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Reconstruction of an Affine Connection in Generalized Fermi Coordinates

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Abstract On a manifold with affine connection, we introduce special pre-semigeodesic charts which generalize Fermi coordinates. We use a version of the Peano's–Picard's-Cauchy-like Theorem on the initial values problem for systems of ODSs. In a fixed pre-semigeodesic chart of a manifold with a symmetric affine connection, we reconstruct, or construct, the connection in some neighborhood from the knowledge of the "initial values", namely the restriction of the components of connection to a fixed surface S and from some of the components of the curvature tensor R in the full coordinate domain. In Riemannian space, analogous methods are used to retrieve (or construct) the metric tensor of a pseudo-Riemannian manifold in a domain of semigeodesic coordinates from the known restriction of the metric to some non-isotropic hypersurface and some of the components of the curvature tensor of type $(0, 4)$ in the ambient space.

Keywords Riemannian manifold · Linear connection · Metric · Fermi coordinates · Semigeodesic coordinates

Mathematics Subject Classification 53B05 · 53B20 · 53B30 · 53B50 · 34A26

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1 Introduction

The problem of finding a Riemannian metric from various pieces of information is of interest from both theoretical and practical point of view. Papers by many authors are devoted to the possibility of finding the metric from the curvature tensor, [7, p. 135–136], or prove existence of metrics with the prescribed Ricci tensor, [2] etc. In general, to solve the problem means to solve a relatively complicated non-linear system of partial differential equations, the coefficients of which are expressed through the components of the Riemannian curvature tensor. One possibility how to simplify the situation is to find a convenient coordinate system with respect to which the system of equations is simplified considerably. Our aim is to present and use such preferable coordinates.

In [6], in a neighborhood of a (positive definite) Riemannian space in which special, semigeodesic, coordinates are given, the metric tensor is calculated from its values on a suitable hypersurface and some of components of the curvature tensor of type $(1,3)$ in the coordinate domain. Semigeodesic coordinates are a generalization of the well-known Fermi coordinates.

In the present paper, we consider a more general situation. We introduce special pre-semigeodesic charts characterized both geometrically and in terms of the connection. Then we apply a version of the Peano's–Picard's-Cauchy theorem on existence and uniqueness of solutions of the initial values problems for systems of first-order ODEs (ordinary differential equations). We use the apparatus in a fixed pre-semigeodesic chart of a manifold equipped with the symmetric affine connection. Our aim is to reconstruct, or construct, the symmetric affine connection in some neighborhood from the knowledge of the "initial conditions": the restriction of the connection to a fixed $(n-1)$-dimensional surface S and some of the components of the curvature tensor R in the ambient space (coordinate domain). By analogous methods, we retrieve (or construct) the metric tensor of type $(0,4)$ of a pseudo-Riemannian manifold in a domain of semigeodesic coordinates from the known restriction of the metric to some non-isotropic hypersurface and some of the components of the curvature tensor in the ambient space. In comparison to the authors of [6], we give shorter proofs of constructive character based on classical results on first-order ODEs.

Recall the so-called Fermi coordinates, named after the Italian physicist Enrico Fermi [3], which were widely used in Minkowskian space, e.g. [9], and play an important role in mechanics, physics, [1,10], and in differential geometry of Riemannian spaces in general. Suppose $\gamma\colon I \to M$ is a geodesic on an n-dimensional Riemannian manifold M, and p a point on γ. Then there exist local coordinates $(t, x^2, \dots, x^n)$ around p such that for small t, $\gamma(t, 0, \dots, 0)$ represents the geodesic near p. The metric tensor is the Euclidean metric along γ, and again (only) along γ, all Christoffel symbols vanish (all the above properties are only valid along the distinguished geodesic). We will consider here a generalization of Fermi coordinates, namely pre-semigeodesic and semigeodesic coordinates, which bring, at the same time, special parametrization for all canonical geodesics in some tubular neighborhood. The celebrated Fermi coordinates can be considered as a particular case.

2 Pre-semigeodesic Chart

Let (M, ∇) be a (differentiable or smooth) n-dimensional manifold M equipped with a symmetric linear connection ∇. Let Γ^h_{ij} denote components of the connection ∇ in a fixed chart $(U, \varphi = (x^1, \dots, x^n))$ in M; $U \subseteq M$ open.

If in the chart $(U, (x^i))$ of M, $\Gamma^h_{11}(x) = 0$ is valid for all $h = 1, \dots, n$, we say that $(U, (x^i))$ is a *pre-semigeodesic chart*[1] related to the coordinate x^1 with respect to the connection ∇ or, that x^1 is a *geodesic coordinate* in U. Obviously, it is quite natural to prefer the first coordinate, and it means no loss of generality.

Let us give a geometric interpretation of the pre-semigeodesic charts. Recall that the equations $\nabla_{\dot c}\dot c = 0$ for canonically paramerized geodesics $c\colon I \to U$ of the connection $\nabla|U$ in local coordinates read ($k = 2, \dots, n$)

$$\begin{aligned} \frac{\mathrm{d}^2 c^1}{\mathrm{d}s^2} + \Gamma^1_{11}\left(\frac{\mathrm{d}c^1}{\mathrm{d}s}\right)^2 + \sum\nolimits_{j=2}^n \Gamma^1_{1j}\frac{\mathrm{d}c^1}{\mathrm{d}s}\frac{\mathrm{d}c^j}{\mathrm{d}s} + \sum\nolimits_{i,j=2}^n \Gamma^1_{ij}\frac{\mathrm{d}c^i}{\mathrm{d}s}\frac{\mathrm{d}c^j}{\mathrm{d}s} = 0,\\ \frac{\mathrm{d}^2 c^k}{\mathrm{d}s^2} + \Gamma^k_{11}\left(\frac{\mathrm{d}c^1}{\mathrm{d}s}\right)^2 + \sum\nolimits_{j=2}^n \Gamma^k_{1j}\frac{\mathrm{d}c^1}{\mathrm{d}s}\frac{\mathrm{d}c^j}{\mathrm{d}s} + \sum\nolimits_{i,j=2}^n \Gamma^k_{ij}\frac{\mathrm{d}c^i}{\mathrm{d}s}\frac{\mathrm{d}c^j}{\mathrm{d}s} = 0. \end{aligned} \tag{1}$$

Lemma 1 *The conditions $\Gamma^h_{11} = 0$, $h = 1, \dots, n$, are satisfied in U if and only if the parametrized curves*

$$c\colon I \to U, \quad c(s) = (s, a_2, \dots, a_n), \quad s \in I,\ a_i \in \mathbb{R}, \quad i = 2, \dots, n, \tag{2}$$

are canonically parametrized geodesics of $\nabla|U$ (I is some interval, a_k are suitable constants chosen so that $c(I) \subset U$).

Proof Let $\Gamma^h_{11} = 0$ hold for $h = 1, \dots, n$. Then the local curves with parametrizations (2) satisfy

$$\frac{\mathrm{d}c(s)}{\mathrm{d}s} = \left(\frac{\partial}{\partial x^1}\right)_{c(s)}, \quad \frac{\mathrm{d}^2 c(s)}{\mathrm{d}s^2} = 0, \tag{3}$$

therefore are solutions to the system (1). Conversely, if the curves (2) are among solutions to (1) then due to (3), we get $\Gamma^h_{11} = 0$ from (1). □

Hence the pre-semigeodesic chart is fully characterized by the condition that the curves $x^1 = s, x^i = \text{const}, i = 2, \dots n$ belong to the geodesics of the given connection in the coordinate neighborhood. The definition domain U of such a chart is "tubular", a tube along geodesics.

3 Reconstruction of Connection

Our aim is to show that a symmetric linear connection in a pre-semigeodesic coordinate domain U (related to x^1) can be uniquely constructed, or retrieved, in some subdomain

[1] Similar coordinates were used e.g. in [12], and called there, in English translation, "almost semigeodesic".

of U if we know the restrictions $\tilde{\Gamma}^h_{ij}(\tilde{x}), \tilde{x} \in S$ of the connection to the surface S defined by $x^1 = 0$ and prescribed components R^h_{i1k} of the curvature tensor in the given tubular domain U. First recall that the components R^h_{ijk} of the curvature tensor are related to the components Γ^h_{ij} of the connection by the classical formula

$$R^h_{ijk} = \partial_j \Gamma^h_{ik} - \partial_k \Gamma^h_{ij} + \Gamma^m_{ik}\Gamma^h_{mj} - \Gamma^m_{ij}\Gamma^h_{mk}. \tag{4}$$

Now suppose that $\Gamma^h_{11}(x) = 0$ is satisfied. Particularly, setting $i = j = 1$ in (4) we get under this assumption

$$\frac{\partial}{\partial x^1}\Gamma^h_{1k} + \sum_m \Gamma^m_{1k}\Gamma^h_{1m} - R^h_{11k} = 0 \tag{5}$$

and plugging $j = 1$ in (4) we get, for each of the indices $i = 2, \dots, n$, the system

$$\frac{\partial}{\partial x^1}\Gamma^h_{ik} + \sum_m \Gamma^m_{ik}\Gamma^h_{m1} - \sum_m \Gamma^m_{i1}\Gamma^h_{mk} - \frac{\partial}{\partial x^k}\Gamma^h_{i1} - R^h_{i1k} = 0. \tag{6}$$

In $\mathbb{R}^n$ with standard coordinates $(x^1, x^2, \dots, x^n)$, let us identify the linear subspace (hypersurface) defined by $x^1 = 0$ with the space $\mathbb{R}^{n-1}$, i.e. $(\tilde{x}) = (x^2, \dots, x^n)$ are standard coordinates in $\mathbb{R}^{n-1}$. Let $\mathcal{J} = (0, 1)$ be the open unit interval and denote by $K_m = \mathcal{J}^m$ the open standard m-cube. Denote

$$D_n(\delta) = \{x = (x^1, \dots, x^n) \in \mathbb{R}^n \mid 0 \le x^1 \le \delta, \quad 0 < x^i < 1, \quad i = 2, \dots, n\}.$$

Hereafter we will deal with the pre-semigeodesic coordinate system in the domain $D_n(\delta)$. The open $(n-1)$-cube $K_{n-1} = \mathcal{J}^{n-1}$, viewed as

$$K_{n-1} = \{\tilde{x} = (x^2, \dots, x^n) \in \mathbb{R}^{n-1} \mid 0 < x^i < 1, \quad i = 2, \dots, n\} \subset \mathbb{R}^{n-1},$$

will be identified with a hypersurface S in $D_n(\delta)$ determined by $x^1 = 0$.

So in what follows let S be a hypersurface in $D_n(\delta)$ defined by $x^1 = 0$. Now let us modify for our purpose the Theorem on existence and uniqueness of solutions of systems of ODEs:

Theorem 1 *Let $\tilde{\nabla}$ be a torsion-free linear connection in S (of the class at least C^2) with components $\tilde{\Gamma}^h_{ij}(\tilde{x})$, $\tilde{x} \in S$, $h, i, j \in \{2, \dots, n\}$, let $\tilde{\Gamma}^h_{1j}(\tilde{x})$ be functions in S (at least C^2), $h, j \in \{1, \dots, n\}$, where $\tilde{\Gamma}^h_{11}(\tilde{x}) = 0$ for $h \in \{1, \dots, n\}$. Let A^h_{ij}, $h, i, j \in \{1, \dots, n\}$ be functions (at least C^0) in $D_n(\delta)$ such that each A^h_{1k} is at least C^1 in each of the variables $x^2, \dots, x^n$ and at least C^0 in x^1. Moreover let the curvature tensor $\tilde{R}$ of $\tilde{\nabla}$ satisfy $\tilde{R}^h_{i1k}(\tilde{x}) = A^h_{ik}(\tilde{x})$ in S. Then there exists a real number $\hat{\delta}$, $0 < \hat{\delta} \le \delta$ and there is a unique torsion-free linear connection ∇ in the neighborhood $D_n(\hat{\delta})$ with components Γ^h_{ij} such that the following holds: $\Gamma^h_{11}(x) = 0$ in $D_n(\hat{\delta})$ for $h = 1, \dots, n$ (i.e. the coordinates are pre-semigeodesic), $\Gamma^h_{ij}(0, \tilde{x}) = \tilde{\Gamma}^h_{ij}(\tilde{x})$ for*

$\tilde{x} \in S, h, i, j \in \{1, \ldots, n\}$ (hence $\nabla|S = \tilde{\nabla}$) and $R^h_{i1k} = A^h_{ik}$ for all $x \in D_n(\hat{\delta})$ where $h, i, k = 1, \ldots, n$.

Proof Let the assumptions be satisfied. Analyzing the system (4) and the consequences mentioned above we find that we can proceed step by step. In three main steps, we find functions Γ^h_{ij} in a certain subdomain of $D_n(\delta)$ such that $\Gamma^h_{ij} = \Gamma^h_{ji}$ and the conclusion of Theorem 1 is satisfied.
Step (1) Let us define $\Gamma^h_{11}(x) = 0$ for $x \in D_n(\delta)$, $h = 1, \ldots, n$.
Step (2) Let us solve the system

$$\frac{\partial}{\partial x^1}\Gamma^h_{1k}(x) = -\sum_m \Gamma^m_{1k}(x)\Gamma^h_{1m}(x) + A^h_{1k}(x) \tag{7}$$

for unknown functions Γ^h_{1k}, $h = 1, \ldots, n, k = 2, \ldots, n$ which we assume as a system of ordinary differential equations of one variable x^1 (while the remaining coordinates $(\tilde{x}) = (x^2, \ldots, x^n) \in K_{n-1} = S$ are considered as parameters) for the initial data

$$\Gamma^h_{1k}(0, x^2, \ldots, x^n) = \tilde{\Gamma}^h_{1k}(x^2, \ldots, x^n) \quad \text{for} \quad (x^2, \ldots, x^n) \in S.$$

According to the theory, there exists δ_1, $0 < \delta_1 \leq \delta$, and there are uniquely determined functions $\Gamma^h_{1k}(x^1, \ldots, x^n)$ of the class at least C^1 in the domain $D_n(\delta_1)$ such that

$$\Gamma^h_{1k}(0, \tilde{x}) = \tilde{\Gamma}^h_{1k}(\tilde{x}), \quad \tilde{x} \in S. \tag{8}$$

These functions together with their derivatives will be used in what follows.
Step (3) Now consider the system

$$\frac{\partial}{\partial x^1}\Gamma^h_{ik} = -\sum_m \Gamma^m_{ik}\Gamma^h_{m1} + \sum_m \Gamma^m_{i1}\Gamma^h_{mk} + \frac{\partial}{\partial x^k}\Gamma^h_{i1} + A^h_{ik} = 0, \tag{9}$$

where we plugged for Γ^h_{i1} from the above; $h = 1, \ldots, n,\ i, k = 2, \ldots, n$. We have again a system of ordinary differential equations of one variable x^1. According to the existence and uniqueness theorem on systems of ODEs there is $\hat{\delta}$, $0 < \hat{\delta} \leq \delta_1$ and there are uniquely determined functions $\Gamma^h_{ik}(x^1, \ldots, x^n)$ of the class at least C^1, $i \neq 1 \neq k$ in the domain $D_n(\hat{\delta})$ which satisfy the initial conditions

$$\Gamma^h_{ik}(0, x^2, \ldots, x^n) = \tilde{\Gamma}^h_{ik}(x^2, \ldots, x^n), \quad (x^2, \ldots, x^n) \in S. \tag{10}$$

Moreover, comparing (5) and (7), (6) and (9) we can see that

$$R^h_{i1k}(x) = A^h_{ik}(x), \quad x \in D_n(\hat{\delta}), \quad h, i, k = 1, \ldots, n, \tag{11}$$

holds as required, and Γ^h_{ik} are components of a connection of the above properties. □

As a consequence, if we use prolongation of the solution, we obtain:

Theorem 2 *Let $(U, \varphi = (x^1, \dots, x^n))$ be a chart in M. Let $S \subset U$ be a submanifold in U defined by $x^1 = 0$. Let $\tilde{\nabla}$ be a torsion-free linear connection in S of the class at least C^2 with the components $\tilde{\Gamma}^h_{ij}$ and the curvature tensor $\tilde{R}$, and let A^h_{ij} be functions in U such that $\tilde{R}^h_{i1k} = A^h_{ik}$ in S, A^h_{ik}, $i = 2, \dots, n$, are of the class at least C^0, A^h_{1k} are continuous in x^1, and A^h_{1k} are at least C^1 in the remaining variables $x^2, \dots, x^n$. Then there is a unique symmetric linear connection ∇ in U with components satisfying $\Gamma^h_{11} = 0$ for $h = 1, \dots, n$ (i.e. the given chart is pre-semigeodesic w.r.t. ∇) such that $\nabla|S = \tilde{\nabla}$, and $R^h_{j1k} = A^h_{jk}$ in U.*

4 Reconstruction of Metric

4.1 Semigeodesic Coordinates

For our purpose, we say that a chart $(U, (x^i))$ of a pseudo-Riemannian manifold (M, g) is *semigeodesic* (or that (x^i) are semigeodesic coordinates) if in this chart, the metric tensor has the coordinate expression

$$g = e\mathrm{d}x^1 \otimes \mathrm{d}x^1 + \tilde{g}_{ij}(x^1, x^2, \dots, x^n)\mathrm{d}x^i \otimes \mathrm{d}x^j, \quad i, j = 2, \dots, n, \tag{12}$$

where $e = \pm 1$ (the plus or minus sign is connected with the square of the integral of the tangent vector to the x^1-coordinate line).

The geometric interpretation is as follows, [12, p. 55].

Lemma 2 *Local coordinates (x^i) in a pseudo-Riemannian manifold are semigeodesic if and only if the 1-net of x^1-coordinate lines is formed by arclength parametrized geodesics which are orthogonal to a non-isotropic hypersurface defined by $x^1 = \mathrm{const}$.*

Note that coordinate hyperplanes defined by $x^j = \mathrm{const}$ are orthogonal to the distinguished system of geodesics. Obviously, semigeodesic coordinates are pre-semigeodesic.

Semigeodesic coordinates can be introduced in a sufficiently small neighborhood of any point of an arbitrary (positive) Riemannian manifold, and is fully characterized by the coordinate form of the metric:

$$g_{ij} = \mathrm{d}x^1 \otimes \mathrm{d}x^1 + \tilde{g}_{ij}(x^1, x^2, \dots, x^n)\mathrm{d}x^i \otimes \mathrm{d}x^j, \quad i, j = 2, \dots, n. \tag{13}$$

E.g. on a cylinder, semigeodesic coordinates can be introduced globally.

Advantages of such coordinates are known since Gauss ([8, p. 201], "Geodätische Parallelkoordinaten"), and are widely used in the two-dimensional case, particularly in applications, [11] and the references therein, [13] etc. Note that geodesic polar coordinates ("Geodätische Polarkoordinaten," [8, pp. 197–204]) can be interpreted as a "limit case" of semigeodesic coordinates (all geodesic coordinate lines $\phi = x^2 = \mathrm{const}$ pass through one point called the pole, corresponding to $r = x^1 = 0$, while $r = x^1 = \mathrm{const}$ are the geodesic circles).

4.2 Reconstruction of Metric in Semigeodesic Coordinates

Recall that the components of the curvature tensor R (in type $(0, 4)$) of the semi-Riemannian manifold $V_n = (M, g)$ are related to the components of the metric by

$$R_{hijk} = \frac{1}{2}\left(\partial_{ij} g_{hk} + \partial_{hk} g_{ij} - \partial_{ik} g_{hj} - \partial_{ij} g_{hk}\right) + g^{rs}\left(\Gamma_{hkr}\Gamma_{ijs} - \Gamma_{hjr}\Gamma_{kjs}\right), \quad (14)$$

where $\Gamma_{ijk} = \frac{1}{2}(\partial_i g_{jk} + \partial_j g_{ik} - \partial_k g_{ij})$ are Christoffel symbols of the first type in V_n, and g^{rs} are components of the dual tensor to g. Hence g^{ij} are functions rational in components g_{ij} of the metric.[2]

Now suppose that components of the metric satisfy $g_{11} = e$, $g_{1j} = 0$. Under these assumptions, setting $h = k = 1$ we obtain from (14)

$$R_{1ij1} = \frac{1}{2}\partial_{11} g_{ij} - \frac{1}{4} g^{rs} \partial_1 g_{ir} \partial_1 g_{js}. \quad (15)$$

Here we can suppose that the indices satisfy $i, j, r, s > 1$. Plugging

$$G_{ij} = \partial_1 g_{ij}, \quad (16)$$

we can write (15) as

$$a_{ij}(x) = R_{1ij1} = \frac{1}{2}\partial_1 G_{ij} - \frac{1}{4} g^{rs} G_{ir} G_{js}. \quad (17)$$

Now we can prove the following.

Theorem 3 *Let a_{ij} be (at least) continuous functions in $D_n(\delta)$, let $\tilde{g}_{ij}$ be functions of the class (at least) C^2 in K_{n-1} and $\tilde{G}_{ij}$ functions of the class (at least) C^1 in K_{n-1}, $i, j = 2, \dots, n$, such that the matrices $(\tilde{g}_{ij})$ and $(\tilde{G}_{ij})$ are symmetric[3] and $\det(\tilde{g}_{ij}) \neq 0$ in K_{n-1}. Fix an element $e \in \{-1, 1\}$. Then there is $\hat{\delta}$, $0 < \hat{\delta} \leq \delta$ and there exists exactly one non-degenerate metric tensor[4] g of the class (at least) C^2 in $D_n(\hat{\delta})$ with components $g_{11} = e$, $g_{1j} = 0$, $j = 2, \dots, n$, such that for $i, j = 2, \dots, n$,*

$$g_{ij}(0, \tilde{x}) = \tilde{g}_{ij}(\tilde{x}), \quad \frac{\partial^+}{\partial x^1} g_{ij}(0, \tilde{x}) = \tilde{G}_{ij}(\tilde{x}), \quad \tilde{x} \in K_{n-1} \quad (18)$$

where $\frac{\partial^+}{\partial x^1}$ means the partial derivative from the right, and

$$a_{ij}(x) = R_{1ij1}(x), \quad x \in D_n(\hat{\delta}). \quad (19)$$

2 $g^{ij} = 1/\det(g_{ij}) \cdot A_{ji}$ where A_{ji} is the algebraic complement of the matrix element g_{ji}.

3 $\tilde{g}_{ji} = \tilde{g}_{ij}$, $\tilde{G}_{ji} = \tilde{G}_{ij}$.

4 $\det(g_{ij}) \neq 0$ in $D_n(\hat{\delta})$.

Proof Step (1) Let us define $g_{11} = e$, $g_{1j} = 0$ in $D_n(\delta)$, $j = 2, \dots, n$.

Step (2) Let us solve the system

$$\partial_1 g_{ij} = G_{ij},$$
$$\partial_1 G_{ij} = \frac{1}{2} g^{rs} G_{ir} G_{js} + 2a_{ij} \tag{20}$$

under the initial values

$$g_{ij}(0, \tilde{x}) = \tilde{g}_{ij}(\tilde{x}), \quad \frac{\partial^+}{\partial x^1} g_{ij}(0, \tilde{x}) = \tilde{G}_{ij}(\tilde{x}), \quad \tilde{x} \in K_{n-1}, \ i, j = 2, \dots, n. \tag{21}$$

Note that since the determinant as well as the algebraic complements are continuous functions in the entries g_{ij}, and we demand $\det(\tilde{g}_{ij})(0, \tilde{x}) = \det(\tilde{g}_{ij})(\tilde{x}) \neq 0$, it is guaranteed that g^{rs} will be well-defined and well-behaved functions of g^{ij}, similarly as in [6]. So (20) can be considered as a system of first-order ordinary differential equations in the variable x^1 for the unknown functions g_{ij} and G_{ij} with the initial values (21); the remaining coordinates $x^2, \dots, x^n \in K_{n-1}$ are supposed to be parameters. The right sides in (20) satisfy the conditions of the existence and uniqueness theorem [4, p. 263] in the domain $D_n(\tilde{\delta})$ and have continuous derivatives with respect to g_{ij} and G_{ij}. The initial value problem (20) and (21) has precisely one solution $g_{ij}(x)$. The functions g_{ij} are components of a metric tensor in $D_n(\tilde{\delta})$, and comparing (20) and (17) we find easily that the components of its curvature tensor satisfy $R_{1ij1}(x) = a_{ij}(x)$ as required.

Since the matrices (g_{ij}) and (G_{ij}) are symmetric we may assume $i \leq j$ in (20) and (21). As a consequence, we get

Theorem 4 *Let a_{ij} be continuous functions in some coordinate neighborhood U, $\tilde{g}_{ij}$ C^2-functions in $\tilde{S} = U \cap S$ where S is the hypersurface $S\colon x^1 = 0$ in R^n, and $\tilde{G}_{ij}$ C^1-functions in $\tilde{S}$, $i, j = 2, \dots, n$ such that the matrices $(\tilde{g}_{ij})$ and $(\tilde{G}_{ij})$ are symmetric and $\det(\tilde{g}_{ij}) \neq 0$ in $\tilde{S}$. Fix an element $e \in \{-1, 1\}$. Then there is $\hat{\delta} > 0$ and there exists precisely one non-degenerate metric tensor g, $\det(g_{ij}) \neq 0$, of the class C^2 in $\tilde{U} = \langle -\hat{\delta}, \hat{\delta} \rangle \times \tilde{S}$ with components $g_{11} = e$, $g_{1j} = 0$, $j = 2, \dots, n$ (i.e. $\tilde{U}$ is semigeodesic) such that for $i, j = 2, \dots, n$,*

$$g_{ij}(0, \tilde{x}) = \tilde{g}_{ij}(\tilde{x}), \quad \frac{\partial^+}{\partial x^1} g_{ij}(0, \tilde{x}) = \tilde{G}_{ij}(\tilde{x}), \quad \tilde{x} \in \tilde{S} \tag{22}$$

and

$$a_{ij}(x) = R_{1ij1}(x), \quad x \in \tilde{U}. \tag{23}$$

Provided $a_{ij}(x) = R_{1ij1}(x)$ the solution of the system (20) answers the problem of finding the metrics with the prescribed components $R_{1ij1}(x)$ of the $(0, 4)$-Riemannian curvature tensor. Substituting the obtained components of metric we get the relationship to the components of the $(1, 3)$-curvature as follows:

$$R_{1ij1} = eR^1_{ij1} = -eR^1_{i1j} = g_{im}R^m_{11j} = -g_{im}R^m_{1j1}. \tag{24}$$

Hence our results generalize the results of [5,6].

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