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Property of Solution to Axially Symmetric Incompressible MHD Equations in Three Dimensions

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Abstract In this paper, we study the property of solutions to axially symmetric incompressible MHD equations in three dimensions. First, we present the three-dimensional axially symmetric incompressible MHD equations. We propose a new one-dimensional model that approximates the MHD equations along the symmetric axis. This one-dimensional model can construct a family of exact solutions to the three-dimensional MHD equations. Second, we give a family of particular solutions to the three-dimensional and a prior estimates of one-dimensional MHD equations. Finally, we construct a family of global smooth solutions to the three-dimensional MHD equations by applying the one-dimensional solutions.

Keywords Incompressible · MHD · Axially asymmetric · Three dimensional

Mathematics Subject Classification 35B07 · 35B65 · 35B30

1 Introduction

Magnetohydrodynamics (MHD) [1] (magneto fluid dynamics or hydromagnetics) is the study of dynamics of electrically conducting fluids. Such fluids include plasmas, liquid metals, salt water, and electrolytes. The word magnetohydrodynamic is derived from magneto-meaning magnetic field, hydro-meaning liquid, and dynamics meaning

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movement. The field of MHD was initiated by Hannes Alfvén, for which he received the Nobel Prize in Physics in 1970.

The fundamental concept behind MHD is that magnetic fields can induce currents in a moving conductive fluid which in turn creates forces on the fluid and also changes the magnetic field itself. The set of equations which describe MHD are a combination of the Navier–Stokes equations of fluid dynamics and Maxwell's equations of electromagnetism.

This paper concerns itself with the incompressible three-dimensional MHD:

$$\begin{cases} \mathbf{u}_t + (\mathbf{u}\cdot\nabla)\mathbf{u} = -\nabla p + \mu\Delta\mathbf{u} + (\nabla\times\mathbf{B})\times\mathbf{B}, \\ \mathbf{B}_t = \nu\triangle\mathbf{B} + \nabla\times(\mathbf{u}\times\mathbf{B}), \\ \nabla\cdot\mathbf{u} = 0, \quad \nabla\cdot\mathbf{B} = 0. \end{cases} \tag{1.1}$$

Here, $\mathbf{B}(\mathbf{x}, t)$ denotes the magnetic field. $\mathbf{u}(\mathbf{x}, t)$ is the velocity field. $p(\mathbf{x}, t)$ is the pressure. $\mu \geq 0$ is the fluid viscocity. $\nu \geq 0$ is the resistivity. First, using the transformation of cylindrical coordinate, we obtain the three-dimensional axially symmetric incompressible MHD equation: (2.13) and (2.14). Second, we get a new one-dimensional model, and this model approximates the three-dimensional asymmetric MHD along the symmetry axis. By expanding the angular velocity u^θ, the angular vorticity ω^θ, the angular stream function ψ^θ, the angular magnetic B^θ, the angular current density j^θ, and the angular magnetic stream function Φ^θ around $r = 0$. We neglect the high-order terms in r and assume that the second partial derivative of $u_1^\theta,\ \omega_1^\theta,\ \psi_1^\theta,\ B_1^\theta,\ j_1^\theta$, and Φ_1^θ with respect to z is much larger than the functions with respect to r. We get the one-dimensional coupled nonlinear partial differential equations are

$$(u_1^\theta)_t - 2(\psi_1^\theta)_z u_1^\theta + 2\psi_1^\theta(u_1^\theta)_z = (u_1^\theta)_{zz} - 2(\Phi_1^\theta)_z B_1^\theta + 2\Phi_1^\theta(B_1^\theta)_z, \tag{1.2}$$

$$(\omega_1^\theta)_t + 2\psi_1^\theta(\omega_1^\theta)_z - [(u_1^\theta)^2]_z = (\omega_1^\theta)_{zz} + 2\Phi_1^\theta(j_1^\theta)_z - [(B_1^\theta)^2]_z, \tag{1.3}$$

$$-(\psi_1^\theta)_{zz} = \omega_1^\theta, \tag{1.4}$$

$$(B_1^\theta)_t + 2\psi_1^\theta(B_1^\theta)_z = (B_1^\theta)_{zz} + 2\Phi_1^\theta(u_1^\theta)_z, \tag{1.5}$$

$$\begin{aligned}(j_1^\theta)_t - (\psi_1^\theta)_z j_1^\theta + 2\psi_1^\theta(j_1^\theta)_z = (j_1^\theta)_{zz} - (\Phi_1^\theta)_z\omega_1^\theta + 2\Phi_1^\theta(\omega_1^\theta)_z \\ +3(\psi_1^\theta)_z(\Phi_1^\theta)_{zz} - 3(\Phi_1^\theta)_z(\psi_1^\theta)_{zz},\end{aligned} \tag{1.6}$$

$$-(\Phi_1^\theta)_{zz} = j_1^\theta. \tag{1.7}$$

Third, we construct a family of exact solutions from the above one-dimensional model. If $(u_1^\theta, \omega_1^\theta, \psi_1^\theta, B_1^\theta, j_1^\theta, \Phi_1^\theta)$ is an exact solution to one-dimensional model, then

$$\begin{aligned} &u^\theta(r,z,t) = ru_1^\theta(z,t), \quad \omega^\theta(r,z,t) = r\omega_1^\theta(z,t) \quad \psi^\theta(r,z,t) = r\psi_1^\theta(z,t), \\ &B^\theta(r,z,t) = rB_1^\theta(z,t), \quad j^\theta(r,z,t) = rj_1^\theta(z,t) \quad \Phi^\theta(r,z,t) = r\Phi_1^\theta(z,t). \end{aligned} \tag{1.8}$$

is an exact solution to the three-dimensional axisymmetric MHD equations (Theorem 2.1). Fourth, we consider a family of particular solutions to three-dimensional axisymmetric MHD equations $u^\theta = 0$ and $B^\theta = 0$, we arrive at

$$\begin{cases} (\bar{\omega})_t + 2\bar{\psi}(\bar{\omega})_z = (\bar{\omega})_{zz} + 2\bar{\Phi}(\bar{j})_z, \\ (\bar{v})_t + 2\bar{\psi}(\bar{v})_z + (\bar{v})^2 = (\bar{v})_{zz} + 2\bar{\Phi}(\bar{l})_z + \bar{l}^2 + e(t), \\ -(\bar{\psi})_{zz} = \bar{\omega}, \\ (\bar{j})_t + \bar{v}\bar{j} + 2\bar{\psi}(\bar{j})_z = (\bar{j})_{zz} + \bar{l}\bar{\omega} + 2\bar{\Phi}(\bar{\omega})_z + 3\bar{v}(\bar{l})_z - 3\bar{l}(\bar{v})_z, \\ (\bar{l})_t + 2\bar{\psi}(\bar{l})_z = (\bar{l})_{zz} + 2\bar{\Phi}(\bar{v})_z + f(t), \\ -(\bar{\Phi})_{zz} = \bar{j}. \end{cases} \tag{1.9}$$

where $\bar{v} = -(\bar{\psi})_z,\ \bar{l} = -(\bar{\Phi})_z, e(t) = 3\int_0^1 \bar{v}^2 \mathrm{d}z + \int_0^1 \bar{B}^2 \mathrm{d}z$, and $d(t) = 0$. Moreover, we get a prior estimates of $\bar{B}(z,t)$ and $\bar{v}(z,t)$ (Theorems 3.2, 3.3). Finally, we construct a family of globally smooth solutions to the three-dimensional MHD using the particular solutions to the one-dimensional model (1.9), then

$$\begin{aligned} u^\theta(r,z,t) &= r[\bar{u}_1(z,t)\phi(r) + u_1(r,z,t)], & \omega^\theta(r,z,t) &= r[\bar{\omega}_1(z,t)\phi(r) + \omega_1(r,z,t)], \\ \psi^\theta(r,z,t) &= r[\bar{\psi}_1(z,t)\phi(r) + \psi_1(r,z,t)], & B^\theta(r,z,t) &= r[\bar{B}_1(z,t)\phi(r) + B_1(r,z,t)], \\ j^\theta(r,z,t) &= r[\bar{j}_1(z,t)\phi(r) + j_1(r,z,t)], & \Phi^\theta(r,z,t) &= r[\bar{\Phi}_1(z,t)\phi(r) + \Phi_1(r,z,t)], \end{aligned}$$

where

$$\begin{aligned} &(u_1^\theta(z,t), \omega_1^\theta(z,t), \psi_1^\theta(z,t), B_1^\theta(z,t), j_1^\theta(z,t), \Phi_1^\theta(z,t)) \\ &\quad = (0, \bar{\omega}_1(z,t), \bar{\psi}_1(z,t), \bar{B}_1(z,t), 0, 0) \end{aligned}$$

is a particular solution to (1.2)–(1.7). $\phi(r)$ is a cut-off function to ensure that the solution has finite energy. We prove that there exist a family of globally smooth functions $u_1(r,z,t),\ \omega_1(r,z,t),\ \psi_1(r,z,t),\ B_1(r,z,t),\ j_1(r,z,t)$, and $\Phi_1(r,z,t)$ such that $\tilde{u}^\theta(r,z,t),\ \tilde{\omega}^\theta(r,z,t),\ \tilde{\psi}^\theta(r,z,t),\ \tilde{B}^\theta(r,z,t),\ \tilde{j}^\theta(r,z,t)$, and $\tilde{\Phi}^\theta(r,z,t)$ are solution to three-dimensional axisymmetric MHD (Theorem 4.1).

Let us mention some important results in the field of incompressible MHD equations. Caflish and Klapper [2] derived analogous conservation results of incompressible ideal MHD (i.e., zero viscocity and resistivity) for both energy and helicity. Moreover, they derived necessary condition for singularity development in ideal MHD generalizing the Beale–Kato–Majda condition for ideal hydrodynamics. He and Xin [3] derived an estimate of Hausdorff dimension on the possible singular set of a suitable weak solution as in the case of pure fluid. Various partial regularity results are obtained as consequences of his blow-up estimates. Zhou and Gala [4] proved that if the velocity field satisfies $u \in L^{\frac{2}{1-\gamma}}(O,T,\dot{X}_\gamma(R^3)), \gamma \in [0,1]$ or the gradient field of velocity satisfies $\nabla u \in L^{\frac{2}{2-\gamma}}(O,T,\dot{X}_\gamma(R^3)), \gamma \in [0,1]$, then the solution remains smooth on [0,T]. Cao and Wu [5,6] established two regularity criteria for the three-dimensional incompressible MHD equations: the first one was in terms of the derivative of the velocity field in one direction, while the second one required suitable boundedness of the derivatives of the pressure in one direction in 2010. They proved global regularity for the two-dimensional MHD equations with mixed partial dissipation and magnetic diffusion in 2011. Wang and Wang

[7] proved if the initial data satisfy $||u_0||_{H^1} + ||b_0||_{H^1} \le \varepsilon$, where ε is a sufficiently small positive number, then the 3D MHD equations with mixed partial dissipation and magnetic diffusion admit global smooth solutions. Tran and Yu [8] proved that if the dissipation terms are $-\nu(-\Delta)^{\alpha}u$ and $-\kappa(-\Delta)^{\beta}b$, smooth solutions are global in three cases: $\alpha \ge \frac{1}{2}, \beta \ge 1$; $0 \le \alpha \le \frac{1}{2}, 2\alpha + \beta > 2$; and $\alpha \ge 2, \beta = 0$. Recently, Lei [9] proved the global regularity of axially symmetric solution to the ideal MHD in three dimension for a family of non-trial magnetic fields.

The rest of this paper is divided into three sections. The second section is devoted to the derivation of the one-dimensional model (2.17)–(2.22) and construct an exact solution to three-dimensional model. The third section presents a family of particular solutions equations: (3.10)–(3.11) and a prior estimates of $\bar{v}(z,t)$ and $\bar{l}(z,t)$. Finally, we construct a family of global smooth solutions to three-dimensional MHD equations using the particular solutions of the one-dimensional model.

2 Derivation of the One-dimensional Model

In this section, we derive the three-dimensional incompressible axially symmetric MHD equations in cylindrical coordinate. Consider the following three-dimensional resistive MHD equations

$$\begin{cases} \mathbf{u}_t + (\mathbf{u}\cdot\nabla)\mathbf{u} = -\nabla p + \mu\Delta\mathbf{u} + (\mathbf{B}\cdot\nabla)\mathbf{B} - \frac{1}{2}\nabla|\mathbf{B}|^2, \\ \mathbf{B}_t + (\mathbf{u}\cdot\nabla)\mathbf{B} = \nu\Delta\mathbf{B} + (\mathbf{B}\cdot\nabla)\mathbf{u}, \\ \mathbf{u}|_{t=0} = \mathbf{u}_0(\mathbf{x}), \quad \mathbf{B}|_{t=0} = \mathbf{B}_0(\mathbf{x}), \quad \mathbf{x} = (x,y,z). \end{cases} \tag{2.1}$$

To simplify, we set the constant μ and ν to be 1. The incompressible and the magnetic divergence constraints are

$$\nabla\cdot\mathbf{u} = 0, \qquad \nabla\cdot\mathbf{B} = 0. \tag{2.2}$$

Let

$$\mathbf{e}_r = \left(\frac{x}{r}, \frac{y}{r}, 0\right), \qquad \mathbf{e}_\theta = \left(-\frac{y}{r}, \frac{x}{r}, 0\right), \qquad \mathbf{e}_z = (0,0,1), \tag{2.3}$$

be three unit orthogonal vectors along the radial, the angular and the axial direction, respectively, $r = \sqrt{x^2+y^2}$.

An axially asymmetric solution to the three-dimensional incompressible MHD (2.1) and (2.2) is a solution $(\mathbf{u}, \mathbf{B}, p)$ which takes the following form

$$\begin{cases} \mathbf{u}(\mathbf{x},t) = u^r(r,z,t)\mathbf{e}_r + u^\theta(r,z,t)\mathbf{e}_\theta + u^z(r,z,t)\mathbf{e}_z, \\ \mathbf{B}(\mathbf{x},t) = B^r(r,z,t)\mathbf{e}_r + B^\theta(r,z,t)\mathbf{e}_\theta + B^z(r,z,t)\mathbf{e}_z, \\ \mathbf{p}(\mathbf{x},t) = p(r,z,t). \end{cases} \tag{2.4}$$

The first equality of (2.1) can be expressed in cylindrical coordinate

$$\begin{cases} (u^r)_t + u^r(u^r)_r - \frac{1}{r}(u^\theta)^2 + u^z(u^r)_z = -p_r + \left(\nabla^2 - \frac{1}{r^2}\right) u^r + B^r(B^r)_r \\ \quad -\frac{1}{r}(B^\theta)^2 + B^z(B^r)_z - [B^r(B^r)_r + B^\theta(B^\theta)_r + B^z(B^z)_r], \\ (u^\theta)_t + u^r(u^\theta)_r + \frac{1}{r}u^\theta u^r + u^z(u^\theta)_z = \left(\nabla^2 - \frac{1}{r^2}\right) u^\theta + B^r(B^\theta)_r \\ \quad +\frac{1}{r}B^\theta B^r + B^z(B^\theta)_z, \\ (u^z)_t + u^r(u^z)_r + u^z(u^z)_z = -p_z + \nabla^2 u^z + B^r(B^z)_r + B^z(B^z)_z \\ \quad -[B^r(B^r)_z + B^\theta(B^\theta)_z + B^z(B^z)_z]. \end{cases} \tag{2.5}$$

The second equality of (2.1) is

$$\begin{cases} (B^r)_t + u^r(B^r)_r + u^z(B^r)_z = \left(\nabla^2 - \frac{1}{r^2}\right) B^r + B^r(u^r)_r + B^z(u^r)_z, \\ (B^\theta)_t + u^r(B^\theta)_r + \frac{1}{r}u^\theta B^r + u^z(B^\theta)_z = \left(\nabla^2 - \frac{1}{r^2}\right) B^\theta + B^r(u^\theta)_r \\ \quad +\frac{1}{r}u^r B^\theta + B^z(u^\theta)_z, \\ (B^z)_t + u^r(B^z)_r + u^z(B^z)_z = \nabla^2 B^z + B^r(u^z)_r + B^z(u^z)_z, \end{cases} \tag{2.6}$$

where $\nabla^2 = \partial_r^2 + \frac{1}{r}\partial r + \partial_z^2$. The incompressible constraints (2.2) are

$$(u^r)_r + \frac{1}{r}u^r + (u^z)_z = 0, \qquad (B^r)_r + \frac{1}{r}B^r + (B^z)_z = 0. \tag{2.7}$$

If the pressure is given by $P = p + |\mathbf{B}|^2/2$, (2.5) can be simplified

$$\begin{cases} (u^r)_t + u^r(u^r)_r - \frac{1}{r}(u^\theta)^2 + u^z(u^r)_z \\ \quad = -P_r + \left(\nabla^2 - \frac{1}{r^2}\right) u^r + B^r(B^r)_r - \frac{1}{r}(B^\theta)^2 + B^z(B^r)_z, \\ (u^\theta)_t + u^r(u^\theta)_r + \frac{1}{r}u^\theta u^r + u^z(u^\theta)_z \\ \quad = \left(\nabla^2 - \frac{1}{r^2}\right) u^\theta + B^r(B^\theta)_r + \frac{1}{r}B^\theta B^r + B^z(B^\theta)_z, \\ (u^z)_t + u^r(u^z)_r + u^z(u^z)_z = -P_z + \nabla^2 u^z + B^r(B^z)_r + B^z(B^z)_z. \end{cases} \tag{2.8}$$

Similarly, the vorticity field $\boldsymbol{\omega}$ and the current density $\mathbf{J}$ can be expressed in cylindrical coordinate

$$\boldsymbol{\omega}(\mathbf{x}, t) = \nabla \times \mathbf{u}(\mathbf{x}, t) = \omega^r(r, z, t)\mathbf{e}_r + \omega^\theta(r, z, t)\mathbf{e}_\theta + \omega^z(r, z, t)\mathbf{e}_z, \tag{2.9}$$

$$\mathbf{J}(\mathbf{x}, t) = \nabla \times \mathbf{B}(\mathbf{x}, t) = j^r(r, z, t)\mathbf{e}_r + j^\theta(r, z, t)\mathbf{e}_\theta + j^z(r, z, t)\mathbf{e}_z, \tag{2.10}$$

where

$$\begin{aligned} &\omega^r = -(u^\theta)_z, \quad \omega^\theta = (u^r)_z - (u^z)_r, \quad \omega^z = \frac{1}{r}u^\theta + (u^\theta)_r, \\ &j^r = -(B^\theta)_z, \quad j^\theta = (B^r)_z - (B^z)_r, \quad j^z = \frac{1}{r}B^\theta + (B^\theta)_r. \end{aligned} \tag{2.11}$$

We can introduce the stream function $\boldsymbol{\psi}$ and the magnetic stream function $\boldsymbol{\Phi}$. u^θ, ω^θ, ψ^θ, B^θ, j^θ, and Φ^θ is called the swirl component of the velocity field $\mathbf{u}$, the vorticity field $\boldsymbol{\omega}$, the stream function $\boldsymbol{\psi}$, the magnetical field $\mathbf{B}$, the current density $\mathbf{J}$, and the magnetic stream function $\boldsymbol{\Phi}$, respectively. Therefore, u^r, u^z, B^r, and B^z can be expressed in terms of ψ^θ and Φ^θ as follows:

$$u^r = -(\psi^\theta)_z, \quad u^z = \frac{1}{r}\psi^\theta + (\psi^\theta)_r, \quad B^r = -(\Phi^\theta)_z, \quad B^z = \frac{1}{r}\Phi^\theta + (\Phi^\theta)_r. \tag{2.12}$$

Therefore, we arrive at

$$\begin{cases} (u^\theta)_t + u^r(u^\theta)_r + \frac{1}{r}u^\theta u^r + u^z(u^\theta)_z = \left(\nabla^2 - \frac{1}{r^2}\right)u^\theta + B^r B^\theta_r + \frac{1}{r}B^\theta B^r + B^z(B^\theta)_z, \\ (\omega^\theta)_t + u^r(\omega^\theta)_r + u^z(\omega^\theta)_z - \frac{1}{r}u^r\omega^\theta - \frac{1}{r}[(u^\theta)^2]_z = \left(\nabla^2 - \frac{1}{r^2}\right)\omega^\theta \\ \quad + B^r(j^\theta)_r + B^z(j^\theta)_z - \frac{1}{r}B^r j^\theta - \frac{1}{r}[(B^\theta)^2]_z, \\ -\left(\nabla^2 - \frac{1}{r^2}\right)\psi^\theta = \omega^\theta, \end{cases} \tag{2.13}$$

and

$$\begin{cases} (B^\theta)_t + u^r(B^\theta)_r + u^z(B^\theta)_z + \frac{1}{r}B^r u^\theta = \left(\nabla^2 - \frac{1}{r^2}\right)B^\theta \\ \quad + B^r(u^\theta)_r + B^z(u^\theta)_z + \frac{1}{r}u^r B^\theta, \\ (j^\theta)_t + u^r(j^\theta)_r + u^z(j^\theta)_z = \left(\nabla^2 - \frac{1}{r^2}\right)j^\theta + B^r(\omega^\theta)_r + B^z(\omega^\theta)_z \\ \quad + [(u^r)_r - (u^z)_z][(B^r)_z + (B^z)_r] + [(B^z)_z - (B^r)_r][(u^r)_z + (u^z)_r], \\ -\left(\nabla^2 - \frac{1}{r^2}\right)\Phi^\theta = j^\theta. \end{cases} \tag{2.14}$$

Any smooth solution to the three-dimensional axisymmetric MHD equations satisfies the following condition at $r = 0$:

$$\begin{aligned} u^\theta(0, z, t) = \omega^\theta(0, z, t) = \psi^\theta(0, z, t) = 0, \\ B^\theta(0, z, t) = j^\theta(0, z, t) = \Phi^\theta(0, z, t) = 0. \end{aligned} \tag{2.15}$$

Moreover, all the even-order derivatives of u^θ, ω^θ, ψ^θ, B^θ, j^θ, and Φ^θ with respect to r at $r = 0$ must vanish. We expand the above solution around $r = 0$ as follows:

$$\begin{aligned} u^\theta(r, z, t) &= r u^\theta_1(0, z, t) + \frac{r^3}{3!}u^\theta_3(0, z, t) + \frac{r^5}{5!}u^\theta_5(0, z, t) + \cdots, \\ \omega^\theta(r, z, t) &= r\omega^\theta_1(0, z, t) + \frac{r^3}{3!}\omega^\theta_3(0, z, t) + \frac{r^5}{5!}\omega^\theta_5(0, z, t) + \cdots, \\ \psi^\theta(r, z, t) &= r\psi^\theta_1(0, z, t) + \frac{r^3}{3!}\psi^\theta_3(0, z, t) + \frac{r^5}{5!}\psi^\theta_5(0, z, t) + \cdots, \\ B^\theta(r, z, t) &= rB^\theta_1(0, z, t) + \frac{r^3}{3!}B^\theta_3(0, z, t) + \frac{r^5}{5!}B^\theta_5(0, z, t) + \cdots, \end{aligned} \tag{2.16}$$

$$j^\theta(r,z,t) = rj_1^\theta(0,z,t) + \frac{r^3}{3!}j_3^\theta(0,z,t) + \frac{r^5}{5!}j_5^\theta(0,z,t) + \cdots,$$
$$\Phi^\theta(r,z,t) = r\Phi_1^\theta(0,z,t) + \frac{r^3}{3!}\Phi_3^\theta(0,z,t) + \frac{r^5}{5!}\Phi_5^\theta(0,z,t) + \cdots,$$

Substituting the above expansions into (2.12) and (2.13), we obtain

$$\begin{aligned}
&r(u_1^\theta)_t - 2r(\psi_1^\theta)_z u_1^\theta + 2r\psi_1^\theta(u_1^\theta)_z \\
&\quad = \frac{4}{3}ru_3^\theta + r(u_1^\theta)_{zz} - 2r(\Phi_1^\theta)_z B_1^\theta + 2r\Phi_1^\theta(B_1^\theta)_z + o(r^3),\\
&r(\omega_1^\theta)_t + 2r\psi_1^\theta(\omega_1^\theta)_z - r(u_1^\theta)_z^2 = \frac{4}{3}r\omega_3^\theta + r(\omega_1^\theta)_{zz} + 2r\Phi_1^\theta(j_1^\theta)_z - r(B_1^\theta)_z^2 + o(r^3),\\
&-\frac{4}{3}r\psi_3^\theta - r(\psi_1^\theta)_{zz} = r\omega_1^\theta + o(r^3),\\
&r(B_1^\theta)_t + 2r\psi_1^\theta(B_1^\theta)_z = \frac{4}{3}rB_3^\theta + r(B_1^\theta)_{zz} + 2r\Phi_1^\theta(u_1^\theta)_z + o(r^3),\\
&r(j_1^\theta)_t - r(\psi_1^\theta)_z j_1^\theta + 2r\psi_1^\theta(j_1^\theta)_z = \frac{4}{3}rj_3^\theta + r(j_1^\theta)_{zz} - r(\Phi_1^\theta)_z\omega_1^\theta + 2r\Phi_1^\theta(\omega_1^\theta)_z\\
&\quad + 3r(\psi_1^\theta)_z(\Phi_1^\theta)_{zz} - 4r(\psi_1^\theta)_z\Phi_3^\theta - 3r(\Phi_1^\theta)_z(\psi_1^\theta)_{zz} + 4r(\Phi_1^\theta)_z\psi_3^\theta + o(r^3),\\
&-\frac{4}{3}r\Phi_3^\theta - r(\Phi_1^\theta)_{zz} = rj_1^\theta + o(r^3),
\end{aligned}$$

where $u_1^\theta = u_r^\theta$, $u_3^\theta = u_{rrr}^\theta$, *and* $u_{1zz}^\theta = u_{rzz}^\theta$. Cancel r from both sides and neglecting the high-order terms in r and assume that the second partial derivative of u_1^θ, ω_1^θ, ψ_1^θ, B_1^θ, j_1^θ, and Φ_1^θ with respect to z are much larger than the second partial derivation of these functions with respect to r, we obtain one-dimensional model

$$(u_1^\theta)_t - 2(\psi_1^\theta)_z u_1^\theta + 2\psi_1^\theta(u_1^\theta)_z = (u_1^\theta)_{zz} - 2(\Phi_1^\theta)_z B_1^\theta + 2\Phi_1^\theta(B_1^\theta)_z, \quad (2.17)$$
$$(\omega_1^\theta)_t + 2\psi_1^\theta(\omega_1^\theta)_z - [(u_1^\theta)^2]_z = (\omega_1^\theta)_{zz} + 2\Phi_1^\theta(j_1^\theta)_z - [B_1^\theta)^2]_z, \quad (2.18)$$
$$-(\psi_1^\theta)_{zz} = \omega_1^\theta, \quad (2.19)$$
$$(B_1^\theta)_t + 2\psi_1^\theta(B_1^\theta)_z = (B_1^\theta)_{zz} + 2\Phi_1^\theta(u_1^\theta)_z, \quad (2.20)$$
$$\begin{aligned}(j_1^\theta)_t - (\psi_1^\theta)_z j_1^\theta + 2\psi_1^\theta(j_1^\theta)_z = (j_1^\theta)_{zz} - (\Phi_1^\theta)_z\omega_1^\theta + 2\Phi_1^\theta(\omega_1^\theta)_z\\
+3(\psi_1^\theta)_z(\Phi_1^\theta)_{zz} - 3(\Phi_1^\theta)_z(\psi_1^\theta)_{zz}, \quad (2.21)\end{aligned}$$
$$-(\Phi_1^\theta)_{zz} = j_1^\theta. \quad (2.22)$$

Theorem 2.1 *Let* $(u_1^\theta(z,t), \omega_1^\theta(z,t), \psi_1^\theta(z,t), B_1^\theta(z,t), j_1^\theta(z,t), \Phi_1^\theta(z,t))$ *be the solution to the one-dimensional model* (2.17)–(2.22)*, and define*

$$\begin{aligned}&u^\theta(r,z,t) = ru_1^\theta(z,t), \quad \omega^\theta(r,z,t) = r\omega_1^\theta(z,t), \quad \psi^\theta(r,z,t) = r\psi_1^\theta(z,t),\\
&B^\theta(r,z,t) = rB_1^\theta(z,t), \quad j^\theta(r,z,t) = rj_1^\theta(z,t), \quad \Phi^\theta(r,z,t) = r\Phi_1^\theta(z,t), \quad (2.23)\end{aligned}$$

then $(u^\theta(r,z,t), \omega^\theta(r,z,t), \psi^\theta(r,z,t), B^\theta(r,z,t), j^\theta(r,z,t), \Phi^\theta(r,z,t))$ *is an exact solution to the three-dimensional axisymmetric MHD equations* (2.13)–(2.14).

Proof By substituting (2.23) into (2.13) and (2.14) and canceling r, respectively, we can obtain (2.17)–(2.22). □

Let

$$\begin{aligned} &\bar{u} = u_1^\theta, \quad \bar{v} = -(\psi_1^\theta)_z, \quad \bar{\omega} = \omega_1^\theta, \quad \bar{\psi} = \psi_1^\theta, \\ &\bar{B} = B_1^\theta, \quad \bar{l} = -(\Phi_1^\theta)_z, \quad \bar{j} = j_1^\theta, \quad \bar{\Phi} = \Phi_1^\theta. \end{aligned} \tag{2.24}$$

By integrating the $\bar{\omega}$ and $\bar{j}$ equations with respect to z and using the relationship (2.19) and (2.22), we get evolution equations for $\bar{v}$ and $\bar{l}$. Now the complete equations of $\bar{u}$, $\bar{v}$, $\bar{\omega}$, $\bar{B}$, $\bar{l}$, and $\bar{j}$ are given by

$$\bar{u}_t + 2\bar{v}\bar{u} + 2\bar{\psi}(\bar{u})_z = (\bar{u})_{zz} + 2\bar{l}\bar{B} + 2\bar{\Phi}(\bar{B})_z, \tag{2.25}$$

$$(\bar{\omega})_t + 2\bar{\psi}(\bar{\omega})_z - (\bar{u}^2)_z = (\bar{\omega})_{zz} + 2\bar{\Phi}(\bar{j})_z - (\bar{B}^2)_z, \tag{2.26}$$

$$(\bar{v})_t + 2\bar{\psi}(\bar{v})_z + (\bar{v})^2 - (\bar{u})^2 = (\bar{v})_{zz} + 2\bar{\Phi}(\bar{l})_z + \bar{l}^2 - \bar{B}^2 + c(t), \tag{2.27}$$

$$-(\bar{\psi})_{zz} = \bar{\omega}, \tag{2.28}$$

$$(\bar{B})_t + 2\bar{\psi}(\bar{B})_z = (\bar{B})_{zz} + 2\bar{\Phi}(\bar{u})_z, \tag{2.29}$$

$$(\bar{j})_t + \bar{v}\bar{j} + 2\bar{\psi}(\bar{j})_z = (\bar{j})_{zz} + \bar{l}\bar{\omega} + 2\bar{\Phi}(\bar{\omega})_z + 3\bar{v}(\bar{l})_z - 3\bar{l}(\bar{v})_z, \tag{2.30}$$

$$(\bar{l})_t + 2\bar{\psi}(\bar{l})_z = (\bar{l})_{zz} + 2\bar{\Phi}(\bar{v})_z + d(t), \tag{2.31}$$

$$-(\bar{\Phi})_{zz} = \bar{j}, \tag{2.32}$$

where $c(t)$ and $d(t)$ are integration constants:

$$\begin{aligned} c(t) = &-\bar{\psi}_{zt}(0,t) - 2\bar{\psi}(0,t)\bar{\psi}_{zz}(0,t) + (\bar{\psi}_z)^2(0,t) - \bar{u}^2(0,t) \\ &+ \bar{\psi}_{zzz}(0,t) + 2\bar{\Phi}(0,t)\bar{\Phi}_{zz}(0,t) - (\bar{\Phi}_z)^2(0,t) + \bar{B}^2(0,t), \\ \mathrm{d}(t) = &-\bar{\Phi}_{zt}(0,t) - \bar{v}(0,t)\bar{\Phi}_z(0,t) - 2\bar{\psi}(0,t)\bar{\Phi}_{zz}(0,t) \\ &+ \bar{\Phi}_{zzz}(0,t) + \bar{l}(0,t)\bar{\psi}_z(0,t) + 2\bar{\Phi}(0,t)\bar{\psi}_{zz}(0,t). \end{aligned}$$

If $\bar{\psi}$ and $\bar{\Phi}$ is periodic with period 1 in z, then $c(t) = 3\int_0^1 \bar{v}^2 - \bar{l}^2 \mathrm{d}z + \int_0^1 \bar{B}^2 - \bar{u}^2 \mathrm{d}z$, $\mathrm{d}(t) = 0$. Note that the equations $\bar{\omega}$ (2.26) and $\bar{j}$ (2.30) are equivalent to that for $\bar{v}$ (2.27) and $\bar{l}$ (2.31), respectively. So it is sufficient to consider the coupled system for $\bar{u}$, $\bar{v}$, $\bar{B}$, and $\bar{l}$.

$$\bar{u}_t + 2\bar{v}\bar{u} + 2\bar{\psi}(\bar{u})_z = (\bar{u})_{zz} + 2\bar{l}\bar{B} + 2\bar{\Phi}(\bar{B})_z, \tag{2.33}$$

$$(\bar{v})_t + 2\bar{\psi}(\bar{v})_z + (\bar{v})^2 - (\bar{u})^2 = (\bar{v})_{zz} + 2\bar{\Phi}(\bar{l})_z + \bar{l}^2 - \bar{B}^2 + c(t), \tag{2.34}$$

$$(\bar{B})_t + 2\bar{\psi}(\bar{B})_z = (\bar{B})_{zz} + 2\bar{\Phi}(\bar{u})_z, \tag{2.35}$$

$$(\bar{l})_t + 2\bar{\psi}(\bar{l})_z = (\bar{l})_{zz} + 2\bar{\Phi}\bar{v}_z + d(t). \tag{2.36}$$

3 A Family of Particular Solutions

In this section, we will present a family of particular solutions to the three-dimensional MHD equations. Consider a family of non-trial solutions with the form

$$\mathbf{u} = u^r(r,z,t)\mathbf{e}_r + u^z(r,z,t)\mathbf{e}_z, \qquad \mathbf{B} = B^r(r,z,t)\mathbf{e}_r + B^z(r,z,t)\mathbf{e}_z. \tag{3.1}$$

It is easy to check (u^θ, B^θ) be zero all for time if they are zero initially, and it is also satisfied the incompressible condition (2.2). In this case, (2.13) and (2.14) are simplified

$$\begin{cases} (\omega^\theta)_t + u^r(\omega^\theta)_r + u^z(\omega^\theta)_z - \frac{1}{r}u^r\omega^\theta \\ \quad = \left(\nabla^2 - \frac{1}{r^2}\right)\omega^\theta + B^r(j^\theta)_r + B^z(j^\theta)_z - \frac{1}{r}B^r j^\theta, \\ -\left(\nabla^2 - \frac{1}{r^2}\right)\psi^\theta = \omega^\theta, \\ (j^\theta)_t + u^r(j^\theta)_r + u^z(j^\theta)_z \\ \quad = \left(\nabla^2 - \frac{1}{r^2}\right) j^\theta + B^r(\omega^\theta)_r + B^z(\omega^\theta)_z \\ \qquad + [(u^r)_r - (u^z)_z][(B^r)_z + (B^z)_r] + [(B^z)_z - (B^r)_r][(u^r)_z + (u^z)_r], \\ -\left(\nabla^2 - \frac{1}{r^2}\right)\Phi^\theta = j^\theta. \end{cases} \tag{3.2}$$

We also obtain one-dimensional model

$$\begin{cases} (\omega_1^\theta)_t + 2\psi_1^\theta(\omega_1^\theta)_z = (\omega_1^\theta)_{zz} + 2\Phi_1^\theta(j_1^\theta)_z, \\ -(\psi_1^\theta)_{zz} = \omega_1^\theta, \\ (j_1^\theta)_t - (\psi_1^\theta)_z j_1^\theta + 2\psi_1^\theta(j_1^\theta)_z \\ \quad = (j_1^\theta)_{zz} - (\Phi_1^\theta)_z\omega_1^\theta + 2\Phi_1^\theta(\omega_1^\theta)_z \\ \qquad + 3(\psi_1^\theta)_z(\Phi_1^\theta)_{zz} - 3(\Phi_1^\theta)_z(\psi_1^\theta)_{zz}, \\ -(\Phi_1^\theta)_{zz} = j_1^\theta. \end{cases} \tag{3.3}$$

Let $\bar\omega = \omega_1^\theta$, $\bar\psi = \psi_1^\theta$, $\bar v = -(\psi_1^\theta)_z$, $\bar j = j_1^\theta, and \bar l = -(\Phi_1^\theta)_z$, we obtain the complete set of evolution equations for $\bar\omega$, $\bar\psi$, $\bar j$ and $\bar\Phi$

$$(\bar\omega)_t + 2\bar\psi(\bar\omega)_z = (\bar\omega)_{zz} + 2\bar\Phi(\bar j)_z, \tag{3.4}$$
$$(\bar v)_t + 2\bar\psi(\bar v)_z + (\bar v)^2 = (\bar v)_{zz} + 2\bar\Phi(\bar l)_z + \bar l^2 + e(t), \tag{3.5}$$
$$-(\bar\psi)_{zz} = \bar\omega, \tag{3.6}$$
$$(\bar j)_t + \bar v\bar j + 2\bar\psi(\bar j)_z = (\bar j)_{zz} + \bar l\bar\omega + 2\bar\Phi(\bar\omega)_z + 3\bar v(\bar l)_z - 3\bar l(\bar v)_z, eq3.7 \tag{3.7}$$
$$(\bar l)_t + 2\bar\psi(\bar l)_z = (\bar l)_{zz} + 2\bar\Phi(\bar v)_z + f(t), \tag{3.8}$$
$$-(\bar\Phi)_{zz} = \bar j, \tag{3.9}$$

where $e(t) = 3\int_0^1 \bar v^2 \mathrm{d}z - \int_0^1 \bar l^2 \mathrm{d}z$, $f(t) = 0$.

Similar to the second section, it is sufficient to consider the coupled system for $\bar B$ and $\bar v$:

$$(\bar v)_t + 2\bar\psi(\bar v)_z + (\bar v)^2 = (\bar v)_{zz} + 2\bar\Phi(\bar l)_z + \bar l^2 + e(t), \tag{3.10}$$
$$(\bar l)_t + 2\bar\psi(\bar l)_z = (\bar l)_{zz} + 2\bar\Phi(\bar v)_z + f(t). \tag{3.11}$$

Theorem 3.1 *Let* $(\omega_1^\theta(z,t), \psi_1^\theta(z,t), j_1^\theta(z,t), \Phi_1^\theta(z,t))$ *be the solution to the one-dimensional model* (3.3)*, and define*

$$\begin{aligned} &\omega^\theta(r,z,t) = r\omega_1^\theta(z,t), \quad \psi^\theta(r,z,t) = r\psi_1^\theta(z,t), \\ &j^\theta(r,z,t) = rj_1^\theta(z,t), \quad \Phi^\theta(r,z,t) = r\Phi_1^\theta(z,t), \end{aligned} \tag{3.12}$$

then $(\omega^\theta(r,z,t), \psi^\theta(r,z,t), j^\theta(r,z,t), \Phi^\theta(r,z,t))$ *is an exact solution to the three-dimensional axisymmetric MHD equations* (3.2).

Proof It is a particular case of Theorem 2.1. □

Theorem 3.2 (A prior estimates of $\bar{v}$ and $\bar{l}$ in H^1) *Consider the one-dimensional axisymmetric incompressible MHD equations* (3.10) *and* (3.11). *Assume that* $\bar{v}_0(z)$, $\bar{l}_0(z) \in H^1[0,1]$ *and for fixed* $T > 0$ *and* $0 < t < T$, *we have*

$$\begin{aligned} &||\bar{v}(z,t)||_{L^2}^2 + ||\bar{l}(z,t)||_{L^2}^2 + 2\int_0^t (||\bar{v}_z(z,t)||_{L^2}^2 + ||\bar{l}_z(z,t)||_{L^2}^2) d\tau \\ &\le 2(||\bar{v}_0(z)||_{L^2}^2 + ||\bar{l}_0(z)||_{L^2}^2). \end{aligned} \tag{3.13}$$

Proof Multiplying (3.10) by $\bar{v}$ and (3.11) by $\bar{l}$, we have

$$\begin{aligned} &\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\bar{v}^2 + \bar{\psi}(\bar{v}^2)_z + \bar{v}^3 = \bar{v}(\bar{v})_{zz} + 2\bar{\Phi}(\bar{l})_z\bar{v} + \bar{l}^2\bar{v} + e(t)\bar{v}, \\ &\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\bar{l}^2 + \bar{\psi}(\bar{l}^2)_z = \bar{l}(\bar{l})_{zz} + 2\bar{\Phi}(\bar{v})_z\bar{l} + f(t)\bar{l}. \end{aligned}$$

Adding the result equations, we obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\bar{v}^2 + \bar{l}^2) + \psi(\bar{v}^2 + \bar{l}^2)_z = \bar{v}\bar{v}_{zz} + \bar{l}\,\bar{l}_{zz} + 2\bar{\Phi}(\bar{l}\bar{v})_z + \bar{l}^2\bar{v} - \bar{v}^3 + e(t)\bar{v} + f(t)\bar{l}.$$

Integrating over z, we arrive at

$$\begin{aligned} &\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 (\bar{v}^2 + \bar{l}^2)\mathrm{d}z + \int_0^1 \psi(\bar{v}^2 + \bar{l}^2)_z\mathrm{d}z \\ &= \int_0^1 \bar{v}\bar{v}_{zz}\mathrm{d}z + \int_0^1 \bar{l}\,\bar{l}_{zz}\mathrm{d}z + \int_0^1 \bar{l}^2\,\bar{v}\mathrm{d}z \\ &\quad - \int_0^1 \bar{v}^3\mathrm{d}z + \int_0^1 2\bar{\Phi}(\bar{l}\bar{v})_z\mathrm{d}z + e(t)\int_0^1 \bar{v}\mathrm{d}z + f(t)\int_0^1 \bar{l}\mathrm{d}z. \end{aligned}$$

Integrating over t, we get (3.13), where we use the integration by parts, Hölder inequality and $-\bar{\psi}_z = \bar{v}$, $-\bar{\Phi}_z = \bar{l}$. This completes the proof of Theorem 3.2. □

Theorem 3.3 (A prior estimates of $\bar{v}$ and $\bar{l}$ in H^2) Consider the one-dimensional axisymmetric incompressible MHD equations (3.10) and (3.11). Assume that $\bar{v}_0(z)$, $\bar{l}_0(z) \in H^2[0,1]$ and for fixed $T > 0$ and $0 < t < T$, we have

$$||\bar{v}_z(z,t)||_{L^2}^2 + ||\bar{l}_z(z,t)||_{L^2}^2 + 2\int_0^t (||\bar{v}_{zz}(z,t)||_{L^2}^2 + ||\bar{l}_{zz}(z,t)||_{L^2}^2)d\tau$$
$$\leq C(||\bar{v}_{0z}(z)||_{L^2}^2 + ||\bar{l}_{0z}(z)||_{L^2}^2). \tag{3.14}$$

Proof Differentiating (3.10) and (3.11) with respect to z, we get

$$(\bar{v})_{zt} + 2(\bar{\psi})_z(\bar{v})_z + 2\bar{\psi}(\bar{v})_{zz} + (\bar{v}^2)_z = (\bar{v})_{zzz} + 2\bar{\Phi}_z\bar{l}_z + 2\bar{\Phi}\bar{l}_{zz} + (\bar{l}^2)_z, \tag{3.15}$$
$$(\bar{l})_{zt} + 2(\bar{\psi})_z(\bar{l})_z + 2\bar{\psi}(\bar{l})_{zz} = (\bar{l})_{zzz} + 2\bar{\Phi}_z\bar{v}_z + 2\bar{\Phi}\bar{v}_{zz}. \tag{3.16}$$

Multiplying (3.15) by $\bar{v}_z$, (3.16) by $\bar{l}_z$, and adding the resulting equations, we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}[(\bar{v}_z)^2 + (\bar{l}_z)^2] + 2(\bar{\psi})_z[(\bar{v}_z)^2 + (\bar{l}_z)^2] + \bar{\psi}[(\bar{v}_z)^2 + (\bar{l}_z)^2]_z + (\bar{v}^2)_z\bar{v}_z$$
$$= (\bar{v})_z(\bar{v})_{zzz} + (\bar{l})_z(\bar{l})_{zzz} + 4\bar{\Phi}_z\bar{l}_z\bar{v}_z + 2\bar{\Phi}\bar{l}_{zz}\bar{v}_z + 2\bar{\Phi}\bar{l}_z\bar{v}_{zz} + (\bar{l}^2)_z\bar{v}_z.$$

Integrating over z, we obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(||\bar{v}_z||_{L^2}^2 + ||\bar{l}_z||_{L^2}^2) + ||(\bar{v})_{zz}||_{L^2}^2 + ||(\bar{l})_{zz}||_{L^2}^2$$
$$\leq \int_0^1 |\bar{v}|(\bar{v}_z)^2\mathrm{d}z + \int_0^1 |\bar{v}|(\bar{l}_z)^2\mathrm{d}z + \int_0^1 \bar{v}^2|\bar{v}_{zz}|\mathrm{d}z$$
$$\leq ||\bar{v}||_{L^2}||\bar{v}_z||_{L^4}^2 + ||\bar{v}||_{L^2}||\bar{l}_z||_{L^4}^2 + ||\bar{v}||_{L^4}^2||\bar{v}_{zz}||_{L^2}$$
$$\leq 2C(\varepsilon)||\bar{v}||_{L^2}^2||\bar{v}_z||_{L^2}^2 + 2\varepsilon||\bar{v}_{zz}||_{L^2}^2 + C(\varepsilon)||\bar{v}||_{L^2}^2||\bar{l}_z||_{L^2}^2 + \varepsilon||\bar{l}_{zz}||_{L^2}^2.$$

Integrating over t, we obtain (3.14), where we use the Theorem 3.2, the second Gagliardo–Nirenberg inequality, the Yong's inequality, the integration by parts, and $-\bar{\psi}_z = \bar{v}$, $-\bar{\Phi}_z = \bar{l}$. The proof is completed. □

Theorem 3.4 (A prior estimates of $\bar{v}$ and $\bar{l}$ in H^3) Consider the one-dimensional axisymmetric incompressible MHD equations (3.9) and (3.10). Assume that $\bar{v}_0, \bar{l}_0 \in H^3[0,1]$ and for fixed $T > 0$ $0 < t < T$, and C is a positive constant, we have

$$||\bar{v}_{zz}(z,t)||_{L^2}^2 + ||\bar{l}_{zz}(z,t)||_{L^2}^2 + 2\int_0^t (||\bar{v}_{zzz}(z,t)||_{L^2}^2 + ||\bar{l}_{zzz}(z,t)||_{L^2}^2)d\tau$$
$$\leq C(||\bar{v}_{0zz}(z)||_{L^2}^2 + ||\bar{l}_{0zz}(z)||_{L^2}^2). \tag{3.17}$$

Proof Similar to the proof of Theorems 3.2 and 3.3, we omit this proof. □

4 Construction of a Family of Global Smooth Solutions to Three-dimensional MHD Equations

In this section, we will construct a family of global smooth solutions to the three-dimensional MHD equations by the one-dimensional model (3.3).

Let $\bar{\omega}_1(z,t)$, $\bar{\psi}_1(z,t)$, $\bar{j}_1(z,t)$, and $\bar{\Phi}_1(z,t)$ be the solutions to the one-dimensional model (3.3). We will construct a family of global smooth solutions to the three-dimensional MHD equations from the solutions to the above one-dimensional model. $\tilde{B}^\theta(r,z,t)$, $\tilde{\omega}^\theta(r,z,t)$, $\tilde{\psi}^\theta(r,z,t)$, $\tilde{B}^\theta(r,z,t)$, $\tilde{j}^\theta(r,z,t)$, and $\tilde{\Phi}^\theta(r,z,t)$ are the solutions to the three-dimensional MHD equations (2.13) and (2.14). We define

$$\begin{aligned}&\tilde{u}_1(r,z,t)=\tilde{u}^\theta(r,z,t)/r,\quad \tilde{\omega}_1(r,z,t)=\tilde{\omega}^\theta(r,z,t)/r,\quad \tilde{\psi}_1(r,z,t)=\tilde{\psi}^\theta(r,z,t)/r,\\&\tilde{B}_1(r,z,t)=\tilde{B}^\theta(r,z,t)/r,\quad \tilde{j}_1(r,z,t)=\tilde{j}^\theta(r,z,t)/r,\quad \tilde{\Phi}_1(r,z,t)=\tilde{\Phi}^\theta(r,z,t)/r.\end{aligned} \tag{4.1}$$

Let $\phi(r)=\phi_0(r/R_0)$ be a smooth cut-off function, where $\phi_0(r)$ satisfies $\phi_0(r)=1$ if $r_0\le r\le 1/2, r_0=1/M^{\frac{1}{32}}$ and $\phi_0(r)=0$ if $0<r\le r_0$ or $r>1/2$. We construct a family of globally smooth functions $u_1(r,z,t)$, $\omega_1(r,z,t)$, $\psi_1(r,z,t)$, $B_1(r,z,t)$, $j_1(r,z,t)$, and $\Phi_1(r,z,t)$, and they are periodic in z such that

$$\tilde{u}^\theta(r,z,t)=r[\bar{u}_1(z,t)\phi(r)+u_1(r,z,t)]=\bar{u}^\theta+u^\theta, \tag{4.2}$$

$$\tilde{\omega}^\theta(r,z,t)=r[\bar{\omega}_1(z,t)\phi(r)+\omega_1(r,z,t)]=\bar{\omega}^\theta+\omega^\theta, \tag{4.3}$$

$$\tilde{\psi}^\theta(r,z,t)=r[\bar{\psi}_1(z,t)\phi(r)+\psi_1(r,z,t)]=\bar{\psi}^\theta+\psi^\theta, \tag{4.4}$$

$$\tilde{B}^\theta(r,z,t)=r[\bar{B}_1(z,t)\phi(r)+B_1(r,z,t)]=\bar{B}^\theta+B^\theta, \tag{4.5}$$

$$\tilde{j}^\theta(r,z,t)=r[\bar{j}_1(z,t)\phi(r)+j_1(r,z,t)]=\bar{j}^\theta+j^\theta, \tag{4.6}$$

$$\tilde{\Phi}^\theta(r,z,t)=r[\bar{\Phi}_1(z,t)\phi(r)+\Phi_1(r,z,t)]=\bar{\Phi}^\theta+\Phi^\theta \tag{4.7}$$

be a solution to the three-dimensional MHD equations. With the above definitions, we can derive the other four velocity and magnetic components $u^r(r,z,t)$, $u^z(r,z,t)$, $B^r(r,z,t)$, and $B^z(r,z,t)$ as

$$\tilde{u}^r(r,z,t)=-(\tilde{\psi}^\theta)_z=-r\bar{\psi}_{1z}\phi-r\psi_{1z}=\bar{u}^r+u^r, \tag{4.8}$$

$$\tilde{u}^z(r,z,t)=(r\tilde{\psi}^\theta)_r/r=(2\bar{\psi}_1\phi+r\bar{\psi}_1\phi_r)+(2\psi_1+r\psi_{1r})=\bar{u}^z+u^z, \tag{4.9}$$

$$\tilde{B}^r(r,z,t)=-(\tilde{\Phi}^\theta)_z=-r\bar{\Phi}_{1z}\phi-r\Phi_{1z}=\bar{B}^r+B^r, \tag{4.10}$$

$$\tilde{B}^z(r,z,t)=(r\tilde{\Phi})_r/r=(2\bar{\Phi}_1\phi+r\bar{\Phi}_1\phi_r)+(2\Phi_1+r\Phi_{1r})=\bar{B}^z+B^z. \tag{4.11}$$

We can rewrite the velocity and magnetic vectors as

$$\tilde{\mathbf{u}}=\bar{\mathbf{u}}+\mathbf{u},\qquad \tilde{\mathbf{B}}=\bar{\mathbf{B}}+\mathbf{B}, \tag{4.12}$$

where

$$\begin{aligned}\tilde{\mathbf{u}}&=(\tilde{u}^r,\tilde{u}^\theta,\tilde{u}^z)=(-r\bar{\psi}_{1z}\phi-r\psi_{1z},ru_1,2\bar{\psi}_1\phi+r\bar{\psi}_1\phi_r+2\psi_1+r\psi_{1r}),\\ \bar{\mathbf{u}}&=(\bar{u}^r,\bar{u}^\theta,\bar{u}^z)=(-r\bar{\psi}_{1z}\phi,0,2\bar{\psi}_1\phi+r\bar{\psi}_1\phi_r),\\ \mathbf{u}&=(u^r,u^\theta,u^z)=(-r\psi_{1z},ru_1,2\psi_1+r\psi_{1r}),\\ \tilde{\mathbf{B}}&=(\tilde{B}^r,\tilde{B}^\theta,\tilde{B}^z)=(-r\bar{\Phi}_{1z}\phi-r\Phi_{1z},rB_1,2\bar{\Phi}_1\phi+r\bar{\Phi}_1\phi_r+2\Phi_1+r\Phi_{1r}),\end{aligned}$$

$$\bar{\mathbf{B}} = (\bar{B}^r, \bar{B}^\theta, \bar{B}^z) = (-r\bar{\Phi}_{1z}\phi, 0, 2\bar{\Phi}_1\phi + r\bar{\Phi}_1\phi_r),$$
$$\mathbf{B} = (B^r, B^\theta, B^z) = (-r\Phi_{1z}, rB_1, 2\Phi_1 + r\Phi_{1r}).$$

We choose the initial data for the one-dimensional model of the following form:

$$\begin{aligned} \bar{v}_0(z) &= \frac{A}{M^3}\bar{\mathcal{V}}(zM), \quad \bar{l}_0(z) = \frac{A}{M^3}\bar{\mathcal{L}}(zM), \\ \bar{\psi}_0(z) &= -\frac{A}{M^4}\bar{\Psi}(zM), \quad \bar{\Phi}_0(z) = -\frac{A}{M^4}\bar{\Phi}(zM), \end{aligned} \tag{4.13}$$

where A and M are some positive constants. $\bar{\mathcal{V}}(y)$, $\bar{\mathcal{L}}(y)$, $\bar{\Psi}(y)$, and $\bar{\Phi}(y)$ are smooth periodic functions in y with period 1. Moreover, we assume that $\bar{\mathcal{L}}(y)$, $\bar{\mathcal{V}}(y)$, $\bar{\Psi}(y)$, and $\bar{\Phi}(y)$ are odd functions in y. We have $-\bar{\Psi}_y = \bar{\mathcal{V}}$, $-\bar{\Phi}_y = \bar{\mathcal{L}}$ and

$$\bar{v}_{0z}(z) = \frac{A}{M^2}\bar{\mathcal{V}}_z(zM), \quad \bar{l}_{0z}(z) = \frac{A}{M^2}\bar{\mathcal{L}}_z(zM),$$
$$\bar{v}_{0zz}(z) = \frac{A}{M}\bar{\mathcal{V}}_{zz}(zM), \quad \bar{l}_{0zz}(z) = \frac{A}{M}\bar{\mathcal{L}}_{zz}(zM).$$

Therefore, we have

$$||\bar{v}_{0zz}(z)||_{L^2}^2 + ||\bar{l}_{0zz}(z)||_{L^2}^2 = A^2 d_0^2/M^2, \tag{4.14}$$

where $||\bar{\mathcal{V}}_{zz}||_{L^2}^2 + ||\bar{\mathcal{L}}_{zz}||_{L^2}^2 = d_0^2$. We arrive at

$$||\bar{v}_{0z}(z)||_{L^2}^2 + ||\bar{l}_{0z}(z)||_{L^2}^2 \le A^2 d_0^2/M^2, \quad ||\bar{v}_0(z)||_{L^2}^2 + ||\bar{l}_0(z)||_{L^2}^2 \le A^2 d_0^2/M^2. \tag{4.15}$$

Let $R_0 = M^{\frac{1}{8}}$. From the definition of $\bar{\mathbf{u}}$ and $\bar{\mathbf{B}}$ we have

$$\begin{aligned} ||\bar{\mathbf{u}}||_{L^2} &= ||(\bar{u}^r, \bar{u}^\theta, \bar{u}^z)||_{L^2} \approx R_0 A/M = A/M^{\frac{7}{8}}, \\ ||\nabla\bar{\mathbf{u}}||_{L^2} &= \left|\left|\left(\frac{\partial\bar{\mathbf{u}}}{\partial r}, \frac{\partial\bar{\mathbf{u}}}{\partial z}\right)\right|\right|_{L^2} \approx R_0 A/M = A/M^{\frac{7}{8}}, \end{aligned} \tag{4.16}$$

$$\begin{aligned} ||\bar{\mathbf{B}}||_{L^2} &= ||(\bar{B}^r, \bar{B}^\theta, \bar{B}^z)||_{L^2} \approx R_0 A/M = A/M^{\frac{7}{8}}, \\ ||\nabla\bar{\mathbf{B}}||_{L^2} &= \left|\left|\left(\frac{\partial\bar{\mathbf{B}}}{\partial r}, \frac{\partial\bar{\mathbf{B}}}{\partial z}\right)\right|\right|_{L^2} \approx R_0 A/M = A/M^{\frac{7}{8}}. \end{aligned} \tag{4.17}$$

We assume that the initial condition for $\tilde{\mathbf{u}}$ and $\tilde{\mathbf{B}}$ satisfying

$$||\tilde{\mathbf{u}}_0||_{L^2} \approx A/M^{\frac{7}{8}}, \quad ||\nabla\tilde{\mathbf{u}}_0||_{L^2} \approx A/M^{\frac{7}{8}}, \quad ||\tilde{\mathbf{B}}_0||_{L^2} \approx A/M^{\frac{7}{8}}, \quad ||\nabla\tilde{\mathbf{B}}_0||_{L^2} \approx A/M^{\frac{7}{8}}. \tag{4.18}$$

Therefore, we have $||\tilde{\mathbf{u}}_0||_{L^2}||\nabla\tilde{\mathbf{u}}_0||_{L^2} \approx A^2/M^{\frac{7}{4}}, \quad ||\tilde{\mathbf{B}}_0||_{L^2}||\nabla\tilde{\mathbf{B}}_0||_{L^2} \approx A^2/M^{\frac{7}{4}}$. By choosing A is much larger than M, the product can be made enough large. Moreover, we have the energy inequality of the three-dimensional incompressible MHD equations (2.1)

$$\begin{aligned} &||\tilde{\mathbf{u}}(r,z,t)||_{L^2}^2 + ||\tilde{\mathbf{B}}(r,z,t)||_{L^2}^2 + 2\int_0^t (||\nabla\tilde{\mathbf{u}}(r,z,t)||_{L^2}^2 + ||\nabla\tilde{\mathbf{B}}(r,z,t)||_{L^2}^2)d\tau \\ &\quad \le ||\tilde{\mathbf{u}}_0(r,z)||_{L^2}^2 + ||\tilde{\mathbf{B}}_0(r,z)||_{L^2}^2. \end{aligned} \tag{4.19}$$

We arrive at

$$||\tilde{\mathbf{u}}(r,z,t)||_{L^2} \le A/M^{\frac{7}{8}}, \quad ||\tilde{\mathbf{B}}(r,z,t)||_{L^2} \le A/M^{\frac{7}{8}}. \tag{4.20}$$

Therefore, we obtain a bound for the perturbed velocity field $\mathbf{u}$ and magnetic $\mathbf{B}$ in L^2 norm:

$$\begin{aligned} ||\mathbf{u}||_{L^2} &= ||\tilde{\mathbf{u}} - \bar{\mathbf{u}}||_{L^2} \le ||\tilde{\mathbf{u}}||_{L^2} + ||\bar{\mathbf{u}}||_{L^2} \le A/M^{\frac{7}{8}}, \\ ||\mathbf{B}||_{L^2} &= ||\tilde{\mathbf{B}} - \bar{\mathbf{B}}||_{L^2} \le ||\tilde{\mathbf{B}}||_{L^2} + ||\bar{\mathbf{B}}||_{L^2} \le A/M^{\frac{7}{8}}. \end{aligned} \tag{4.21}$$

We have the following prior estimates for the three-dimensional and one-dimensional MHD equations

$$\begin{aligned} &||\tilde{u}^r(r,z,t)||_{L^2} \le A/M^{\frac{7}{8}}, \qquad ||\tilde{u}^z(r,z,t)||_{L^2} \le A/M^{\frac{7}{8}}, \\ &||\tilde{B}^r(r,z,t)||_{L^2} \le A/M^{\frac{7}{8}}, \qquad ||\tilde{B}^z(r,z,t)||_{L^2} \le A/M^{\frac{7}{8}}, \end{aligned} \tag{4.22}$$

$$\begin{aligned} &||\bar{v}_1(z,t)||_{L^\infty} \le Ad_0/M, \qquad ||\bar{l}_1(z,t)||_{L^\infty} \le Ad_0/M, \\ &||\bar{v}_{1z}(z,t)||_{L^2} \le Ad_0/M, \qquad ||\bar{l}_{1z}(z,t)||_{L^2} \le Ad_0/M, \\ &||\bar{v}_{1zz}(z,t)||_{L^2} \le Ad_0/M, \qquad ||\bar{l}_{1zz}(z,t)||_{L^2} \le Ad_0/M, \\ &||\bar{\psi}_1||_{L^\infty} \le Ad_0/M, \qquad ||\bar{\psi}_{1z}||_{L^\infty} \le Ad_0/M, \\ &||\bar{\Phi}_1||_{L^\infty} \le Ad_0/M, \qquad ||\bar{\Phi}_{1z}||_{L^\infty} \le Ad_0/M \end{aligned} \tag{4.23}$$

$$||u^r(r,z,t)||_{L^2} \le \frac{A}{M^{7/8}}(1+d_0), \qquad ||u^z(r,z,t)||_{L^2} \le \frac{A^{1/2}}{M^{7/8}} + \frac{Ad_0}{M}(2+C_1), \tag{4.24}$$

$$\begin{aligned} &||\psi_{1z}||_{L^2} \le \frac{A^{1/2}}{M^{7/16}}||\omega_1||_{L^2}^{1/2} + \frac{AC_2^{1/2}d_0^{1/2}}{M^{17/16}}, \\ &||\Phi_{1z}||_{L^2} \le \frac{A^{1/2}}{M^{7/16}}||j_1||_{L^2}^{1/2} + \frac{AC_2^{1/2}d_0^{1/2}}{M^{17/16}}, \end{aligned} \tag{4.25}$$

$$||\psi_{1zz}||_{L^2} + ||\psi_{1zr}||_{L^2} + ||\psi_{1rr}||_{L^2} + 3||\psi_{1r}/r||_{L^2} \le ||\omega_1||_{L^2} + \frac{AC_2d_0}{M^{5/4}}, \tag{4.26}$$

$$||\Phi_{1zz}||_{L^2} + ||\Phi_{1zr}||_{L^2} + ||\Phi_{1rr}||_{L^2} + 3||\Phi_{1r}/r||_{L^2} \le ||j_1||_{L^2} + \frac{AC_2d_0}{M^{5/4}}, \tag{4.27}$$

$$||\psi_{1zzz}||_{L^2} + ||\psi_{1zzr}||_{L^2} + ||\psi_{1zrr}||_{L^2} + 3||\psi_{1zr}/r||_{L^2} \le ||\omega_{1z}||_{L^2} + \frac{AC_2d_0}{M^{5/4}}, \tag{4.28}$$

$$||\Phi_{1zzz}||_{L^2} + ||\Phi_{1zzr}||_{L^2} + ||\Phi_{1zrr}||_{L^2} + 3||\Phi_{1zr}/r||_{L^2} \le ||j_{1z}||_{L^2} + \frac{AC_2d_0}{M^{5/4}}, \tag{4.29}$$

where (4.24)–(4.29) will be proved in Appendix. We define

$$H^2(t) = \int (u_1^2 + \omega_1^2)\, r\mathrm{d}r\mathrm{d}z, \tag{4.30}$$

$$E^2(t) = \int (|\nabla u_1|^2 + |\nabla \omega_1|^2)\, r\mathrm{d}r\mathrm{d}z, \tag{4.31}$$

$$G^2(t) = \int (B_1^2 + j_1^2)\, r\mathrm{d}r\mathrm{d}z, \tag{4.32}$$

$$K^2(t) = \int (|\nabla B_1|^2 + |\nabla j_1|^2)\, r\mathrm{d}r\mathrm{d}z, \tag{4.33}$$

where the integration of r and z is over $[0, 1] \times [0, \infty)$. We further assume that the initial conditions for $u_1, \omega_1, \psi_1, B_1, j_1$, and Φ_1 are odd functions of z, then we get $\tilde{u}_1, \tilde{\omega}_1, \tilde{\psi}_1, \tilde{B}_1, \tilde{j}_1$ and $\tilde{\Phi}_1$ are odd functions of z for all time. Since $\bar{u}_1, \bar{\omega}_1, \bar{B}_1, \bar{j}_1$, and $\bar{\Psi}_1$ are odd functions of z, we can get $u_1, \omega_1, \psi_1, B_1, j_1$, and Φ_1 are odd functions of z. It follows by the Poincaré inequality and we have

$$\int u_1^2\, r\mathrm{d}r\mathrm{d}z \le \int (u_{1z})^2\, r\mathrm{d}r\mathrm{d}z \le \int |\nabla u_1|^2\, r\mathrm{d}r\mathrm{d}z, \tag{4.34}$$

$$\int \omega_1^2\, r\mathrm{d}r\mathrm{d}z \le \int (\omega_{1z})^2\, r\mathrm{d}r\mathrm{d}z \le \int |\nabla \omega_1|^2\, r\mathrm{d}r\mathrm{d}z, \tag{4.35}$$

$$\int B_1^2\, r\mathrm{d}r\mathrm{d}z \le \int (B_{1z})^2\, r\mathrm{d}r\mathrm{d}z \le \int |\nabla B_1|^2\, r\mathrm{d}r\mathrm{d}z, \tag{4.36}$$

$$\int j_1^2\, r\mathrm{d}r\mathrm{d}z \le \int (j_{1z})^2\, r\mathrm{d}r\mathrm{d}z \le \int |\nabla j_1|^2\, r\mathrm{d}r\mathrm{d}z, \tag{4.37}$$

and $H \le E, G \le K$.

Theorem 4.1 *Assume that the initial conditions for $u_1(r, z, t), \omega_1(r, z, t), \psi_1(r, z, t)$, $B_1(r, z, t), j_1(r, z, t)$, and $\Phi_1(r, z, t)$ are smooth functions of compact support and odd in z. For any given $A > 1, d_0 > 1$, there exists $C(A, d_0) > 0$ such that if $M > C(A, d_0)$, $H(0) \le 1, G(0) \le 1$, then the solutions to the three-dimensional MHD equations given by* (4.2)–(4.7) *remain smooth for all times.*

Proof We use (2.13), (2.14), and (4.1) to derive the corresponding evolution for $\tilde{u}_1(r, z, t), \tilde{\omega}_1(r, z, t), \tilde{\psi}_1(r, z, t), \tilde{B}_1(r, z, t), \tilde{j}_1(r, z, t)$, and $\tilde{\Phi}_1(r, z, t)$ as follows:

$$\tilde{u}_{1t} - 2\tilde{\psi}_{1z}\tilde{u}_1 + \tilde{u}^r\tilde{u}_{1r} + \tilde{u}^z\tilde{u}_{1z} = \left(\tilde{u}_{1zz} + \tilde{u}_{1rr} + \frac{3}{r}\tilde{u}_{1r}\right)$$
$$-2\tilde{\Phi}_{1z}\tilde{B}_1 + \tilde{B}^r\tilde{B}_{1r} + \tilde{B}^z\tilde{B}_{1z}, \tag{4.38}$$

$$\tilde{\omega}_{1t} + \tilde{u}^r \tilde{\omega}_{1r} + \tilde{u}^z \tilde{\omega}_{1z} - (\tilde{u}_1^2)_z = \left(\tilde{\omega}_{1zz} + \tilde{\omega}_{1rr} + \frac{3}{r}\tilde{\omega}_{1r}\right)$$
$$+\tilde{B}^r \tilde{j}_{1r} + \tilde{B}^z \tilde{j}_{1z} - (\tilde{B}_1^2)_z, \tag{4.39}$$
$$-(\tilde{\psi}_{1zz} + \tilde{\psi}_{1rr} + \frac{3}{r}\tilde{\psi}_{1r}) = \tilde{\omega}_1, \tag{4.40}$$
$$\tilde{B}_{1t} + \tilde{u}^r \tilde{B}_{1r} + \tilde{u}^z \tilde{B}_{1z} = \left(\tilde{B}_{1zz} + \tilde{B}_{1rr} + \frac{3}{r}\tilde{B}_{1r}\right) + \tilde{B}^r \tilde{u}_{1r} + \tilde{B}^z \tilde{u}_{1z}, \tag{4.41}$$
$$\tilde{j}_{1t} - \tilde{\psi}_{1z}\tilde{j}_1 + \tilde{u}^r \tilde{j}_{1r} + \tilde{u}^z \tilde{j}_{1z} = (\tilde{j}_{1zz} + \tilde{j}_{1rr} + \frac{3}{r}\tilde{j}_{1r}) - \tilde{\Phi}_{1z}\tilde{\omega}_1 + \tilde{B}^r \tilde{\omega}_{1r}$$
$$+\tilde{B}^z \tilde{\omega}_{1z} + \frac{1}{r}(\tilde{B}^z_z - \tilde{B}^r_r)(\tilde{u}^r_z + \tilde{u}^z_r) + \frac{1}{r}(\tilde{u}^r_r - \tilde{u}^z_z)(\tilde{B}^r_z + \tilde{B}^z_r), \tag{4.42}$$
$$-\left(\tilde{\Phi}_{1zz} + \tilde{\Phi}_{1rr} + \frac{3}{r}\tilde{\Phi}_{1r}\right) = \tilde{j}_1. \tag{4.43}$$

In the following, we will discuss the estimates of the velocity equation u_1 (Part I), the vorticity equation ω_1 (Part II), the magnetic B_1 (Part III), and the current density j_1 (Part IV).

4.1 Part I The estimates of the velocity equation u_1

Substituting (4.2) into (4.38) and using (4.1), we obtain an evolution equation of u_1

$$u_{1t} + \tilde{u}^r u_{1r} + \tilde{u}^z u_{1z} = \Delta u_1 + 2(\bar{\psi}_{1z}\phi + \psi_{1z})u_1$$
$$- 2(\bar{\Phi}_{1z}\phi + \Phi_{1z})B_1 + \tilde{B}^r B_{1r} + \tilde{B}^z B_{1z}, \tag{4.44}$$

where $\Delta u_1 = u_{1rr} + u_{1zz} + \frac{3}{r}u_{1r}$. On the other hand, (2.17) of one-dimensional model vanish since $\bar{u}_1(z,t) = 0$, $\bar{B}_1(z,t) = 0$. Multiplying (4.44) by u_1 and integrating over $[0,1] \times [0,\infty)$, we obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int u_1^2\, r\mathrm{d}r\mathrm{d}z \le -\int |\nabla u_1|^2\, r\mathrm{d}r\mathrm{d}z + 2\int (\bar{\psi}_{1z}\phi + \psi_{1z})u_1^2\, r\mathrm{d}r\mathrm{d}z$$
$$-2\int (\bar{\Phi}_{1z}\phi + \Phi_{1z})B_1 u_1\, r\mathrm{d}r\mathrm{d}z + \int (\tilde{B}^r B_{1r} + \tilde{B}^z B_{1z})u_1\, r\mathrm{d}r\mathrm{d}z$$
$$= -\int |\nabla u_1|^2\, r\mathrm{d}r\mathrm{d}z + I_{(a)} + I_{(b)} + I_{(c)}. \tag{4.45}$$

We need the relational expressions:

$$(1)\ \int u_1(\tilde{u}^r u_{1r} + \tilde{u}^z u_{1z})\, r\mathrm{d}r\mathrm{d}z = \int u_1\tilde{u}^r\, r\mathrm{d}u_1\mathrm{d}z + \int u_1\tilde{u}^z\, r\mathrm{d}r\mathrm{d}u_1$$
$$= -\int u_1^2\left[(\tilde{u}^r)_r + \frac{1}{r}\tilde{u}^r + (\tilde{u}^z)_z\right] r\mathrm{d}r\mathrm{d}z$$
$$-\int u_1(\tilde{u}^r u_{1r} + \tilde{u}^z u_{1z})\, r\mathrm{d}r\mathrm{d}z, \tag{4.46}$$

we have $\int u_1(\tilde{u}^r u_{1r} + \tilde{u}^z u_{1z})\, r\mathrm{d}r\mathrm{d}z = 0$.

$$(2)\ \int u_1 \Delta u\, r\mathrm{d}r\mathrm{d}z \leq \ -\int |\nabla u_1|^2\, r\mathrm{d}r\mathrm{d}z\, r\mathrm{d}r\mathrm{d}z, \tag{4.47}$$

where we apply the integration by parts and the incompressible constrains (2.7). In the following, we will estimate the right-hand of (4.45): I$_{(a)}$-I$_{(b)}$.

$$\begin{aligned} I_{(a)} &= 2\int (\bar{\psi}_{1z}\phi + \psi_{1z})u_1^2\, r\mathrm{d}r\mathrm{d}z \leq 2(||\bar{v}_1||_{L^2}||\phi||_{L^\infty} + ||\psi_{1z}||_{L^2})||u_1||_{L^4}^2 \\ &\leq 2\left(\frac{Ad_0}{M} + \frac{A^{1/2}}{M^{7/16}}H^{1/2} + \frac{AC_2^{1/2}d_0^{1/2}}{M^{17/16}}\right)H^{1/2}E^{3/2}, \end{aligned} \tag{4.48}$$

where we use $\bar{\psi}_{1z} = \bar{v}_1$, (4.8), (4.22)–(4.25), the Hölder inequality and the sobolev interpolation inequality $||f||_{L^4} \leq ||f||_{L^2}^{1/4}||\nabla f||_{L^2}^{3/4}$.

$$\begin{aligned} I_{(b)} &= 2\int (\bar{\Phi}_{1z}\phi + \Phi_{1z})u_1 B_1\, r\mathrm{d}r\mathrm{d}z \\ &\leq 2(||\bar{l}_1||_{L^2}||\phi||_{L^\infty} + ||\Phi_{1z}||_{L^2})||u_1||_{L^4}||B_1||_{L^4} \\ &\leq 2\left(\frac{Ad_0}{M} + \frac{A^{1/2}}{M^{7/16}}G^{1/2} + \frac{AC_2^{1/2}d_0^{\frac{1}{2}}}{M^{17/16}}\right)H^{1/4}E^{3/4}G^{1/4}K^{3/4}. \end{aligned} \tag{4.49}$$

Therefore, we get

$$\begin{aligned} \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int u_1^2\, r\mathrm{d}r\mathrm{d}z \leq &-\int |\nabla u_1|^2\, r\mathrm{d}r\mathrm{d}z \\ &+2\left(\frac{Ad_0}{M} + \frac{A^{1/2}}{M^{7/16}}H^{1/2} + \frac{AC_2^{1/2}d_0^{1/2}}{M^{17/16}}\right)H^{1/2}E^{3/2} \\ &+2\left(\frac{Ad_0}{M} + \frac{A^{1/2}}{M^{7/16}}G^{1/2} + \frac{AC_2^{1/2}d_0^{\frac{1}{2}}}{M^{17/16}}\right) \\ &\times H^{1/4}E^{3/4}G^{1/4}K^{3/4} + I_{(c)}. \end{aligned} \tag{4.50}$$

Note: It is not necessary to estimate I$_{(c)}$ since the sum of I$_{(c)}$ and $III_{(a)}$ is zero.

4.2 Part II The estimates of the vorticity equation ω_1

Substituting (4.3) into (4.39) and using (4.1), we obtain an evolution equation of ω_1

$$\begin{aligned} &\omega_{1t} + \tilde{u}^r\omega_{1r} + \tilde{u}^z\omega_{1z} \\ &\quad = -\bar{\omega}_{1t}\phi + \Delta\omega_1 + \Delta(\bar{\omega}_1\phi) - \tilde{u}^r\bar{\omega}_1\phi_r - \tilde{u}^z\bar{\omega}_{1z}\phi \\ &\qquad + \tilde{B}^r\bar{j}_1\phi_r + \tilde{B}^z\bar{j}_{1z}\phi + \tilde{B}^r j_{1r} + \tilde{B}^z j_{1z} + (u_1^2)_z - (B_1^2)_z. \end{aligned} \tag{4.51}$$

Moreover, $\bar{\omega}_1(z,t)$ satisfies the one-dimensional model equation (3.4)

$$(\bar{\omega}_1)_t + 2\bar{\psi}_1(\bar{\omega}_1)_z = (\bar{\omega}_1)_{zz} + 2\bar{\Phi}_1(\bar{j}_1)_z. \tag{4.52}$$

Multiplying (4.52) with ϕ and subtracting the result equation from (4.51), we obtain

$$\begin{aligned}\omega_{1t} + (\tilde{u}^r\omega_{1r} + \tilde{u}^z\omega_{1z}) = \Delta\omega_1 + \bar{\omega}_1\Delta_r\phi - \tilde{u}^r\bar{\omega}_1\phi_r + \tilde{B}^r\bar{j}_1\phi_r + \tilde{B}^z\bar{j}_{1z}\phi \\ + \bar{\omega}_{1z}\phi(2\bar{\psi}_1 - \tilde{u}^z) + (\tilde{B}^r j_{1r} + \tilde{B}^z j_{1z}) \\ - 2\bar{\Phi} j_{1z}\phi + (u_1^2)_z - (B_1^2)_z.\end{aligned} \tag{4.53}$$

Multiplying (4.53) by ω_1 and integrating over $[0,1]\times[0,\infty)$, we have

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int \omega_1^2\, r\mathrm{d}r\mathrm{d}z \\ &\leq -\int |\nabla\omega_1|^2\, r\mathrm{d}r\mathrm{d}z + \int \bar{\omega}_1(\Delta_r\phi)\omega_1\, r\mathrm{d}r\mathrm{d}z \\ &\quad + \int \omega_1\bar{\omega}_{1z}\phi\{\bar{\psi}_1[r\phi_r + 2(\phi-1)] + u^z\}\, r\mathrm{d}r\mathrm{d}z \\ &\quad + \int (\tilde{B}^r\bar{j}_1 - \tilde{u}^r\bar{\omega}_1)\phi_r\omega_1\, r\mathrm{d}r\mathrm{d}z + \int \tilde{B}^z\bar{j}_{1z}\phi\omega_1\, r\mathrm{d}r\mathrm{d}z \\ &\quad + \int (\tilde{B}^r j_{1r} + \tilde{B}^z j_{1z})\omega_1\, r\mathrm{d}r\mathrm{d}z + \int 2\bar{\Phi} j_{1z}\phi\omega_1\, r\mathrm{d}r\mathrm{d}z \\ &\quad + \int [(u_1^2)_z - (B_1^2)_z]\omega_1\, r\mathrm{d}r\mathrm{d}z \\ &= -\int |\nabla\omega_1|^2\, r\mathrm{d}r\mathrm{d}z + II_{(a)} + II_{(b)} + II_{(c)} + II_{(d)} + II_{(e)} + II_{(f)} + II_{(g)}.\end{aligned} \tag{4.54}$$

We need the relational expressions:

$$\begin{aligned}&(1) \int \omega_1(\tilde{u}^r\omega_{1r} + \tilde{u}^z\omega_{1z})\, r\mathrm{d}r\mathrm{d}z \\ &= \int \omega_1\tilde{u}^r\, rd\omega_1\mathrm{d}z + \int \omega_1\tilde{u}^z\, r\mathrm{d}rd\omega_1 \\ &= -\int \omega_1^2\left[(\tilde{u}^r)_r + \frac{1}{r}\tilde{u}^r + (\tilde{u}^z)_z\right] r\mathrm{d}r\mathrm{d}z - \int \omega_1(\tilde{u}^r\omega_{1r} + \tilde{u}^z\omega_{1z})\, r\mathrm{d}r\mathrm{d}z \\ &= -\int \omega_1(\tilde{u}^r\omega_{1r} + \tilde{u}^z\omega_{1z})\, r\mathrm{d}r\mathrm{d}z,\end{aligned} \tag{4.55}$$

we have $\int \omega_1(\tilde{u}^r\omega_{1r} + \tilde{u}^z\omega_{1z})\, r\mathrm{d}r\mathrm{d}z = 0$.

$$(2) \int \omega_1\Delta\omega_1\, r\mathrm{d}r\mathrm{d}z \leq -\int |\nabla\omega_1|^2\, r\mathrm{d}r\mathrm{d}z, \tag{4.56}$$

where we use the integration by parts, the incompressible constrains (2.7) and the Green formula. In the following, we will estimate the right-hand of (4.54): $II_{(a)}$-$\mathrm{II}_{(g)}$.

$$II_{(a)} = \int \bar{\omega}_1(\Delta_r\phi)\omega_1 \, r\mathrm{d}r\mathrm{d}z \le ||\Delta_r\phi||_{L^\infty}||\bar{\omega}_1||_{L^2}||\omega_1||_{L^2} \le \frac{AC_2d_0}{M^{5/4}}H. \quad (4.57)$$

We estimate $II_{(b)}$ by applying $\bar{\omega}_1 = \bar{v}_{1z}$, the Höler inequality, the integration by parts, Appendix (5.2)–(5.5) and

$$\begin{aligned} ||\nabla u^z||_{L^2} &= ||(3\psi_{1r} + r\psi_{1rr}, 2\psi_{1z} + r\psi_{1rz})||_{L^2} \\ &\le 3||\psi_{1r}||_{L^2} + ||r\psi_{1rr}||_{L^2} + 2||\psi_{1z}||_{L^2} + ||r\psi_{1zr}||_{L^2}, \\ ||u^z_z||_{L^2} &= ||2\psi_{1z} + r\psi_{1zr}||_{L^2} \le 2||\psi_{1z}||_{L^2} + ||r\psi_{1zr}||_{L^2}. \end{aligned} \quad (4.58)$$

$$\begin{aligned} II_{(b)} &= -\int \omega_1\bar{\omega}_{1z}\phi\{\bar{\psi}_1[r\phi_r + 2(\phi-1)] + u^z\} \, r\mathrm{d}r\mathrm{d}z \\ &= \int \omega_{1z}\bar{\omega}_1\phi\{\bar{\psi}_1[r\phi_r + 2(\phi-1)] + u^z\} \, r\mathrm{d}r\mathrm{d}z \\ &\quad + \int \omega_1\bar{\omega}_1\phi\{\bar{\psi}_{1z}[r\phi_r + 2(\phi-1)] + u^z_z\} \, r\mathrm{d}r\mathrm{d}z = II_{(b1)} + II_{(b2)}. \end{aligned} \quad (4.59)$$

$$\begin{aligned} II_{(b1)} &\le ||\phi||_{L^\infty}||r\phi_r + 2(\phi-1)||_{L^\infty}\int \omega_{1z}\bar{\omega}_1\bar{\psi}_1 \, r\mathrm{d}r\mathrm{d}z + ||\phi||_{L^\infty} \\ &\quad \int \omega_{1z}\bar{\omega}_1 u^z \, r\mathrm{d}r\mathrm{d}z \\ &\le ||\nabla\omega_1||_{L^2}[(2+C_1)||\bar{v}_{1z}||_{L^2}||\bar{\psi}_1||_{L^\infty} \\ &\quad + ||\bar{v}_{1z}||^{1/4}_{L^2}||\bar{v}_{1zz}||^{3/4}_{L^2}||u^z||^{1/4}_{L^2}||\nabla u^z||^{3/4}_{L^2}] \\ &\le \frac{(2+C_1)A^2d_0^2}{M^2}E + E\left[\frac{A^{1/4}}{M^{7/32}} + \frac{A^{1/4}d_0^{1/4}(2+C_1)^{1/4}}{M^{1/4}}\right] \\ &\quad \cdot\left[3^{3/4}\left(\frac{Ad_0}{M^{29/32}}H^{3/4} + \frac{A^{7/4}C_2^{3/4}d_0^{7/4}}{M^{59/32}}\right)\right. \\ &\quad \left. + 2^{3/4}\left(\frac{A^{11/8}H^{3/8}d_0}{M^{85/64}} + \frac{A^{11/4}C_2^{3/8}d_0^{11/8}}{M^{115/64}}\right)\right]. \end{aligned} \quad (4.60)$$

$$\begin{aligned} II_{(b2)} &\le ||\phi||_{L^\infty}||r\phi_r + 2(\phi-1)||_{L^\infty}\int \omega_1\bar{\omega}_1\bar{\psi}_{1z} \, r\mathrm{d}r\mathrm{d}z + ||\phi||_{L^\infty} \\ &\quad \times\int \omega_1\bar{\omega}_1 u^z_z \, r\mathrm{d}r\mathrm{d}z \\ &\le (2+C_1)||\bar{\psi}_{1z}||_{L^\infty}||\omega_1||_{L^2}||\bar{\omega}_1||_{L^2} + ||\omega_1||_{L^4}||\bar{\omega}_1||_{L^2}||u^z_z||_{L^2} \\ &\le \frac{(2+C_1)A^2d_0^2}{M^2}H + \left[2\left(\frac{A^{3/2}d_0}{M^{23/16}}H^{1/2} + \frac{A^2C_2^{1/2}d_0^{3/2}}{M^{33/16}}\right)\right. \end{aligned}$$

$$+\left(\frac{Ad_0}{M^{7/8}}H+\frac{A^2C_2d_0^2}{M^{11/8}}\right)\Bigg]H^{1/4}E^{3/4}. \tag{4.61}$$

$$\begin{aligned} II_{(c)} &= -\int(\tilde{B}^r\bar{j}_1-\tilde{u}^r\bar{\omega}_1)\phi_r\omega_1\ r\mathrm{d}r\mathrm{d}z \\ &\le \|\phi_r\|_{L^\infty}\|\omega_1\|_{L^2}(\|\tilde{\mathbf{B}}\|_{L^2}\|\bar{l}_{1z}\|_{L^2}+\|\tilde{\mathbf{u}}\|_{L^2}\|\bar{v}_{1z}\|_{L^2})\le\frac{2A^2C_1d_0}{M^2}H. \end{aligned} \tag{4.62}$$

$$\begin{aligned} II_{(d)} &= -\int\tilde{B}^z\bar{j}_{1z}\phi\omega_1\ r\mathrm{d}r\mathrm{d}z\le\|\tilde{B}^z\|_{L^4}\|\bar{l}_{1zz}\|_{L^2}\|\omega_1\|_{L^4} \\ &\le \|\tilde{B}^z\|_{L^2}^{1/4}\|\nabla\tilde{B}^z\|_{L^2}^{3/4}\|\bar{l}_{1zz}\|_{L^2}\|\omega_1\|_{L^2}^{1/4}\|\nabla\omega_1\|_{L^2}^{3/4} \\ &\le \Bigg[\frac{A^2d_0^{7/4}(3C_1+C_2)^{3/4}}{M^{33/16}}+5^{3/4}\left(\frac{A^{5/4}d_0}{M^{9/8}}G^{3/4}+\frac{A^2C_2^{3/4}d_0^{7/4}}{M^{33/16}}\right) \\ &\quad +\frac{A^2d_0^{7/4}(2+C_1)^{3/4}}{M^{15/8}} \\ &\quad +\frac{2^{3/4}A^2d_0(1+d_0)^{3/4}}{M^{237/128}}\Bigg]H^{1/4}G^{3/4}, \end{aligned} \tag{4.63}$$

where we need the following facts:

$$\begin{aligned} \|\nabla\tilde{B}^z\|_{L^2} &= \|\nabla(2\bar{\Phi}_1\phi+r\bar{\Phi}_1\phi_r+2\Phi_1+r\Phi_{1r})\|_{L^2} \\ &= \|(3\bar{\Phi}_1\phi_r+r\bar{\Phi}_1\phi_{rr}+3\Phi_{1r}+r\Phi_{1rr},2\bar{\Phi}_{1z}\phi \\ &\quad +r\bar{\Phi}_{1z}\phi_r+2\Phi_{1z}+r\Phi_{1zr})\|_{L^2} \\ &\le 3\|\phi_r\|_{L^\infty}\|\bar{\Phi}_1\|_{L^\infty}+3\|r\phi_{rr}\|_{L^\infty}\|\bar{\Phi}_1\|_{L^\infty}+3\|\Phi_{1r}\|_{L^2}+\|r\Phi_{1rr}\|_{L^2} \\ &\quad +\|\phi\|_{L^\infty}\|\bar{\Phi}_{1z}\|_{L^2}+\|\bar{\Phi}_{1z}\|_{L^2}\|r\phi_r\|_{L^\infty}+2\|\bar{\Phi}_{1z}\|_{L^2}+\|r\Phi_{1zr}\|_{L^2}. \end{aligned} \tag{4.64}$$

$$II_{(f)}=\int2\bar{\Phi}_1j_{1z}\phi\omega_1\ r\mathrm{d}r\le2\|\bar{\Phi}_1\|_{L^\infty}\|j_{1z}\|_{L^2}\|\omega_1\|_{L^2}\le\frac{2Ad_0}{M}EH. \tag{4.65}$$

$$\begin{aligned} II_{(g)} &= -\int[(u_1^2)_z-(B_1^2)_z]\omega_1\ r\mathrm{d}r\mathrm{d}z=\int(u_1^2-B_1^2)\omega_{1z}\ r\mathrm{d}r\mathrm{d}z \\ &\le (\|u_1\|_{L^4}^2+\|B_1\|_{L^4}^2)\|\omega_{1z}\|_{L^2}\le\frac{2A}{M^{27/32}}E^{7/4}. \end{aligned} \tag{4.66}$$

Therefore, we get

$$\begin{aligned} &\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int\omega_1^2\ r\mathrm{d}r\mathrm{d}z \\ &\quad\le -\int|\nabla\bar{\omega}_1|^2\ r\mathrm{d}r\mathrm{d}z \\ &\qquad +\frac{AC_2d_0}{M^{5/4}}H+\frac{(2+C_1)A^2d_0^2}{M^2}E+E\left[\frac{A^{1/4}}{M^{7/32}}+\frac{A^{1/4}d_0^{1/4}(2+C_1)^{1/4}}{M^{1/4}}\right] \end{aligned}$$

$$\cdot\left[3^{3/4}\left(\frac{Ad_0}{M^{29/32}}H^{3/4}+\frac{A^{7/4}C_2^{3/4}d_0^{7/4}}{M^{59/32}}\right)+2^{3/4}\left(\frac{A^{11/8}H^{3/8}d_0}{M^{85/64}}\right.\right.$$
$$\left.\left.+\frac{A^{11/4}C_2^{3/8}d_0^{11/8}}{M^{115/64}}\right)\right]$$
$$+\frac{(2+C_1)A^2d_0^2}{M^2}H+\left[2\left(\frac{A^{3/2}d_0}{M^{23/16}}H^{1/2}+\frac{A^2C_2^{1/2}d_0^{3/2}}{M^{33/16}}\right)\right.$$
$$\left.+\left(\frac{Ad_0}{M^{7/8}}H+\frac{A^2C_2d_0^2}{M^{11/8}}\right)\right]H^{1/4}E^{3/4}+\frac{2A^2C_1d_0}{M^2}H$$
$$+\left[\frac{A^2d_0^{7/4}(3C_1+C_2)^{3/4}}{M^{33/16}}+5^{3/4}\left(\frac{A^{5/4}d_0}{M^{9/8}}G^{3/4}\right.\right.$$
$$\left.+\frac{A^2C_2^{3/4}d_0^{7/4}}{M^{33/16}}\right)+\frac{A^2d_0^{7/4}(2+C_1)^{3/4}}{M^{15/8}}$$
$$\left.+\frac{2^{3/4}A^2d_0(1+d_0)^{3/4}}{M^{237/128}}\right]H^{1/4}G^{3/4}+II_{(e)}+\frac{2Ad_0}{M}EH+\frac{2A}{M^{27/32}}E^{7/4}. \tag{4.67}$$

Note: It is not necessary to estimate $II_{(e)}$ since the sum of $II_{(e)}$ and $IV_{(g)}$ is zero.

4.3 Part III The estimates of the magnetic equation B_1

Substituting (4.5) into (4.41) and using (4.1), we obtain an evolution equation of B_1

$$B_{1t}+\tilde{u}^rB_{1r}+\tilde{u}^zB_{1z}=\Delta B_1+\tilde{B}^ru_{1r}+\tilde{B}^zu_{1z}. \tag{4.68}$$

Moreover, (2.20) of one-dimensional model vanish since $\bar{u}_1(z,t)=0$, $\bar{B}_1(z,t)=0$. Multiplying (4.68) by B_1 and integrate over $[0,1]\times[0,\infty)$, we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int B_1^2\,r\mathrm{d}r\mathrm{d}z\le-\int|\nabla B_1|^2\,r\mathrm{d}r\mathrm{d}z+III_{(a)}. \tag{4.69}$$

We need the relational expressions:

$$(1)\int B_1(\tilde{u}^rB_{1r}+\tilde{u}^zB_{1z})\,r\mathrm{d}r\mathrm{d}z$$
$$=-\int B_1^2\left[(\tilde{u}^r)_r+\frac{1}{r}\tilde{u}^r+(\tilde{u}^z)_z\right]r\mathrm{d}r\mathrm{d}z-\int B_1(\tilde{u}^rB_{1r}+\tilde{u}^zB_{1z})\,r\mathrm{d}r\mathrm{d}z. \tag{4.70}$$

Therefore, we have $\int B_1(\tilde{u}^r u_{1r} + \tilde{u}^z u_{1z})\, r\mathrm{d}r\mathrm{d}z = 0.$

$$(2) \int B_1 \Delta B_1\, r\mathrm{d}r\mathrm{d}z \le -\int |\nabla B_1|^2\, r\mathrm{d}r\mathrm{d}z. \tag{4.71}$$

In the following, we will estimate the right-hand of (4.74): $III_{(a)}$–$III_{(d)}$ using the Hölder inequality and the sobolev interpolation inequality .

$$\begin{aligned} III_{(a)} &= \int B_1(\tilde{B}^r u_{1r} + \tilde{B}^z u_{1z})\, r\mathrm{d}r\mathrm{d}z \\ &= -\int u_1 B_1 \left[(\tilde{B}^r)_r + \frac{1}{r}\tilde{B}^r + (\tilde{B}^z)_z \right] r\mathrm{d}r\mathrm{d}z \\ &\quad - \int u_1(B_{1r}\tilde{B}^r + B_{1z}\tilde{B}^z)\, r\mathrm{d}r\mathrm{d}z. \end{aligned} \tag{4.72}$$

Therefore, we arrive at

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int B_1^2\, r\mathrm{d}r\mathrm{d}z \le -\int |\nabla B_1|^2\, r\mathrm{d}r\mathrm{d}z + III_{(a)}. \tag{4.73}$$

Note: It is not necessary to estimate $III_{(a)}$ since the sum of $III_{(a)}$ and $\mathrm{I}_{(c)}$ is zero.

4.4 Part IV The estimates of j_1

Substituting (4.6) into (4.42) and using (4.1), we obtain an evolution equation of j_1:

$$\begin{aligned} j_{1t} + \tilde{u}^r j_{1r} + \tilde{u}^z j_{1z} = &-\bar{j}_{1t}\phi + (\bar{\psi}_{1z}\phi + \psi_{1z})\bar{j}_1\phi - (\bar{\Phi}_{1z}\phi + \Phi_{1z})\bar{\omega}_1\phi \\ &+ (\bar{\psi}_{1z}\phi + \psi_{1z}) j_1 - (\bar{\Phi}_{1z}\phi + \Phi_{1z})\omega_1 - \tilde{u}^r \bar{j}_1\phi_r + \tilde{B}^r\bar{\omega}_1\phi_r \\ &- \tilde{u}^z \bar{j}_{1z}\phi + \tilde{B}^z\bar{\omega}_{1z}\phi + \tilde{B}^r\omega_{1r} + \tilde{B}^z\omega_{1z} + \Delta j_1 + \bar{j}_1(\Delta_r\phi) \\ &+ \frac{1}{r}(\tilde{u}^r_r - \tilde{u}^z_z)(\tilde{B}^r_z + \tilde{B}^z_r) + \frac{1}{r}(\tilde{B}^z_z - \tilde{B}^r_r)(\tilde{u}^r_z + \tilde{u}^z_r). \end{aligned} \tag{4.74}$$

Moreover, $\bar{j}_1(z,t)$ satisfies the one-dimensional model equation (3.7)

$$(\bar{j}_1)_t + \bar{v}_1\bar{j}_1 + 2\bar{\psi}_1(\bar{j}_1)_z = (\bar{j}_1)_{zz} + \bar{l}_1\bar{\omega}_1 + 2\bar{\Phi}_1(\bar{\omega}_1)_z + 3\bar{v}_1(\bar{l}_1)_z - 3\bar{l}_1(\bar{v}_1)_z. \tag{4.75}$$

Multiplying (4.75) with ϕ and subtracting the result equation from (4.74), we obtain

$$\begin{aligned} &j_{1t} + \tilde{u}^r j_{1r} + \tilde{u}^z j_{1z} \\ &\quad = \Delta j_1 + \bar{j}_1(\Delta_r\phi) + (\bar{\psi}_{1z}\phi + \psi_{1z})\bar{j}_1\phi - (\bar{\Phi}_{1z}\phi + \Phi_{1z})\bar{\omega}_1\phi \\ &\qquad + (\bar{\psi}_{1z}\phi + \psi_{1z}) j_1 - (\bar{\Phi}_{1z}\phi + \Phi_{1z})\omega_1 + \tilde{u}^r\bar{j}_1\phi_r + \tilde{B}^r\bar{\omega}_1\phi_r + \bar{j}_{1z}\phi(2\bar{\psi}_1 - \tilde{u}^z) \\ &\qquad + \bar{\omega}_{1z}\phi(\tilde{B}^z - 2\bar{\Phi}_1) + \tilde{B}^r\omega_{1r} + \tilde{B}^z\omega_{1z} - \bar{l}_1\bar{\omega}_1\phi + \bar{v}_1\bar{j}_1\phi - 3\bar{v}_1(\bar{l}_1)_z\phi \\ &\qquad + 3\bar{l}_1(\bar{v}_1)_z\phi + \frac{1}{r}(\tilde{u}^r_r - \tilde{u}^z_z)(\tilde{B}^r_z + \tilde{B}^z_r) + \frac{1}{r}(\tilde{B}^z_z - \tilde{B}^r_r)(\tilde{u}^r_z + \tilde{u}^z_r). \end{aligned} \tag{4.76}$$

Multiplying (4.76) by j_1 and integrating over $[0, 1] \times [0, \infty)$, we get

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int j_1^2\, r\mathrm{d}r\mathrm{d}z \\
&\quad \le -\int |\nabla j_1|^2\, r\mathrm{d}r\mathrm{d}z \\
&\qquad + \int \left[(\bar{\psi}_{1z}\phi + \psi_{1z})\bar{j}_1\phi - (\bar{\Phi}_{1z}\phi + \Phi_{1z})\bar{\omega}_1\phi\right] j_1\, r\mathrm{d}r\mathrm{d}z \\
&\qquad + \int [(\bar{\psi}_{1z}\phi + \psi_{1z}) j_1 - (\bar{\Phi}_{1z}\phi + \Phi_{1z})\omega_1] j_1\, r\mathrm{d}r\mathrm{d}z \\
&\qquad + \int (-\tilde{u}^r \bar{j}_1\phi_r + \tilde{B}^r \bar{\omega}_1\phi_r) j_1\, r\mathrm{d}r\mathrm{d}z + \int \bar{j}_{1z}\phi(2\bar{\psi}_1 - \tilde{u}^z) j_1\, r\mathrm{d}r\mathrm{d}z \\
&\qquad + \int \bar{\omega}_{1z}\phi(\tilde{B}^z - 2\bar{\Phi}_1) j_1\, r\mathrm{d}r\mathrm{d}z + \int \bar{j}_1(\Delta_r\phi) j_1\, r\mathrm{d}r\mathrm{d}z \\
&\qquad + \int (\tilde{B}^r\omega_{1r} + \tilde{B}^z\omega_{1z}) j_1\, r\mathrm{d}r\mathrm{d}z + \int (-\bar{l}_1\bar{\omega}_1\phi + \bar{v}_1\bar{j}_1\phi) j_1\, r\mathrm{d}r\mathrm{d}z \\
&\qquad + \int (-3\bar{v}_1\bar{l}_{1z}\phi + 3\bar{l}_1\bar{v}_{1z}\phi) j_1\, r\mathrm{d}r\mathrm{d}z + \int \frac{1}{r}(\tilde{u}^r_r - \tilde{u}^z_z)(\tilde{B}^r_z + \tilde{B}^z_r) j_1\, r\mathrm{d}r\mathrm{d}z \\
&\qquad + \int \frac{1}{r}(\tilde{B}^z_z - \tilde{B}^r_r)(\tilde{u}^r_z + \tilde{u}^z_r) j_1\, r\mathrm{d}r\mathrm{d}z \\
&\quad = -\int |\nabla j_1|^2\, r\mathrm{d}r\mathrm{d}z\, r\mathrm{d}r\mathrm{d}z + IV_{(a)} + IV_{(b)} + IV_{(c)} + IV_{(d)} + IV_{(e)} \\
&\qquad + IV_{(f)} + IV_{(g)} + IV_{(h)} + +IV_{(i)} + +IV_{(j)} + +IV_{(k)}.
\end{aligned} \tag{4.77}$$

We need the relational expressions:

$$\begin{aligned}
&(1) \int j_1(\tilde{u}^r j_{1r} + \tilde{u}^z j_{1z})\, r\mathrm{d}r\mathrm{d}z \\
&\quad = -\int j_1^2\left[(\tilde{u}^r)_r + \frac{1}{r}\tilde{u}^r + (\tilde{u}^z)_z\right] r\mathrm{d}r\mathrm{d}z - \int j_1(\tilde{u}^r j_{1r} + \tilde{u}^z j_{1z})\, r\mathrm{d}r\mathrm{d}z.
\end{aligned} \tag{4.78}$$

We have $\int j_1(\tilde{u}^r j_{1r} + \tilde{u}^z j_{1z})\, r\mathrm{d}r\mathrm{d}z = 0$.

$$(2) \int j_1\Delta j_1\, r\mathrm{d}r\mathrm{d}z \le -\int |\nabla j_1|^2\, r\mathrm{d}r\mathrm{d}z. \tag{4.79}$$

In the following, we will estimate the right-hand of (4.77): $IV_{(a)}$–$IV_{(g)}$.

$$\begin{aligned}
IV_{(a)} = &\int [(\bar{\psi}_{1z}\phi + \psi_{1z})\bar{j}_1\phi - (\bar{\Phi}_{1z}\phi + \Phi_{1z})\bar{\omega}_1\phi] j_1\, r\mathrm{d}r\mathrm{d}z \\
&\le ||\phi||^2_{L^\infty}||j_1||_{L^2}(||\bar{v}_1||_{L^\infty}||\bar{l}_{1z}||_{L^2} + ||\bar{l}_1||_{L^\infty}||\bar{v}_{1z}||_{L^2}) \\
&\quad + ||\phi||_{L^\infty}||j_1||^{1/4}_{L^2}||\nabla j_1||^{1/4}_{L^2} \\
&\quad \cdot (||\psi_{1z}||_{L^2}||\bar{l}_{1z}||^{1/4}_{L^2}||\bar{l}_{1zz}||^{3/4}_{L^2} + ||\Phi_{1z}||_{L^2}||\bar{v}_{1z}||^{1/4}_{L^2}||\bar{v}_{1zz}||^{3/4}_{L^2})
\end{aligned}$$

$$\leq 2\frac{A^2d_0^2}{M^2}G+\left(\frac{2A^{3/2}d_0}{M^{23/16}}+\frac{A^2C_2^{1/2}d_0^{3/2}}{M^{33/16}}\right)G^{1/4}K^{3/4}. \tag{4.80}$$

$$\begin{aligned}IV_{(b)} &= \int[(\bar\psi_{1z}\phi+\psi_{1z})j_1-(\bar\Phi_{1z}\phi+\Phi_{1z})\omega_1]j_1\ r\mathrm{d}r\mathrm{d}z\\ &\leq (||\bar\psi_{1z}||_{L^2}+||\psi_{1z}||_{L^2})||j_1||_{L^4}^2+(||\bar\Phi_{1z}||_{L^2}+||\Phi_{1z}||_{L^2})||\omega_1||_{L^4}||j_1||_{L^4}\\ &\leq +2\left(\frac{Ad_0}{M}+\frac{A^{1/2}}{M^{7/16}}H^{1/2}+\frac{AC_2^{1/2}d_0^{1/2}}{M^{17/16}}\right)G^{1/4}K^{3/4}\\ &\quad +2\left(\frac{Ad_0}{M}+\frac{A^{1/2}}{M^{7/16}}G^{1/2}+\frac{AC_2^{1/2}d_0^{1/2}}{M^{17/16}}\right)H^{1/4}E^{3/4}G^{1/4}K^{3/4}.\end{aligned} \tag{4.81}$$

$$\begin{aligned}IV_{(c)} &= \int(-\tilde u^r\bar j_1\phi_r+\tilde B^r\bar\omega_1\phi_r)j_1\ r\mathrm{d}r\mathrm{d}z\\ &\leq ||\phi_r||_{L^\infty}(||\tilde u^r||_{L^2}||\bar l_{1z}||_{L^4}||j_1||_{L^4}+||\tilde B^r||_{L^2}||\bar v_{1z}||_{L^4}||j_1||_{L^4})\\ &\leq \frac{2A^2C_1d_0}{M^2}G^{1/4}K^{3/4}.\end{aligned} \tag{4.82}$$

$$\begin{aligned}IV_{(d)} &= \int \bar j_{1z}\phi(2\bar\psi_1-\tilde u^z)j_1\ r\mathrm{d}r\mathrm{d}z\\ &= -\int j_1\bar j_{1z}\phi\{\bar\psi_1[r\phi_r+2(\phi-1)]+u^z\}\ r\mathrm{d}r\mathrm{d}z\\ &= \int j_{1z}\bar j_1\phi\{\bar\psi_1[r\phi_r+2(\phi-1)]+u^z\}\ r\mathrm{d}r\mathrm{d}z\\ &\quad +\int j_1\bar j_1\phi\{\bar\psi_{1z}[r\phi_r+2(\phi-1)]+u^z_z\}\ r\mathrm{d}r\mathrm{d}z = IV_{(d1)}+IV_{(d2)}.\end{aligned} \tag{4.83}$$

$$\begin{aligned}IV_{(d1)} &\leq ||\phi||_{L^\infty}||r\phi_r+2(\phi-1)||_{L^\infty}\int j_{1z}\bar j_1\bar\psi_1\ r\mathrm{d}r\mathrm{d}z+||\phi||_{L^\infty}\int j_{1z}\bar j_1u^z\ r\mathrm{d}r\mathrm{d}z\\ &\leq ||\nabla j_1||_{L^2}[(2+C_1)||\bar l_{1z}||_{L^2}||\bar\psi_1||_{L^\infty}+||\bar l_{1z}||_{L^2}^{1/4}||\bar l_{1zz}||_{L^2}^{3/4}||u^z||_{L^2}^{1/4}||\nabla u^z||_{L^2}^{3/4}]\\ &\leq \frac{(2+C_1)A^2d_0^2}{M^2}K+K\left[\frac{A^{1/4}}{M^{7/32}}+\frac{A^{1/4}d_0^{1/4}(2+C_1)^{1/4}}{M^{1/4}}\right]\\ &\quad\cdot\left[3^{3/4}\left(\frac{Ad_0}{M^{29/32}}H^{3/4}+\frac{A^{7/4}C_2^{3/4}d_0^{7/4}}{M^{59/32}}\right)\right.\\ &\quad\left.+2^{3/4}\left(\frac{A^{11/8}H^{3/8}d_0}{M^{85/64}}+\frac{A^{11/4}C_2^{3/8}d_0^{11/8}}{M^{115/64}}\right)\right].\end{aligned} \tag{4.84}$$

$$\begin{aligned}IV_{(d2)} &\leq ||\phi||_{L^\infty}||r\phi_r+2(\phi-1)||_{L^\infty}\int j_1\bar\omega_1\bar\psi_{1z}\ r\mathrm{d}r\mathrm{d}z+||\phi||_{L^\infty}\int \omega_1\bar j_1u^z_z\ r\mathrm{d}r\mathrm{d}z\\ &\leq (2+C_1)||\bar\psi_{1z}||_{L^\infty}||j_1||_{L^2}||\bar j_1||_{L^2}+||j_1||_{L^4}||\bar j_1||_{L^2}||u^z_z||_{L^2}\\ &\leq \frac{(2+C_1)A^2d_0^2}{M^2}G+\left[2\left(\frac{A^{3/2}d_0}{M^{23/16}}H^{1/2}+\frac{A^2C_2^{1/2}d_0^{3/2}}{M^{33/16}}\right)\right.\end{aligned}$$

$$+\left(\frac{Ad_0}{M^{7/8}}H+\frac{A^2C_2d_0^2}{M^{11/8}}\right)\Bigg]G^{1/4}K^{3/4}. \tag{4.85}$$

$$\begin{aligned}IV_{(e)}&=\int\bar{\omega}_{1z}\phi(\tilde{B}^z-2\bar{\Phi}_1)j_1\ r\mathrm{d}r\mathrm{d}z\\&=\int j_1\bar{\omega}_{1z}\phi\{\bar{\Phi}_1[r\phi_r+2(\phi-1)]+B^z\}\ r\mathrm{d}r\mathrm{d}z\\&=\int j_{1z}\bar{\omega}_1\phi\{\bar{\Phi}_1[r\phi_r+2(\phi-1)]+B^z\}\ r\mathrm{d}r\mathrm{d}z\\&\quad+\int j_1\bar{\omega}_1\phi\{\bar{\Phi}_{1z}[r\phi_r+2(\phi-1)]+B^z_z\}\ r\mathrm{d}r\mathrm{d}z=IV_{(e1)}+IV_{(e2)}.\end{aligned} \tag{4.86}$$

$$\begin{aligned}IV_{(e1)}&\le||\phi||_{L^\infty}||r\phi_r+2(\phi-1)||_{L^\infty}\int j_{1z}\bar{\omega}_1\bar{\Phi}_1\ r\mathrm{d}r\mathrm{d}z\\&\quad+||\phi||_{L^\infty}\int j_{1z}\bar{\omega}_1B^z\ r\mathrm{d}r\mathrm{d}z\\&\le||\nabla j_1||_{L^2}[(2+C_1)||\bar{v}_{1z}||_{L^2}||\bar{\Phi}_1||_{L^\infty}\\&\quad+||\bar{v}_{1z}||^{1/4}_{L^2}||\bar{v}_{1zz}||^{3/4}_{L^2}||B^z||^{1/4}_{L^2}||\nabla B^z||^{3/4}_{L^2}]\\&\le\frac{(2+C_1)A^2d_0^2}{M^2}K+K\left[\frac{A^{1/4}}{M^{7/32}}+\frac{A^{1/4}d_0^{1/4}(2+C_1)^{1/4}}{M^{1/4}}\right]\\&\quad\cdot\left[3^{3/4}\left(\frac{Ad_0}{M^{29/32}}G^{3/4}+\frac{A^{7/4}C_2^{3/4}d_0^{7/4}}{M^{59/32}}\right)+2^{3/4}\right.\\&\quad\left.\times\left(\frac{A^{11/8}d_0}{M^{85/64}}G^{3/8}+\frac{A^{11/4}C_2^{3/8}d_0^{11/8}}{M^{115/64}}\right)\right],\end{aligned} \tag{4.87}$$

$$\begin{aligned}IV_{(e2)}&\le||\phi||_{L^\infty}||r\phi_r+2(\phi-1)||_{L^\infty}\int j_1\bar{\omega}_1\bar{\Phi}_{1z}\ r\mathrm{d}r\mathrm{d}z+||\phi||_{L^\infty}\\&\quad\times\int j_1\bar{\omega}_1B^z_z\ r\mathrm{d}r\mathrm{d}z\\&\le(2+C_1)||\bar{\Phi}_{1z}||_{L^\infty}||j_1||_{L^2}||\bar{\omega}_1||_{L^2}+||j_1||_{L^4}||\bar{\omega}_1||_{L^2}||B^z_z||_{L^2}\\&\le\frac{(2+C_1)A^2d_0^2}{M^2}G+\left[2\left(\frac{A^{3/2}d_0}{M^{23/16}}G^{1/2}+\frac{A^2C_2^{1/2}d_0^{3/2}}{M^{33/16}}\right)\right.\\&\quad\left.+\left(\frac{Ad_0}{M^{7/8}}G+\frac{A^2C_2d_0^2}{M^{11/8}}\right)\right]G^{1/4}K^{3/4}.\end{aligned} \tag{4.88}$$

$$IV_{(f)}=\int\bar{j}_1(\Delta_r\phi)j_1\ r\mathrm{d}r\mathrm{d}z\le||\Delta_r\phi||_{L^\infty}||\bar{l}_{1z}||_{L^2}||j_1||_{L^2}\le\frac{AC_2d_0}{M^{5/4}}. \tag{4.89}$$

$$\begin{aligned}IV_{(g)}&=\int(\tilde{B}^r\omega_{1r}+\tilde{B}^z\omega_{1z})j_1\ r\mathrm{d}r\mathrm{d}z\\&=-\int\omega_1j_1[(\tilde{B}^r)_r+\tilde{B}^r/r+(\tilde{B}^z)_z]\ r\mathrm{d}r\mathrm{d}z\end{aligned}$$

$$-\int \omega_1(\tilde{B}^r j_{1r} + \tilde{B}^z j_{1z})\, r\mathrm{d}r\mathrm{d}z. \tag{4.90}$$

$$\begin{aligned} IV_{(h)} &= \int (-\bar{l}_1\bar{\omega}_1\phi + \bar{v}_1\bar{j}_1\phi) j_1\, r\mathrm{d}r\mathrm{d}z\mathrm{d}r\mathrm{d}z \\ &\le (||\bar{l}_1||_{L^\infty}||\bar{v}_{1z}||_{L^2} + ||\bar{v}_1||_{L^\infty}||\bar{l}_{1z}||_{L^2})||j_1||_{L^2} \le \frac{2A^2d_0^2}{M^2}G. \end{aligned} \tag{4.91}$$

$$\begin{aligned} IV_{(i)} &= \int (-3\bar{v}_1\bar{l}_{1z}\phi + 3\bar{l}_1\bar{v}_{1z}\phi) j_1\, r\mathrm{d}r\mathrm{d}z \\ &\le 3(||\bar{v}_1||_{L^\infty}||\bar{l}_{1z}||_{L^2} + ||\bar{l}_1||_{L^\infty}||\bar{v}_{1z}||_{L^2})||j_1||_{L^2} \le \frac{6A^2d_0^2}{M^2}G. \end{aligned} \tag{4.92}$$

As for $IV_{(j)}$ and $IV_{(k)}$, we apply the Hölder inequality, the sobolev interpolation inequality, (4.8)–(4.11), Appendix (5.3)–(5.8), and the following fact:

$$\begin{aligned} \tilde{u}^r_r &= -\bar{\psi}_{1z}\phi - r\bar{\psi}_{1z}\phi_r + u^r_r, \quad \tilde{u}^r_z = -r\bar{\psi}_{1zz}\phi + u^r_z, \\ \tilde{u}^z_r &= 3\bar{\psi}_1\phi_r + r\bar{\psi}_1\phi_{rr} + u^z_r, \quad \tilde{u}^z_z = 2\bar{\psi}_{1z}\phi + r\bar{\psi}_{1z}\phi_r + u^z_z, \\ \tilde{B}^r_r &= -\bar{\Phi}_{1z}\phi - r\bar{\Phi}_{1z}\phi_r + B^r_r, \quad \tilde{B}^r_z = -r\bar{\Phi}_{1zz}\phi + B^r_z, \\ \tilde{B}^z_r &= 3\bar{\Phi}_1\phi_r + r\bar{\Phi}_1\phi_{rr} + B^z_r, \quad \tilde{B}^z_z = 2\bar{\Phi}_{1z}\phi + r\bar{\Phi}_{1z}\phi_r + B^z_z. \\ u^r_r &= -\psi_{1z} - r\psi_{1rz}, \qquad u^r_z = -\psi_{1zz}, \\ u^z_r &= 3\psi_{1r} + r\psi_{1rr}, \qquad u^z_z = 2\psi_{1z} + r\psi_{1rz}, \\ B^r_r &= -\Phi_{1z} - r\Phi_{1rz}, \qquad B^r_z = -\Phi_{1zz}, \\ B^z_r &= 3\Phi_{1r} + r\Phi_{1rr}, \qquad B^z_z = 2\Phi_{1z} + r\Phi_{1rz}. \end{aligned} \tag{4.93}$$

$$\begin{aligned} IV_{(j)} &= \int \frac{1}{r} j_1(\tilde{u}^r_r - \tilde{u}^z_z)(\tilde{B}^r_z + \tilde{B}^z_r) j_1\, r\mathrm{d}r\mathrm{d}z \\ &= \int \frac{1}{r}(-3\bar{\psi}_{1z}\phi - 2r\bar{\psi}_{1z}\phi_r + u^r_r - u^z_z) \\ &\quad \cdot(-r\bar{\Phi}_{1zz}\phi + 3\bar{\Phi}_1\phi_r + r\bar{\Phi}_1\phi_{rr} + B^r_z + B^z_r) j_1\, r\mathrm{d}r\mathrm{d}z \\ &= \int \frac{1}{r}(-3\bar{\psi}_{1z}\phi - 2r\bar{\psi}_{1z}\phi_r)(-r\bar{\Phi}_{1zz}\phi + 3\bar{\Phi}_1\phi_r + r\bar{\Phi}_1\phi_{rr}) j_1\, r\mathrm{d}r\mathrm{d}z \\ &\quad + \int \frac{1}{r}(-3\bar{\psi}_{1z}\phi - 2r\bar{\psi}_{1z}\phi_r)(B^r_z + B^z_r) j_1\, r\mathrm{d}r\mathrm{d}z \\ &\quad + \int \frac{1}{r}(u^r_r - u^z_z)(-r\bar{\Phi}_{1zz}\phi + 3\bar{\Phi}_1\phi_r + r\bar{\Phi}_1\phi_{rr}) j_1\, r\mathrm{d}r\mathrm{d}z \\ &\quad + \int \frac{1}{r}(u^r_r - u^z_z)(B^r_z + B^z_r) j_1\, r\mathrm{d}r\mathrm{d}z \\ &= IV_{(j1)} + IV_{(j2)} + IV_{(j3)} + IV_{(j4)}. \end{aligned} \tag{4.94}$$

In the following, we will estimate $IV_{(j1)}$ and $IV_{(j4)}$

$$\begin{aligned} IV_{(j1)} \le\ & (3||\bar{v}_1||_{L^\infty}||\phi||_{L^\infty} + 2||\bar{v}_1||_{L^\infty}||\phi_r||_{L^\infty}) \\ & \cdot(||\phi||_{L^\infty}||\bar{l}_z||_{L^2} + ||\bar{\Phi}_1||_{L^\infty}||\phi_r/r||_{L^\infty} + ||\phi_{rr}||_{L^\infty}||\bar{\Phi}_1||_{L^\infty})||j_1||_{L^2} \end{aligned}$$

$$\leq \left(\frac{3Ad_0}{M} + \frac{2Ad_0C_1}{M^{9/8}}\right)\left(\frac{Ad_0}{M} + \frac{Ad_0C_1}{M^{9/8}} + \frac{AC_2d_0}{M^{5/4}}\right)G. \tag{4.95}$$

$$\begin{aligned} IV_{(j2)} &\leq ||\bar{v}_1||_{L^\infty}(3||\phi||_{L^\infty} + 2||\phi_r||_{L^\infty})(||\Phi_{1zz}||_{L^2} + 3||\Phi_{1r}/r||_{L^2} \\ &\quad +||\Phi_{1rr}||_{L^2})||j_1||_{L^2} \\ &\leq \left(\frac{3Ad_0}{M} + \frac{2Ad_0C_1}{M^{9/8}}\right)(G^2 + \frac{AC_2d_0}{M^{5/4}}G). \end{aligned} \tag{4.96}$$

$$\begin{aligned} IV_{(j3)} &\leq ||j_1||_{L^4}||\bar{l}_{1z}||_{L^4}||\phi||_{L^\infty}(3||\psi_{1z}||_{L^2} + 2||r\psi_{1rz}||_{L^2}) \\ &\quad +||\bar{\Phi}_1||_{L^\infty}(3||\phi_r/r||_{L^\infty} + ||\phi_{rr}||_{L^\infty})(3||\psi_{1z}||_{L^2} + 2||r\psi_{1rz}||_{L^2})||j_1||_{L^2}. \\ &\leq \left[3\left(\frac{A^{1/2}}{M^{11/16}}H^{1/2} + \frac{AC_2^{1/2}d_0^{1/2}}{M^{21/16}}\right) + 2\frac{1}{M^{1/8}}H + 2\frac{AC_2d_0}{M^{19/16}}\right] \\ &\quad \cdot\left[\frac{A^2d_0^2}{M^{7/4}}G^{1/4}K^{3/4} + \frac{3Ad_0C_1}{M} + \frac{AC_2d_0}{M}\right]. \end{aligned} \tag{4.97}$$

$$\begin{aligned} IV_{(j4)} &\leq (||\Phi_{1zz}||_{L^2} + 3||\Phi_{1r}/r||_{L^2} + ||\Phi_{1rr}||_{L^2})(3||\psi_{1z}||_{L^2}||j_1||_{L^4} \\ &\quad +2||r\psi_{1rz}||_{L^2}||j_1||_{L^2}) \\ &\leq (G + \frac{AC_2d_0}{M^{5/4}})\left[\left(\frac{A^{7/8}C_2^{3/4}d_0^{3/4}}{M^{67/64}}H^{1/8} + \frac{AC_2^{7/8}d_0^{7/8}}{M^{77/64}}\right)\right. \\ &\quad (2H+1)^{3/4}G^{1/4}K^{3/4} \\ &\quad \left. + \frac{2A^{1/2}C_2^{1/2}d_0^{1/2}}{M^{5/8}}(2H+1)^{1/2}K^{1/2} + \frac{2AC_2d_0}{M^{5/4}}(2H+1)^{1/2}\right]G. \end{aligned} \tag{4.98}$$

$$\begin{aligned} IV_{(k)} &= \int \frac{1}{r}j_1(\tilde{B}^z_z - \tilde{B}^r_r)(\tilde{u}^r_z + \tilde{u}^z_r)\, r\mathrm{d}r\mathrm{d}z \\ &= \int \frac{1}{r}j_1(3\bar{\Phi}_{1z}\phi + 2r\bar{\Phi}_{1z}\phi_r + B^z_z + B^r_r) \\ &\quad \cdot(-r\bar{\psi}_{1zz}\phi + 3\bar{\psi}_1\phi_r + r\bar{\psi}_1\phi_{rr} + u^r_z + u^z_r)\, r\mathrm{d}r\mathrm{d}z \\ &= \int \frac{1}{r}(3\bar{\Phi}_{1z}\phi + 2r\bar{\Phi}_{1z}\phi_r)(-r\bar{\psi}_{1zz}\phi + 3\bar{\psi}_1\phi_r + r\bar{\psi}_1\phi_{rr})j_1\, r\mathrm{d}r\mathrm{d}z \\ &\quad + \int \frac{1}{r}(3\bar{\Phi}_{1z}\phi + 2r\bar{\Phi}_{1z}\phi_r)(u^r_z + u^z_r)j_1\, r\mathrm{d}r\mathrm{d}z \\ &\quad + \int \frac{1}{r}(B^z_z + B^r_r)(-r\bar{\psi}_{1zz}\phi + 3\bar{\psi}_1\phi_r + r\bar{\psi}_1\phi_{rr})j_1\, r\mathrm{d}r\mathrm{d}z \\ &\quad + \int \frac{1}{r}(B^z_z + B^r_r)(u^r_z + u^z_r)j_1\, r\mathrm{d}r\mathrm{d}z = IV_{(k1)} + IV_{(k2)} + IV_{(k3)} + IV_{(k4)}. \end{aligned} \tag{4.99}$$

In the following, we estimate from $IV_{(k1)}$ to $IV_{(k4)}$:

$$\begin{aligned} IV_{(k1)} &\leq (3||\bar{\Phi}_{1z}||_{L^2}||\phi||_{L^\infty} + 2||\bar{\Phi}_{1z}||_{L^2}||r\phi_r||_{L^\infty}) \\ &\quad \cdot(||\bar{\psi}_{1zz}||_{L^2}||\phi||_{L^\infty} + 3||\bar{\psi}_1||_{L^2}||\phi_r/r||_{L^\infty} + ||\bar{\psi}_1||_{L^2}||\phi_{rr}||_{L^\infty})||j_1||_{L^2} \end{aligned}$$

$$\leq \frac{Ad_0}{M}(3+2C_1)\left[\frac{A}{M}d_0 + \frac{Ad_0(3C_1+C_2)}{M^{5/4}}\right]G. \tag{4.100}$$

$$\begin{aligned} IV_{(k2)} &\leq ||\bar{\Phi}_{1z}||_{L^2}(3||\phi||_{L^\infty} + 2||r\phi_r||_{L^\infty})(||\psi_{1zz}||_{L^2} + 3||\psi_{1r}/r||_{L^2} \\ &\quad + ||\psi_{1rr}||_{L^2})||j_1||_{L^2} \leq \frac{Ad_0(3+2C_1)}{M}G^2 + \frac{A^2C_2(3+2C_1)d_0^2}{M^{9/4}}G. \end{aligned} \tag{4.101}$$

$$\begin{aligned} IV_{(k3)} &\leq (3||\Phi_{1z}||_{L^2} + 2||r\Phi_{1zr}||_{L^2})[||\bar{v}||_{L^4}||\phi||_{L^\infty}||j_1||_{L^4} \\ &\quad + (3||\bar{\psi}_1||_{L^\infty}||\phi_r/r||_{L^\infty} + ||\bar{\psi}_1||_{L^\infty}||\phi_{rr}||_{L^\infty})||j_1||_{L^2}] \\ &\leq \left[3\left(\frac{A^{1/2}}{M^{11/16}}G^{1/2} + \frac{AC_2^{1/2}d_0^{1/2}}{M^{21/16}}\right) + 2\frac{1}{M^{1/8}}G + 2\frac{AC_2d_0}{M^{19/16}}\right] \\ &\quad \cdot\left[\frac{A^2d_0^2}{M^{7/4}}G^{1/4}K^{3/4} + \frac{3Ad_0C_1}{M} + \frac{AC_2d_0}{M}\right]. \end{aligned} \tag{4.102}$$

As for $IV_{(k4)}$, we need the following facts:

$$||\Phi_{1z}||_{L^4} \leq ||\Phi_{1z}||_{L^2}^{1/4}(||\Phi_{1zz}||_{L^2} + ||\Phi_{1zr}||_{L^2})^{3/4}. \tag{4.103}$$

$$\begin{aligned} IV_{(k4)} &= \int (3\Phi_{1z} + 2r\Phi_{1rz})(-\psi_{1zz} + 3\psi_{1r}/r + \psi_{1rr})rj_1\, r\mathrm{d}r\mathrm{d}z \\ &\leq [3||\Phi_{1z}||_{L^2}^{1/4}(||\Phi_{1zr}||_{L^2} + ||\Phi_{1zz}||_{L^2})^{3/4} + ||r\Phi_{1zr}||_{L^2}] \\ &\quad \cdot(||\psi_{1zz}||_{L^2} + ||3\psi_{1r}/r||_{L^2} + ||\psi_{1zr}||_{L^2})||j_1||_{L^2}^{1/4}||\nabla j_1||_{L^2}^{3/4} \\ &\leq 3\left(\frac{A^{9/8}C_2d_0}{M^{87/64}}\right)(2H+1)G^{9/8}K^{3/4} + \frac{AC_2d_0}{M^{5/4}}(2H+1)G^{3/4}K^{5/4}. \end{aligned} \tag{4.104}$$

By adding the estimates for $\int u_1^2 r\mathrm{d}r\mathrm{d}z$, $\int \omega_1^2 r\mathrm{d}r\mathrm{d}z$, $\int B_1^2 r\mathrm{d}r\mathrm{d}z$, and $\int j_1^2 r\mathrm{d}r\mathrm{d}z$, we obtain $\frac{\mathrm{d}}{\mathrm{d}t}(H^2+G^2)$. Each term in our estimates from I_a to IV_g can be bounded by

$$C\varepsilon(E^2+K^2) + \varepsilon g(H,G),$$

where C is a constant, $g(H,G)$ is polynomial of H and G with positive rational exponents and positive coefficients that depend on $C_0, A, r_0, d_0, M, \epsilon = 1/M^\gamma\,(\gamma > 0)$. Putting all the estimates together, we obtain

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}(H^2+G^2) &\leq -1/2(E^2+K^2) + \varepsilon g(H,G) \\ &\leq -1/2(H^2+G^2) + \varepsilon g(H,G) \end{aligned} \tag{4.105}$$

since $H \leq E, G \leq K$. For given $A > 1, d_0 > 0$, we can choose M is large enough so that

$$-1/2(1+1) + \varepsilon g(1,1) \leq 0.$$

If the initial condition for $u_1, \omega_1, \psi_1, B_1, j_1$, and Φ_1 are chosen such that $H(0) \leq 1$, $G(0) \leq 1$, then we have

$$H(t) \leq 1, \quad G(t) \leq 1, \qquad \text{for all } t > 0.$$

This completes the proof of the Theorem 4.1. □

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Appendix

In this appendix, we will present the L^2 norm of $u^r,\ u^z$, the derivatives of $\psi_1,\ \Phi_1$ of ω_1 and j_1.

$$||u^r(r,z,t)||_{L^2} \leq \frac{A}{M^{7/8}}(1+d_0); \tag{5.1}$$

$$||u^z(r,z,t)||_{L^2} \leq \frac{A}{M^{7/8}} + \frac{Ad_0}{M}(2+C_1); \tag{5.2}$$

$$||\psi_{1z}||_{L^2} \leq \frac{A^{1/2}}{M^{7/16}}||\omega_1||_{L^2}^{1/2} + \frac{AC_2^{1/2}d_0^{1/2}}{M^{17/16}}; \tag{5.3}$$

$$||\Phi_{1z}||_{L^2} \leq \frac{A^{1/2}}{M^{7/16}}||j_1||_{L^2}^{1/2} + \frac{AC_2^{1/2}d_0^{1/2}}{M^{17/16}}; \tag{5.4}$$

$$||\psi_{1zz}||_{L^2} + ||\psi_{1zr}||_{L^2} + ||\psi_{1rr}||_{L^2} + 3||\psi_{1r}/r||_{L^2} \leq ||\omega_1||_{L^2} + \frac{AC_2d_0}{M^{5/4}}; \tag{5.5}$$

$$||\Phi_{1zz}||_{L^2} + ||\Phi_{1zr}||_{L^2} + ||\Phi_{1rr}||_{L^2} + 3||\Phi_{1r}/r||_{L^2} \leq ||j_1||_{L^2} + \frac{AC_2d_0}{M^{5/4}}; \tag{5.6}$$

$$||\Phi_{1zzz}||_{L^2} + ||\Phi_{1zzr}||_{L^2} + ||\Phi_{1zrr}||_{L^2} + 3||\Phi_{1zr}/r||_{L^2} \leq ||j_{1z}||_{L^2} + \frac{AC_2d_0}{M^{5/4}}; \tag{5.7}$$

$$||\Phi_{1zzz}||_{L^2} + ||\Phi_{1zzr}||_{L^2} + ||\Phi_{1zrr}||_{L^2} + 3||\Phi_{1zr}/r||_{L^2} \leq ||j_{1z}||_{L^2} + \frac{AC_2d_0}{M^{5/4}}. \tag{5.8}$$

Proof 1) proof of (5.1) and (5.2)

$$\begin{aligned}
||u^r(r,z,t)||_{L^2} &= ||\tilde{u}^r(r,z,t) - \bar{u}^r(r,z,t)||_{L^2} \leq ||\tilde{u}^r(r,z,t)||_{L^2} + ||\bar{u}^r(r,z,t)||_{L^2} \\
&\leq \frac{A}{M^{7/8}} + ||r\bar{\psi}_{1z}\phi||_{L^2} \leq \frac{A}{M^{7/8}} + R_0||\phi||_{L^\infty}||\bar{\psi}_{1z}||_{L^2} \\
&\leq \frac{A}{M^{7/8}}(1+d_0), \\
||u^z(r,z,t)||_{L^2} &= ||\tilde{u}^z(r,z,t) - \bar{u}^z(r,z,t)||_{L^2} \leq ||\tilde{u}^z(r,z,t)||_{L^2} + ||\bar{u}^z(r,z,t)||_{L^2}
\end{aligned}$$

$$\leq \frac{A}{M^{7/8}} + 2||\phi||_{L^\infty}||\bar{\psi}_1||_{L^\infty} + R_0||\bar{\psi}_1||_{L^\infty}||\phi_r||_{L^\infty}$$
$$\leq \frac{A}{M^{7/8}} + \frac{Ad_0}{M}(2 + C_1),$$

where we use (4.8), (4.9), (4.8) and (4.23).
2) proof of (5.3) and (5.4)

$$||\psi_{1z}||_{L^2}^2 = \int (\psi_{1z})^2 d(r^2/2)\mathrm{d}z = -\int r\psi_{1z}\psi_{1zr}\, r\mathrm{d}r\mathrm{d}z \leq ||r\psi_{1z}||_{L^2}||\psi_{1rz}||_{L^2}$$
$$\leq ||u^r||_{L^2}||\psi_{1rz}||_{L^2} \leq ||\mathbf{u}||_{L^2}||\psi_{1rz}||_{L^2} \leq \frac{A}{M^{7/8}}\left(||\omega_1||_{L^2} + \frac{AC_2d_0}{M^{5/4}}\right).$$
$$||\Phi_{1z}||_{L^2}^2 = \int (\Phi_{1z})^2 d(r^2/2)\mathrm{d}z = -\int r\Phi_{1z}\Phi_{1zr}\, r\mathrm{d}r\mathrm{d}z \leq ||r\Phi_{1z}||_{L^2}||\Phi_{1rz}||_{L^2},$$
$$\leq ||B^r||_{L^2}||\Phi_{1rz}||_{L^2} \leq ||\mathbf{B}||_{L^2}||\Phi_{1rz}||_{L^2} \leq \frac{A}{M^{7/8}}\left(||j_1||_{L^2} + \frac{AC_2d_0}{M^{5/4}}\right).$$

we arrive at (5.3) and (5.4), where we use (4.8)–(4.10), (5.5)–(5.6) and (4.21).
3) proof of (5.5)
From (4.40), we have

$$-\left(\tilde{\psi}_{1zz} + \tilde{\psi}_{1rr} + \frac{3}{r}\tilde{\psi}_{1r}\right) = \tilde{\omega}_1. \tag{5.9}$$

Using the definition of $\tilde{\psi}_1 = \bar{\psi}_1\phi + \psi_1$ and $\tilde{\omega}_1 = \bar{\omega}_1\phi + \omega_1$, we rewrite (5.9) as

$$-\omega_1 = \Delta\psi_1 + \bar{\psi}_1(\Delta_r\phi), \tag{5.10}$$

where $\Delta\psi_1 = \psi_{1zz} + \psi_{1rr} + \frac{3}{r}\psi_{1r}$, $\Delta_r\phi = \phi_{rr} + \frac{3}{r}\phi_r$. Multiplying (5.10) by ψ_{1zz}, and integrating over $[0, 1] \times [0, \infty)$, we have

$$||\omega_1||_{L^2}||\psi_{1zz}||_{L^2} \geq \int \omega_1\psi_{1zz}\, r\mathrm{d}r\mathrm{d}z \geq \int [(\Delta\psi_1)\psi_{1zz} - \bar{\psi}_1(\Delta_r\phi)\psi_{1zz}]\, r\mathrm{d}r\mathrm{d}z$$
$$= \int \left[\left(\psi_{1zz} + \psi_{1rr} + \frac{3}{r}\psi_{1r}\right)\psi_{1zz} - \bar{\psi}_1(\Delta_r\phi)\psi_{1zz}\right] r\mathrm{d}r\mathrm{d}z$$
$$\geq ||\psi_{1zz}||_{L^2}^2 + ||\psi_{1rz}||_{L^2}^2 - \frac{AC_2d_0}{M^{5/4}}||\psi_{1zz}||_{L^2}.$$

We arrive at

$$||\psi_{1zz}||_{L^2} + ||\psi_{1rz}||_{L^2} \leq ||\omega_1||_{L^2} + \frac{AC_2d_0}{M^{5/4}}, \tag{5.11}$$

where

a) $\int \psi_{1rr}\psi_{1zz}\, r\mathrm{d}r\mathrm{d}z = ||\psi_{1rz}||_{L^2}^2 + \int \psi_{1z}\psi_{1rz}\,\mathrm{d}r\mathrm{d}z$,

b) $\int \frac{3}{r}\psi_{1r}\psi_{1zz} r\mathrm{d}r\mathrm{d}z = -3\int \psi_{1z}\psi_{1rz}\ \mathrm{d}r\mathrm{d}z$,
c) $\int (\Delta_r\phi)\bar{\psi}_1\psi_{1zz}\ r\mathrm{d}r\mathrm{d}z \le ||\Delta_r\phi||_{L^\infty}\int |\bar{\psi}_1|\ |\psi_{1zz}|\ r\mathrm{d}r\mathrm{d}z \le \frac{AC_2d_0}{M^{5/4}}||\psi_{1zz}||_{L^2}$.
Multiplying (5.10) by $\Delta_r\psi_1$ and integrating over $[0, 1] \times [0, \infty)$, we get

$$\begin{aligned}
||\omega_1||_{L^2}||\Delta_r\psi_1||_{L^2} &\ge \int \omega_1(\Delta_r\psi_1)\ r\mathrm{d}r\mathrm{d}z \\
&\ge \int [(\Delta\psi_1)\Delta_r\psi_1 - \bar{\psi}_1(\Delta_r\psi_1)\Delta_r\phi]\ r\mathrm{d}r\mathrm{d}z \\
&= \int [\psi_{1zz}\psi_{1rr} + \frac{3}{r}\psi_{1zz}\psi_{1r} + (\Delta_r\psi_1)^2]\ r\mathrm{d}r\mathrm{d}z \\
&\ge ||\psi_{1rz}||_{L^2}^2 + ||\Delta_r\psi_1||_{L^2}^2 - \frac{AC_2d_0}{M^{5/4}}||\Delta_r\psi_1||_{L^2},
\end{aligned}$$

where
d) $\int (\Delta\psi_1)\Delta_r\psi_1\ r\mathrm{d}r\mathrm{d}z = ||\psi_{1rz}||_{L^2}^2 + ||\Delta_r\psi_1||_{L^2}^2$,
e) $\int \bar{\psi}_1(\Delta_r\psi_1)\Delta_r\phi\ r\mathrm{d}r\mathrm{d}z \le \frac{AC_2d_0}{M^{5/4}}||\Delta_r\psi_1||_{L^2}$.
Moreover,

$$||\Delta_r\psi_1||_{L^2}^2 = ||\psi_{1rr}||_{L^2}^2 + ||3\psi_{1r}/r||_{L^2}^2. \tag{5.12}$$

According to the above inequalities, we obtain

$$||\psi_{1rr}||_{L^2}^2 + ||3\psi_{1r}/r||_{L^2}^2 \le ||\omega_1||_{L^2} + \frac{AC_2d_0}{M^{5/4}}. \tag{5.13}$$

Combining (5.11) and (5.13), we arrive at (5.5).
3) proof of (5.6)
From (4.1), (4.7) and (4.43), we have

$$-j_1 = \Delta\Phi_1 + \bar{\Phi}_1(\Delta_r\phi). \tag{5.14}$$

Multiplying (5.14) by Φ_{1zz} and integrating over $[0, 1] \times [0, \infty)$, we have

$$\begin{aligned}
||j_1||_{L^2}||\Phi_{1zz}||_{L^2} &\ge \int j_1\Phi_{1zz}\ r\mathrm{d}r\mathrm{d}z \ge \int [(\Delta\Phi_1)\Phi_{1zz} - \bar{\Phi}_1(\Delta_r\phi)\Phi_{1zz}]\ r\mathrm{d}r\mathrm{d}z \\
&= \int [(\Phi_{1zz} + \Phi_{1rr} + 3\Phi_{1r}/r)\Phi_{1zz} - \bar{\Phi}_1(\Delta_r\phi)\Phi_{1zz}]\ r\mathrm{d}r\mathrm{d}z \\
&\ge ||\Phi_{1zz}||_{L^2}^2 + ||\Phi_{1rz}||_{L^2}^2 - \frac{AC_2d_0}{M^{5/4}}||\Phi_{1zz}||_{L^2}.
\end{aligned}$$

We arrive at

$$||\Phi_{1zz}||_{L^2} + ||\Phi_{1rz}||_{L^2} \le ||j_1||_{L^2} + \frac{AC_2d_0}{M^{5/4}}, \tag{5.15}$$

where

a) $\int \Phi_{1rr}\Phi_{1zz}\ r\mathrm{d}r\mathrm{d}z = ||\Phi_{1rz}||_{L^2}^2 + \int \Phi_{1z}\Phi_{1rz}\ \mathrm{d}r\mathrm{d}z$,
b) $\int \frac{3}{r}\Phi_{1r}\Phi_{1zz} r\mathrm{d}r\mathrm{d}z = -3\int \Phi_{1z}\Phi_{1rz}\ \mathrm{d}r\mathrm{d}z$,
c) $\int (\Delta_r\phi)\bar{\Phi}_1\Phi_{1zz}\ r\mathrm{d}r\mathrm{d}z \le ||\Delta_r\phi||_{L^\infty}\int |\bar{\Phi}_1|\ |\Phi_{1zz}|\ r\mathrm{d}r\mathrm{d}z \le \frac{AC_2d_0}{M^{5/4}}||\Phi_{1zz}||_{L^2}$.
Multiplying (5.14) by $\Delta_r\Phi_1$ and integrating over $[0, 1] \times [0, \infty)$, we get

$$\begin{aligned} ||j_1||_{L^2}||\Delta_r\Phi_1||_{L^2} &\ge \int j_1(\Delta_r\Phi_1)\ r\mathrm{d}r\mathrm{d}z \\ &\ge \int [(\Delta\Phi_1)\Delta_r\Phi_1 - \bar{\Phi}_1(\Delta_r\Phi_1)\Delta_r\phi]\ r\mathrm{d}r\mathrm{d}z \\ &= \int [\Phi_{1zz}\Phi_{1rr} + \frac{3}{r}\Phi_{1zz}\Phi_{1r} + (\Delta_r\Phi_1)^2]\ r\mathrm{d}r\mathrm{d}z \\ &\ge ||\Phi_{1rz}||_{L^2}^2 + ||\Delta_r\Phi_1||_{L^2}^2 - \frac{AC_2d_0}{M^{5/4}}||\Delta_r\Phi_1||_{L^2}, \end{aligned}$$

where d) $\int (\Delta\Phi_1)\Delta_r\Phi_1\ r\mathrm{d}r\mathrm{d}z = ||\Phi_{1rz}||_{L^2}^2 + ||\Delta_r\Phi_1||_{L^2}^2$,
e) $\int \bar{\Phi}_1(\Delta_r\Phi_1)\Delta_r\phi\ r\mathrm{d}r\mathrm{d}z \le \frac{AC_2d_0}{M^{5/4}}||\Delta_r\Phi_1||_{L^2}$.
Moreover,

$$||\Delta_r\Phi_1||_{L^2}^2 = ||\Phi_{1rr}||_{L^2}^2 + ||3\Phi_{1r}/r||_{L^2}^2. \tag{5.16}$$

According to the above inequalities, we obtain

$$||\Phi_{1rr}||_{L^2}^2 + ||3\Phi_{1r}/r||_{L^2}^2 \le ||\omega_1||_{L^2} + \frac{AC_2d_0}{M^{5/4}}. \tag{5.17}$$

Combining (5.15) and (5.17), we arrive at (5.6).
4) proof of (5.7) and (5.8)

Different (5.9), (5.14) with respect to z, respectively. The process of proof is similar to the proof of (5.5) and (5.6). We complete the proof of Appendix. □

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