-1)^{m-1}}{m} \int_{-1}^0 v^m \rho^{(m)}(v) \ln|v|\, dv \\
&= -(m-1)! \left[c(\rho) + \frac{1}{2}\phi(m) \right],
\end{aligned} \tag{22}$$

and

$$\begin{aligned}
&\int_{-1}^0 v^m \rho^{(m)}(v) \int_{-1}^0 u^{m-1} \ln(1-u)\, du\, dv \\
&= \frac{1}{2}(-1)^m (m-1)! \int_{-1}^0 \ln(1-u) d(u^m - 1) \\
&= \frac{1}{2}[(-1)^m - 1](m-1)! \ln 2 - \frac{1}{2}(-1)^m (m-1)! \int_{-1}^0 \frac{u^m - 1}{u-1}\, du \\
&= \frac{1}{2}[(-1)^m - 1](m-1)! \ln 2 + \frac{1}{2}(-1)^m (m-1)! \sum_{i=1}^m \frac{(-1)^i}{i}.
\end{aligned} \tag{23}$$

It follows from Eqs. (21), (22), and (23) that

$$\mathrm{N}-\lim_{n\to\infty} J_3 = \frac{[1-(-1)^m](m-1)!}{2\mu}\ln 2 + \frac{(m-1)!}{2\mu}\phi(m) + \\ -\frac{(m-1)!}{2\mu}\sum_{i=1}^{m}\frac{(-1)^i}{i} + \frac{(m-1)!}{\mu}c(\rho). \quad (24)$$

Further

$$\begin{aligned}
J_2 &= \frac{1}{\mu}\int_0^1 \rho^{(m)}(v)\int_v^1 y^{m-1}\,[\ln y + \ln(1-v/y)]\,dy\,dv \\
&= \frac{1}{\mu}\int_0^1 \rho^{(m)}(v)\int_v^1 y^{m-1}\ln y\,dy\,dv - \frac{1}{\mu}\sum_{i=1}^{\infty}\frac{1}{i}\int_0^1 v^i\rho^{(m)}(v)\int_v^1 y^{m-i-1}\,dy\,dv \\
&= \frac{(-1)^m(m-1)!}{2\mu m} - \frac{1}{\mu m^2}\int_0^1 \rho^{(m)}(v)\,dv + \\
&\quad -\frac{1}{\mu}\sum_{i=1,\,i\neq m}^{\infty}\frac{1}{i(m-i)}\int_0^1 (v^i - v^m)\rho^{(m)}(v)\,dv \\
&= \frac{(-1)^m(m-1)!}{2\mu m} + \frac{\rho^{(m-1)}(0)}{\mu m^2} + \frac{(-1)^m(m-1)!}{2\mu}\,[2\phi(m-1)-\phi(m)] \\
&\quad -\frac{1}{\mu}\sum_{i=1,\,i\neq m}^{\infty}\frac{1}{i(m-i)}\int_0^1 v^i\rho^{(m)}(v)\,dv \\
&= \frac{\rho^{(m-1)}(0)}{\mu m^2} + \frac{(-1)^m(m-1)!}{2\mu}\phi(m-1) + \\
&\quad -\frac{1}{\mu}\sum_{i=1,\,i\neq m}^{\infty}\frac{1}{i(m-i)}\int_0^1 v^i\rho^{(m)}(v)\,dv \qquad (25)
\end{aligned}$$

because

$$\sum_{i=1,\,i\neq m}^{\infty}\frac{1}{i(m-i)} = \frac{2\phi(m-1)-\phi(m)}{m} = \frac{\phi(m-1)}{m} - \frac{1}{m^2}.$$

Finally, we have

$$\begin{aligned}
J_4 &= \frac{1}{\mu}\int_{-1}^0 \rho^{(m)}(v)\int_{-v}^1 y^{m-1}\,[\ln y + \ln(1-v/y)]\,dy\,dv \\
&= \frac{1}{\mu}\int_{-1}^0 \rho^{(m)}(v)\int_{-v}^1 y^{m-1}\ln y\,dy\,dv - \frac{1}{\mu}\sum_{i=1}^{\infty}\frac{1}{i}\int_{-1}^0 v^i\rho^{(m)}(v)\int_{-v}^1 y^{m-i-1}\,dy\,dv \\
&= \frac{1}{\mu}\int_{-1}^0 \left[\frac{(-v)^m-1}{m^2} - \frac{(-v)^m\ln|v|}{m}\right]\rho^{m)}(v)\,dv + \frac{1}{\mu m}\int_{-1}^0 v^m\ln|v|\rho^{(m)}(v)\,dv
\end{aligned}$$

$$
\begin{aligned}
&-\frac{1}{\mu}\sum_{i=1,\,i\neq m}^{\infty}\frac{1}{i(m-i)}\int_{-1}^{0}\left[v^i-(-1)^{m-i}v^m\right]\rho^{(m)}(v)\,dv \\
&=\frac{(m-1)!}{2\mu m}-\frac{\rho^{(m-1)}(0)}{\mu m^2}-\frac{[1-(-1)^m]m!}{2\mu}[\phi(m)+2c(\rho)]+ \\
&\quad-\frac{1}{\mu}\sum_{i=1,\,i\neq m}^{\infty}\frac{1}{i(m-i)}\int_{-1}^{0}v^i\rho^{(m)}(v)\,dv-\frac{(-1)^m(m-1)!}{2\mu m} \\
&\quad+\frac{(m-1)!}{2\mu}\sum_{i=1}^{m-1}\frac{(-1)^{m+i}}{i}-\frac{[1-(-1)^m](m-1)!\ln 2}{2\mu}
\end{aligned}
\tag{26}
$$

because

$$
\sum_{i=1,\,i\neq m}^{\infty}\frac{1}{i(m-i)}=-\frac{(-1)^m}{m^2}+\frac{1}{m}\sum_{i=1}^{m-1}\frac{(-1)^{m+i}}{i}-\frac{[1-(-1)^m]}{m}\ln 2.
$$

Next, we similarly have

$$
\begin{aligned}
I_2&=\frac{1}{\mu}\int_{-1}^{1}\rho^{(m)}(v)\int_{1}^{n}y^{m-1}\left[\ln y+\ln(1-v/y)\right]dy\,dv \\
&=\frac{1}{\mu}\int_{-1}^{1}\rho^{(m)}(v)\int_{1}^{n}y^{m-1}\ln(1-v/y)\,dy\,dv \\
&=-\frac{1}{\mu}\sum_{i=1}^{\infty}\frac{1}{i}\int_{-1}^{1}v^i\rho^{(m)}(v)\int_{1}^{n}y^{m-i-1}\,dy\,dv \\
&=-\frac{(-1)^m(m-1)!\ln n}{\mu}-\frac{1}{\mu}\sum_{i=m+1}^{\infty}\frac{1}{i(i-m)}\int_{-1}^{1}(n^{m-i}-1)v^i\rho^{(m)}(v)\,dv,
\end{aligned}
$$

and so

$$
\mathrm{N}-\lim_{n\to\infty} I_2=-\frac{1}{\mu}\sum_{i=m+1}^{\infty}\frac{1}{i(i-m)}\int_{-1}^{1}v^i\rho^{(m)}(v)\,dv. \tag{27}
$$

It now follows from Eqs. (8), (9), (16), (20), and (24)–(27) that

$$
\begin{aligned}
\mathrm{N}-\lim_{n\to\infty}\int_{-1}^{1}x^{s-1}\left[(x_+^{\mu})^{-m}\right]_n dx&=\frac{(-1)^m(m-1)!}{\mu}[2c(\rho)+\phi(m-1)] \\
&=\frac{(-1)^m m!}{s}[2c(\rho)+\phi(m-1)].
\end{aligned}
\tag{28}
$$

We now consider the case $k = s$. If $x < 0$ and ψ is an arbitrary continuous function, then

$$\begin{aligned}(-1)^{m-1}(m-1)! \int_{-1}^{0} x^s \left[(x_+^\mu)^{-m}\right]_n \psi(x)\, dx \\ = \int_{-1}^{0} x^s \psi(x) \int_{-1/n}^{1/n} \ln|t| \delta_n^{(m)}(t)\, dt\, dx \\ = n^m \int_{-1}^{0} x^s \psi(x)\, dx \int_{-1}^{1} \ln|v/n| \rho^{(m)}(v)\, dv,\end{aligned}$$

where $v = nt$. Thus

$$\mathrm{N} - \lim_{n\to\infty} \int_{-1}^{0} x^s \left[(x_+^\mu)^{-m}\right]_n \psi(x)\, dx = 0. \tag{29}$$

Next with $k = s$ in I_1, we see that

$$\int_0^{n^{-1/\mu}} x^s \left[(x_+^\mu)^{-m}\right]_n\, dx = \frac{n^{-1/\mu}}{\mu} \int_{-1}^{1} \rho^{(m)}(v) \int_0^1 y^{m-1+1/\mu} \ln|(y-v)/n|\, dy\, dv$$

and if ψ is any continuous function, then

$$\lim_{n\to\infty} \int_0^{n^{-1/\mu}} x^s \left[(x_+^\mu)^{-m}\right]_n \psi(x)\, dx = 0. \tag{30}$$

If $x^\mu \geq \frac{1}{n}$, then we have

$$\begin{aligned}(-1)^{m-1}(m-1)! \left[(x_+^\mu)^{-m}\right]_n &= \int_{-1/n}^{1/n} \ln|x^\mu - t| \delta_n^{(m)}(t)\, dt \\ &= n^m \int_{-1}^{1} \ln|x^\mu - v/n| \rho^{(m)}(v)\, dv \\ &= n^m \int_{-1}^{1} \left[\ln|x^\mu - \sum_{i=1}^{\infty} \frac{v^i}{in^i x^{\mu i}}\right] \rho^{(m)}(v)\, dv \\ &= -\sum_{i=m}^{\infty} \int_{-1}^{1} \frac{v^i}{in^{i-m} x^{\mu i}} \rho^{(m)}(v)\, dv,\end{aligned}$$

and it follows that

$$\begin{aligned}\left|(m-1)! \left[(x_+^\mu)^{-m}\right]_n\right| &\leq \sum_{i=m}^{\infty} \int_{-1}^{1} \frac{|v|^i}{in^{i-m} x^{\mu i}} |\rho^{(m)}(v)|\, dv \\ &\leq \sum_{i=m}^{\infty} \frac{\kappa_m}{in^{i-m} x^{\mu i}},\end{aligned}$$

where $\kappa_m = \int_{-1}^{1} |\rho^{(m)}(v)|\, dv$ for $m = 1, 2, \ldots$.

If now $n^{-1/\mu} < \eta < 1$

$$(m-1)! \int_{n^{-1/\mu}}^{\eta} \left| x^s \left[(x_+^\mu)^{-m}\right]_n \right| dx \leq \kappa_m \sum_{i=m}^{\infty} \frac{n^{m-i}}{i} \int_{n^{-1/\mu}}^{\eta} x^{s-\mu i}\, dx$$
$$= \kappa_m \sum_{i=m}^{\infty} \frac{n^{-1/\mu}}{\mu i} \int_{1}^{n\eta^\mu} y^{m-i+1/\mu-1}\, dy$$
$$= \begin{cases} \kappa_m \sum_{i=m}^{\infty} \frac{n^{-1/\mu}}{\mu i(m-i+1/\mu)} \left[(n\eta^\mu)^{m-i+1/\mu} - 1\right], & \mu \neq 1 \\ \kappa_m \sum_{i=m, i\neq m+1}^{\infty} \frac{n^{-1}}{i(m-i+1)} \left[(n\eta)^{m-i+1} - 1\right] + \frac{\kappa_m n^{-1} \ln(n\eta)}{m+1}, & \mu = 1. \end{cases}$$

It follows that

$$\lim_{n\to\infty} \left| \left[(x_+^\mu)^{-m}\right]_n \right| = O(\eta)$$

for $m = 1, 2, \ldots$ and if ψ is any continuous function then

$$\lim_{n\to\infty} \left| \int_{n^{-1/\mu}}^{\eta} x^s \left[(x_+^\mu)^{-m}\right]_n \psi(x)\, dx \right| = O(\eta) \tag{31}$$

for $m = 1, 2, \ldots$. Now let $\varphi(x) \in \mathcal{D}[-1, 1]$. By Taylor's theorem, we have

$$\varphi(x) = \sum_{k=0}^{s-1} \frac{\varphi^{(k)}(0)}{k!} x^k + \frac{x^s}{s!} \varphi^{(s)}(\xi x) \qquad (0 < \xi < 1).$$

Then

$$\langle \left[(x_+^\mu)^{-m}\right]_n, \varphi(x) \rangle = \sum_{k=0}^{s-1} \frac{\varphi^{(k)}(0)}{k!} \int_{-1}^{1} x^k \left[(x_+^\mu)^{-m}\right]_n dx$$
$$+ \frac{1}{s!} \int_{-1}^{0} x^s \left[(x_+^\mu)^{-m}\right]_n \varphi^{(s)}(\xi x)\, dx$$
$$+ \frac{1}{s!} \int_{0}^{n^{-1/\mu}} x^s \left[(x_+^\mu)^{-m}\right]_n \varphi^{(s)}(\xi x)\, dx$$
$$+ \frac{1}{s!} \int_{n^{-1/\mu}}^{\eta} x^s \left[(x_+^\mu)^{-m}\right]_n \varphi^{(s)}(\xi x)\, dx$$
$$+ \frac{1}{s!} \int_{\eta}^{1} x^s \left[(x_+^\mu)^{-m}\right]_n \varphi^{(s)}(\xi x)\, dx.$$

Using Eqs. (15) and (28)–(31) and noting that the sequence $\{\big[(x_+^\mu)^{-m}\big]_n\}$ converges uniformly to x^{-s} on the interval $[\eta, 1]$, it follows that

$$\mathrm{N}-\lim_{n\to\infty}\langle\big[(x_+^\mu)^{-m}\big]_n, \varphi(x)\rangle = \frac{(-1)^m m!}{s!}\,[2c(\rho)+\phi(m-1)]\,\varphi^{(s-1)}(0) - \sum_{k=0}^{s-2}\frac{\varphi^{(k)}(0)}{k!(s-k-1)} + \int_\eta^1 \varphi^{(s)}(\xi x)\,dx + O(\eta).$$

Because η can be arbitrarily small, it follows that

$$\begin{aligned}
&\mathrm{N}-\lim_{n\to\infty}\langle\big[(x_+^\mu)^{-m}\big]_n, \varphi(x)\rangle \\
&= \frac{(-1)^m m!}{s!}\,[2c(\rho)+\phi(m-1)]\,\varphi^{(s-1)}(0) - \sum_{k=0}^{s-2}\frac{\varphi^{(k)}(0)}{k!(s-k-1)} \\
&\quad + \int_\eta^1 \varphi^{(s)}(\xi x)\,dx + O(\eta). \\
&= \int_0^1 x^{-s}\left[\varphi(x) - \sum_{k=0}^{s-1}\frac{\varphi^{(k)}(0)}{k!}x^k\right]dx - \sum_{k=0}^{s-2}\frac{\varphi^{(k)}(0)}{k!(s-k-1)} \\
&\quad + \frac{(-1)^m m!}{s!}\,[2c(\rho)+\phi(m-1)]\,\varphi^{(s-1)}(0) \\
&= \langle x_+^{-s}, \varphi(x)\rangle + \\
&\quad -(-1)^s\,\frac{(-1)^m m!\,[2c(\rho)+\phi(m-1)] + s\phi(s-1)}{s!}\langle\delta^{(s-1)}(x), \varphi(x)\rangle
\end{aligned}$$

on using Eq. (1). This proves Eq. (5) on the interval $[-1, 1]$. However, Eq. (5) clearly holds on any interval not containing the origin, and the proof is complete. □

Note that if $\mu \in \mathbb{Z}^+$, then Theorem 1 is in agreement with the Theorem 3 given in [7].

Corollary 2.3 *The distribution $(x_-^\mu)^{-m}$ exists and*

$$(x_-^\mu)^{-m} = x_-^{-s} + \frac{(-1)^m m!\,[2c(\rho)+\phi(m-1)] + s\phi(s-1)}{s!}\delta^{(s-1)}(x) \qquad (32)$$

for $\mu > 0, \quad m = 1, 2, \ldots$ *and* $\mu m = s \in \mathbb{Z}^+$.

Proof Equation (32) follows after replacing x by $-x$ in Eq. (5). □

Acknowledgements The second author was supported by Tubitak. This work was supported by the research project "Functional spaces, topological and statistical aspects and their application in electrical engineering" financed by State University "Goce Delcev"—Stip, Republic of Macedonia.

Authors' Contributions All authors contributed equally and significantly in writing this paper. All authors read and approved the final manuscript.

Compliance with Ethical Standards

Conflict of interest The authors declare that they have no conflict of interest.

References

1. Antosik, P., Mikusinski, J., Sikorski, R.: Theory of Distributions. The Sequential Approach. PWN-Elsevier, Amsterdam (1973)
2. van der Corput, J.G.: Introduction to the neutrix calculus. J. Anal. Math. **7**, 291–398 (1959)
3. Fisher, B.: On defining the change of variable in distributions. Rostock. Math. Kolloq. **28**, 75–86 (1985)
4. Fisher, B., Jolevska-Tuneska, B., Özçağ, E.: Further results on the compositions of distributions. Integral Transforms Spec. Funct. **13**, 109–116 (2002)
5. Gel'fand, I.M., Shilov, G.E.: Generalized Functions, vol. I. Academic Press, New York (1964)
6. Kou, H., Fisher, B.: On composition of distributions. Publ. Math. Debr. **40**(3–4), 279–290 (1992)
7. Nicholas, J.D., Fisher, B.: The distribution composition $(x_+^r)^{-s}$. J. Math. Anal. Appl. **258**, 131–145 (2001)
8. Özçağ, E.: On the composition of the distributions x^{-s} and $|x|^\mu$, (submitted)
9. Özçağ, E., Ege, I., Gürçay, H.: On powers of the heaviside function for negative integers. J. Math. Anal. Appl. **326**, 101–107 (2007)