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One-Step Iterations for a Finite Family of Generalized Nonexpansive Mappings in CAT(0) Spaces

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Abstract We study a one-step iterative scheme to establish strong convergence theorems and $\triangle$-convergence theorems for a finite family of generalized nonexpansive mappings on a nonlinear domain. Our results generalize and extend several relevant results in the literature.

Keywords CAT(0) space · Generalized nonexpansive mapping · One-step iterations · Common fixed point · $\triangle$-convergence · Strong convergence

Mathematics Subject Classification 47H09 · 47H10

1 Introduction and Preliminaries

Let (X, d) be a metric space and $x, y \in X$ with $l = d(x, y)$. A *geodesic path* from x to y is a mapping $g : [0, l] \to X$ such that $g(0) = x, g(l) = y$, and $d\left(g(t), g\left(t'\right)\right) = \left|t - t'\right|$ for all $t, t' \in [0, l]$. The image of a geodesic path is called a geodesic segment. A metric space X is a (uniquely) geodesic space if every two points in X are joined by a unique geodesic segment.

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A *geodesic triangle* $\Delta(x_1, x_2, x_3)$ in a geodesic metric space X consists of three points $x_1, x_2, x_3 \in X$ and a geodesic segment between each pair of these points. A *comparison triangle* for geodesic triangle $\Delta(x_1, x_2, x_3)$ in X is a triangle $\overline{\Delta}(x_1, x_2, x_3) := \Delta(\bar{x}_1, \bar{x}_2, \bar{x}_3)$ in $\mathbb{R}^2$ such that $d_{\mathbb{R}^2}\left(\bar{x}_i, \bar{x}_j\right) = d\left(x_i, x_j\right)$ for all $i, j = 1, 2, 3$.

A geodesic space X is a CAT(0) space if for each Δ in X and $\overline{\Delta}$ in $\mathbb{R}^2$, the inequality

$$d(x, y) \leq d_{\mathbb{R}^2}(\bar{x}, \bar{y})$$

holds for all $x, y \in \Delta$ and $\bar{x}, \bar{y} \in \overline{\Delta}$.

The complex Hilbert ball with the hyperbolic metric is an example of a CAT(0) space [16]. It is worth mentioning that fixed point theorems in CAT(0) spaces (especially in $\mathbb{R}-$trees) can be applied to graph theory (see e.g., [7,15]). A thorough discussion of these spaces and their important role in various branches of mathematics can be found in [2,3].

In this paper, we write $(1-\alpha)x \oplus \alpha y$ for the unique point z on the geodesic segment from x to y such that

$$d(z, x) = \alpha d(x, y) \text{ and } d(z, y) = (1-\alpha) d(x, y).$$

Also denote by $[x, y]$, the geodesic segment from x to y, that is,

$$[x, y] = \{(1-\alpha)x \oplus \alpha y : \alpha \in [0, 1]\}.$$

A subset C of a CAT(0) space is convex if $[x, y] \subset C$ for all $x, y \in C$.

Let $\{x_n\}$ be a bounded sequence in a metric space X and $x \in X$. Set

$$r(x, \{x_n\}) = \limsup_{n\to\infty} d(x_n, x).$$

The asymptotic radius $r(\{x_n\})$ of $\{x_n\}$ is defined by

$$r(\{x_n\}) = \inf\{r(x, \{x_n\}) : x \in X\},$$

and the asymptotic center $A(\{x_n\})$ of $\{x_n\}$ is the set

$$A(\{x_n\}) = \{x \in X \mid r(x, \{x_n\}) = r(\{x_n\})\}.$$

It is known that $A(\{x_n\})$ consists of exactly one point in a CAT(0) space [5].

A sequence $\{x_n\}$ in X is said to be Δ-convergent to $x \in X$ if x is the unique asymptotic center of every subsequence of $\{x_n\}$. We write $\Delta\text{-}\lim_{n\to\infty} x_n = x$ and call x as Δ-limit of $\{x_n\}$. Given $\{x_n\} \subset X$ such that $\{x_n\}$ is Δ-convergent to x and given $y \in X$ with $y \neq x$, we have

$$\limsup_{n\to\infty} d(x_n, x) < \limsup_{n\to\infty} d(x_n, y),$$

a similar condition to the Opial's property in Banach spaces [16].

Let $x_i \in X$ and $\lambda_i \in [0, 1]$ for $i = 1, 2, ..., n$ such that $\sum_{i=1}^{n} \lambda_i = 1$. Following the definition of unique point $(1-\alpha)\, x \oplus \alpha y$ on a geodesic segment $[x, y]$, we build the following notations:

$$\bigoplus_{i=1}^{2} \lambda_i x_i = \frac{\lambda_1}{\lambda_1 + \lambda_2} x_1 \oplus \frac{\lambda_2}{\lambda_1 + \lambda_2} x_2.$$

For $n = 3$, we have to find $\bigoplus_{i=1}^{3} \lambda_i x_i$ with $\sum_{i=1}^{3} \lambda_i = 1$. Note that

$$\begin{aligned}\bigoplus_{i=1}^{3} \lambda_i x_i &= (1-\lambda_3) \bigoplus_{i=1}^{2} \lambda_i x_i \oplus \lambda_3 x_3 \\ &= (1-\lambda_3)\left(\frac{\lambda_1}{\lambda_1 + \lambda_2} x_1 \oplus \frac{\lambda_2}{\lambda_1 + \lambda_2} x_2\right) \oplus \lambda_3 x_3 \\ &= (1-\lambda_3)\left(\frac{\lambda_1}{1-\lambda_3} x_1 \oplus \frac{\lambda_2}{1-\lambda_3} x_2\right) \oplus \lambda_3 x_3.\end{aligned}$$

By induction, we can write

$$\bigoplus_{i=1}^{n} \lambda_i x_i = (1-\lambda_n)\left(\frac{\lambda_1}{1-\lambda_n} x_1 \oplus \frac{\lambda_2}{1-\lambda_n} x_2 \oplus ... \oplus \frac{\lambda_{n-1}}{1-\lambda_n} x_{n-1}\right) \oplus \lambda_n x_n.$$

Let C be a nonempty subset of a metric space X and $F(T)$ be the set of fixed points of a mapping T on C. The mapping T is (i) generalized nonexpansive if $d(Tx, Ty) \le ad(x, y) + b\,[d(x, Tx) + d(y, Ty)] + c\,[d(x, Ty) + d(y, Tx)]$ for all $x, y \in C$, where $a + 2b + 2c \le 1$; in particular, nonexpansive if $a = 1, b = 0 = c$, (ii) quasi-nonexpansive if $F(T) \ne \emptyset$ and $d(Tx, y) \le d(x, y)$ for $x \in C, y \in F(T)$.

Example 1 Let $X = \mathbb{R}$, $d(x, y) = |x - y|$ and $T : [0, 1] \to [0, 1]$ be defined by

$$Tx = \begin{cases} \frac{109}{60}x - \frac{3}{2} \text{ if } x \in \left[\frac{10}{11}, 1\right], \\ \frac{1}{6}x \text{ if } x \in [0, \frac{10}{11}). \end{cases}$$

Obviously, T has a fixed point at 0.

For $x, y \in [0, 1]$, choose $a = \frac{1}{6}, b = c = 0$ when both $x, y \in \left[\frac{10}{11}, 1\right]$ or both $x, y \in [0, \frac{10}{11})$ and choose $a = c = 0, b = \frac{19}{41}$ when x (or y) $\in \left[\frac{10}{11}, 1\right]$. It is easy to verify that T is a generalized nonexpansive mapping but not nonexpansive (see also [30]).

For $x \in C$ and $p \in F(T)$, we calculate

$$\begin{aligned} d(Tx, p) &\le a\, d(x, p) + b\, d(x, Tx) + c\,(d(x, p) + d(p, Tx)) \\ &\le a\, d(x, p) + b\,(d(x, p) + d(Tx, p)) + c\,(d(x, p) + d(p, Tx)) \\ &= (a + b + c)\, d(x, p) + (b + c)\, d(Tx, p). \end{aligned}$$

That is, $d(Tx, p) \leq \dfrac{a+b+c}{1-b-c} d(x, p)$. Since $\dfrac{a+b+c}{1-b-c} \leq 1$, therefore

$$d(Tx, p) \leq d(x, p).$$

Above example and calculations show that every generalized nonexpansive mapping is quasi-nonexpansive and the notion of generalized nonexpansiveness is weaker than nonexpansiveness and stronger than quasi-nonexpansiveness.

Recently, Fukhar-ud-din, Khan and Akhter [11] have shown that the above defined generalized nonexpansive mapping T (which is also continuous) has a fixed point in a convex metric space; in particular, CAT(0) space.

It is well-known that Picard iterates of nonexpansive mappings fail to converge to its fixed point even on a Banach space. Therefore, Mann [20] iterates were introduced to approximate fixed points of nonexpansive mappings. Mann iterates were not adequate for the approximation of fixed points of pseudocontractive mappings and this led to the introduction of Ishikawa iterates [12]. The Ishikawa iterative scheme has been frequently used to construct common fixed point of certain nonlinear mappings on Banach spaces and metric spaces. Many authors have studied the two mappings case of Ishikawa iterative scheme for different types of mappings, see for example [4,9, 10,13,14,19,27] . Recently, Kumam et al. [18] studied modified S–iterative scheme (two mappings case) in the setting of CAT(0) space for a class of mappings which is wider than that of asymptotically nonexpansive mappings (see also [8,23]). It has been noticed in [25] that two mappings case has a direct connection with the minimization problem.

Finding common fixed points of a finite family of mappings acting on a Hilbert space is a problem that often arises in applied mathematics. In fact, many algorithms have been introduced for different classes of mappings with a nonempty set of common fixed points. Unfortunately, the existence results of common fixed points of a family of mappings are not known in many situations. Therefore, it is natural to consider approximation results for these classes of mappings. Approximating common fixed points of a finite family of nonexpansive mappings by iteration has been studied by many authors (see for instance [17,22,26]).

In this paper, we introduce a new one-step iterative scheme for approximating common fixed points of a family $\{T_i : i = 1, 2, ..., m\}$ of generalized nonexpansive mappings and study the $\triangle-$convergence and strong convergence of such iterative scheme in a CAT(0) space.

For arbitrary $x_1 \in C$, let the sequence $\{x_n\}$ in C be defined as follows:

$$x_{n+1} = \bigoplus_{i=1}^{m+1} a_{n,i} T_{i-1} x_n, \tag{1}$$

where $T_0 = I$ (the identity mapping) and $\{a_{n,i}\}$ are $(m+1)$ sequences in $[0, 1]$ such that $\sum_{i=1}^{m+1} a_{n,i} = 1$.

If $a_{n,i} = 0$ for $i \geq 2$ or $T_i = T$ for each i, it becomes Mann iterative scheme in a CAT(0) space.

Throughout the paper, we assume that $F = \bigcap_{i=1}^{m} F(T_i) \neq \emptyset$ for a finite family $\{T_i : i = 1, 2, ..., m\}$ of generalized nonexpansive mappings.

We need the following results for our convergence analysis in the main section.

Lemma 1 [29] *Let X be a CAT(0) space with $x, x_i \in X$ and $\lambda_i \in [0, 1]$ for $i = 1, 2, ..., n$ such that $\sum_{i=1}^{n} \lambda_i = 1$. Then*

(a) $d\left(\bigoplus_{i=1}^{n} \lambda_i x_i, x\right) \leq \sum_{i=1}^{n} \lambda_i d(x_i, x)$;

(b) $d\left(\bigoplus_{i=1}^{n} \lambda_i x_i, x\right)^2 \leq \sum_{i=1}^{n} \lambda_i d(x_i, x)^2 - \lambda_i \lambda_j d(x_i, x_j)^2$, *for* $i, j \in \{1, 2, ..., n\}$.

Lemma 2 [16] *Every bounded sequence in a complete CAT(0) space has a Δ-convergent subsequence.*

Lemma 3 [6] *If C is a closed convex subset of a complete CAT(0) space and if $\{x_n\}$ is a bounded sequence in C, then the asymptotic center of $\{x_n\}$ lies in C.*

2 Convergence Analysis

We start with the following lemma.

Lemma 4 *Let C be a nonempty, closed and convex subset of a CAT(0) space X and $\{T_i : i = 1, 2, ..., m\}$ be a family of generalized nonexpansive mappings on C. If $\{x_n\}$ is a sequence generated by (1), with $T_0 = I$ (the identity mapping) and $\{a_{n,i}\}$ are $m + 1$ sequences in $[0, 1]$ such that $\sum_{i=1}^{m+1} a_{n,i} = 1$, then $\lim_{n\to\infty} d(x_n, p)$ exists for all $p \in F$.*

Proof For any $p \in F$, we apply Lemma 1 (a) to (1) and get that

$$\begin{aligned} d(x_{n+1}, p) &= d\left(\bigoplus_{i=1}^{m+1} a_{n,i} T_{i-1} x_n, p\right) \\ &\leq \sum_{i=1}^{m+1} a_{n,i} d\left(T_{i-1} x_n, p\right) \\ &\leq \sum_{i=1}^{m+1} a_{n,i} d\left(x_n, p\right) \\ &= d\left(x_n, p\right). \end{aligned}$$

Thus, $\{d(x_n, p)\}$ is a decreasing sequence of real numbers which is bounded below. Hence, $\lim_{n\to\infty} d(x_n, p)$ exists. □

Lemma 5 *Let C be a nonempty, closed and convex subset of a CAT(0) space X and $\{T_i : i = 1, 2, ..., m\}$ be a family of generalized nonexpansive mappings on C. If $\{x_n\}$ is a sequence in (1), with $T_0 = I$ (the identity mapping) and $\{a_{n,i}\}$ are $m+1$ sequences* *in* $[\delta, 1-\delta]$ *for some* $\delta \in \left(0, \frac{1}{2}\right)$ *with* $\sum_{i=1}^{m+1} a_{n,i} = 1$, *then* $\lim_{n\to\infty} d(x_n, T_\ell x_n) = 0$ *for* $\ell \in \{1, 2, ..., m\}$.

Proof For any $p \in F$ and $\ell \in \{1, 2, ..., m\}$, we apply Lemma 1 (b) to (1) and proceed as

$$\begin{aligned} d(x_{n+1}, p)^2 &= d\left(\bigoplus_{i=1}^{m+1} a_{n,i} T_{i-1} x_n, p\right)^2 \\ &\le \sum_{i=1}^{m+1} a_{n,i} d(T_{i-1}x_n, p)^2 - a_{n,1}a_{n,\ell} d\,(x_n, T_\ell x_n)^2 \\ &\le \sum_{i=1}^{m+1} a_{n,i} d(x_n, p)^2 - a_{n,1}a_{n,\ell} d\,(x_n, T_\ell x_n)^2 \\ &= d(x_n, p)^2 - a_{n,1}a_{n,\ell} d\,(x_n, T_\ell x_n)^2 . \end{aligned}$$

That is,

$$\delta^2 d\,(x_n, T_\ell x_n)^2 \le d(x_n, p)^2 - d(x_{n+1}, p)^2.$$

It follows that for any positive integer $N \ge 1$,

$$\begin{aligned} \delta^2 \sum_{n=1}^{N} d\,(x_n, T_\ell x_n)^2 &\le d(x_1, p)^2 - d(x_{N+1}, p)^2 \\ &\le d(x_1, p)^2 < \infty. \end{aligned}$$

That is,

$$\delta^2 \sum_{n=1}^{N} d\,(x_n, T_\ell x_n)^2 < \infty. \tag{2}$$

When $N \to \infty$ in (2), we get that $\delta^2 \sum_{n=1}^{\infty} d\,(x_n, T_\ell x_n)^2 < \infty$ and hence $\lim_{n\to\infty} d\,(x_n, T_\ell x_n) = 0$ for $\ell \in \{1, 2, ..., m\}$. □

Now we obtain demiclosed principle for generalized nonexpansive mappings.

Theorem 1 *Let C be a nonempty, closed and convex subset of a CAT*(0) *space X and* $T : C \to C$ *be a generalized nonexpansive mapping. If* $\Delta\text{-}\lim_n x_n = x$ *and* $\lim_{n\to\infty} d(x_n, Tx_n) = 0$, *then* $x \in C$ *and* $Tx = x$.

Proof Note that

$$\begin{aligned} d(Tx, x_n) \le\ & d(Tx, Tx_n) + d(Tx_n, x_n) \\ &\le a\, d(x, x_n) + b\, d(x, Tx) + c\, (d(x, Tx_n) + d(x_n, Tx)) + d(Tx_n, x_n) \\ &\le (a+b+c)\, d(x_n, x) + (1+b+c)\, d(x_n, Tx_n) + (b+c)\, d(Tx, x_n). \end{aligned}$$

That is,

$$d(Tx, x_n) \leq \left(\frac{a+b+c}{1-b-c}\right) d(x_n, x) + \left(\frac{1+b+c}{1-b-c}\right) d(x_n, Tx_n)$$
$$\leq d(x_n, x) + \left(\frac{1+b+c}{1-b-c}\right) d(x_n, Tx_n). \tag{3}$$

Taking lim sup on both sides in (3), we have

$$r(Tx, \{x_n\}) = \limsup_{n\to\infty} d(Tx, x_n) \leq \limsup_{n\to\infty} \; d(x, x_n) = r(x, \{x_n\}).$$

The conclusion follows from the uniqueness of the asymptotic center. □

Theorem 2 *Let C be a nonempty closed convex subset of a complete CAT(0) space X and $\{T_1, T_2, .., T_m\}$ be a family of continuous and generalized nonexpansive mappings on C. If $\{x_n\}$ is a sequence generated by (1) with $T_0 = I$ (the identity mapping) and $\{a_{n,i}\}$ are $m+1$ sequences in $[\delta, 1-\delta]$ for some $\delta \in \left(0, \frac{1}{2}\right)$ with $\sum_{i=1}^{m+1} a_{n,i} = 1$, then $\{x_n\}$ is Δ-convergent to a common fixed point of T_i $(i = 1, 2, ..., m)$.*

Proof By Lemma 4, $\lim_{n\to\infty} d(x_n, p)$ exists for each $p \in F$ and therefore $\{x_n\}$ is bounded. Let $v \in W_\omega(\{x_n\}) = \cup A(\{u_n\})$, where the union is taken over all subsequences $\{u_n\}$ of $\{x_n\}$. There exists a subsequence $\{v_n\}$ of $\{x_n\}$ such that $A(\{v_n\}) = \{v\}$. By Lemmas 2 and 3, there must be a subsequence $\{z_n\}$ of $\{v_n\}$ such that $\Delta - \lim_n z_n = z \in C$. Since $\lim_{n\to\infty} d(x_n, T_\ell x_n) = 0$ for $\ell \in \{1, 2, ..., m\}$, in view of Lemma 5 and continuity of T_ℓ, we have $T_\ell z = z$ for each $\ell \in \{1, 2, ..., m\}$. That is, $z \in F$. Now we claim that $z = v$. Suppose $z \neq v$, then we have

$$\begin{aligned}
\limsup_{n\to\infty} d(z_n, z) &< \limsup_{n\to\infty} d(z_n, v) \\
&\leq \limsup_{n\to\infty} d(v_n, v) \\
&< \limsup_{n\to\infty} d(v_n, z) \\
&= \limsup_{n\to\infty} d(x_n, z) \\
&= \limsup_{n\to\infty} d(z_n, z),
\end{aligned}$$

which is a contradiction and hence $z = v \in F$.

To prove that the set $W_\omega(\{x_n\})$ is a singleton, let $A(\{x_n\}) = \{x\}$. Now suppose $z \neq x$. From Lemma 4, we know that $\{d(x_n, z)\}$ is convergent. By the uniqueness of the asymptotic center, we obtain

$$\begin{aligned}\limsup_{n\to\infty} d(z_n, z) &< \limsup_{n\to\infty} d(z_n, x)\\ &\le \limsup_{n\to\infty} d(x_n, x)\\ &< \limsup_{n\to\infty} d(x_n, z)\\ &= \limsup_{n\to\infty} d(z_n, z),\end{aligned}$$

which is a contradiction. Hence $z = x$. This completes the proof. □

Theorem 3 *Let C be a nonempty closed convex subset of a complete CAT(0) space X and $\{T_i : i = 1, 2, ..., m\}$ be a family of generalized nonexpansive mappings on C. If $\{x_n\}$ is a sequence generated by (1) with $T_0 = I$ (the identity mapping) and $\{a_{n,i}\}$ are $m+1$ sequences in $[\delta, 1-\delta]$ for some $\delta \in \left(0, \frac{1}{2}\right)$ with $\sum_{i=1}^{m+1} a_{n,i} = 1$, then $\{x_n\}$ converges to a common fixed point of $T_1, T_2, .., T_m$ if and only if* $\liminf_{n\to\infty} d(x_n, F) = 0$ *or* $\limsup_{n\to\infty} d(x_n, F) = 0$.

Proof Suppose $x_n \to p \in F$. So for a given $\varepsilon > 0$, there exists $n_0 \ge 1$ such that $d(x_n, p) < \varepsilon$ for all $n \ge n_0$. Taking the inf over $p \in F$, we get that $d(x_n, F) < \varepsilon$ for all $n \ge n_0$. That is, $\lim_{n\to\infty} d(x_n, F) = 0$. This means that

$$\liminf_{n\to\infty} d(x_n, F) = 0 = \limsup_{n\to\infty} d(x_n, F).$$

Conversely, suppose that

$$\liminf_{n\to\infty} d(x_n, F) = 0 \text{ or } \limsup_{n\to\infty} d(x_n, F) = 0.$$

By Lemma 4,

$$d(x_{n+1}, p) \le d(x_n, p) \text{ for any } p \in F,$$

so that

$$d(x_{n+1}, F) \le d(x_n, F).$$

Hence, $\lim_{n\to\infty} d(x_n, F)$ exists. Since

$$\liminf_{n\to\infty} d(x_n, F) = 0 \text{ or } \limsup_{n\to\infty} d(x_n, F) = 0,$$

we have

$$\lim_{n\to\infty} d(x_n, F) = 0.$$

Let $\varepsilon > 0$. There exists $n_0 \ge 1$ such that

$$d(x_n, F) < \frac{\varepsilon}{3}, \text{ for all } n \ge n_0.$$

In particular, $d(x_{n_0}, F) < \frac{\varepsilon}{3}$. Thus, there exists $p \in F$ such that

$$d(x_{n_0}, p) < \frac{\varepsilon}{2}.$$

For $m, n \geq n_0$,

$$d(x_{m+n}, x_n) \leq d(x_{m+n}, p) + d(x_n, p) < 2d(x_{n_0}, p) < \varepsilon.$$

Thus, $\{x_n\}$ is a Cauchy sequence in the closed subset C of X. Therefore, $x_n \to x \in C$. Since T_ℓ is a generalized nonexpansive mapping, we have

$$\begin{aligned} d(x, T_\ell x) &\leq d(x, x_n) + d(x_n, T_\ell x_n) + d(T_\ell x_n, T_\ell x) \\ &\leq d(x_n, x) + d(x_n, T_\ell x_n) + ad(x_n, x) + b\left[d(x_n, T_\ell x_n) + d(x, T_\ell x)\right] \\ &\quad + c\left[d(x_n, T_\ell x) + d(x, T_\ell x_n)\right]. \end{aligned}$$

That is,

$$d(x, T_\ell x) \leq \left(\frac{a + 2c + 1}{1 - b - c)}\right) d(x_n, x) + \left(\frac{1 + b + c}{1 - b - c}\right) d(x_n, T_\ell x_n) \to 0.$$

Thus, $T_\ell x = x$ for each $\ell \in \{1, 2, ..., m\}$. Hence x is a common fixed point of $T_1, T_2, ..., T_m$. □

The following condition is due to Senter and Dotson [24].

A mapping $T : C \to C$, where C is a subset of X, is said to satisfy condition (A) if there is a nondecreasing function $f : [0, \infty) \to [0, \infty)$ with $f(0) = 0$ and $f(r) > 0$ for $r > 0$ such that

$$d(x, Tx) \geq f(d(x, F)) \quad \text{for all } x \in C,$$

where $d(x, F) = \inf_{p \in F} d(x, p)$.

Senter and Dotson [24] approximated fixed points of a nonexpansive mapping T by Mann iterates [20] under condition (A). Later on, Maiti and Ghosh [21] and Tan and Xu [28] studied the approximation of fixed points of a nonexpansive mapping T by Ishikawa iterates under the same condition which is weaker than the requirement that T is demicompact. This condition has been modified for a family of mappings and different modifications of condition (A) are available in the literature. One of the modifications is the following.

A family of mappings $T_i : C \to C (i = 1, 2, ..., m)$, where C is a subset of X, is said to satisfy condition (AR) if there exists a nondecreasing function $f : [0, \infty) \to [0, \infty)$ with $f(0) = 0$ and $f(r) > 0$ for $r > 0$ and $\lambda_i \in [0, 1]$ such that

$$\sum_{i=1}^{n} \lambda_i d(x, T_i x) \geq f(d(x, F)) \quad \text{for all } x \in C,$$

where $\sum_{i=1}^{n} \lambda_i = 1$ and $d(x, F) = \inf_{p \in F} d(x, p)$.

Note that condition (AR) reduces to condition (A) if $T_i = T$ for each $i = 1, 2, ..., m$. We shall use condition (AR) instead of compactness to study the strong convergence of $\{x_n\}$ defined in (1). It is worth to mention that in case of a family of generalized nonexpansive mappings $T_i : C \to C$ $(i = 1, 2, ..., m)$, condition (AR) is weaker than the compactness.

Next we approximate common fixed points using condition (AR) by the following strong convergence theorem.

Theorem 4 *Let C be a nonempty, closed and convex subset of a complete CAT(0) space X and $\{T_i : i = 1, 2, ..., m\}$ be a family of generalized nonexpansive mappings on C satisfying condition (AR). If $\{x_n\}$ is a sequence generated by (1) with $T_0 = I$(the identity mapping) and $\{a_{n,i}\}$ are $m + 1$ sequences in $[\delta, 1 - \delta]$ for some $\delta \in \left(0, \frac{1}{2}\right)$ with $\sum_{i=1}^{m+1} a_{n,i} = 1$, then $\{x_n\}$ strongly converges to a common fixed point of T_i $(i = 1, 2, ..., m)$.*

Proof As a consequence of Theorem 4, $\lim_{n\to\infty} d(x_n, F)$ exists. So by condition (AR), we obtain

$$\lim_{n\to\infty} f(d(x_n, F)) \leq \sum_{i=1}^{n} a_{n,i} \lim_{n\to\infty} d(x_n, T_i x_n) = 0.$$

Since f is a nondecreasing function and $f(0) = 0$, $\lim_{n\to\infty} d(x_n, F) = 0$. The conclusion follows from Theorem 3. □

Since every nonexpansive mapping is a generalized nonexpansive and continuous, therefore the following results due to Abbas and Khan [1] are the immediate consequences of our theorems.

Corollary 1 [1] *Let C be a nonempty, closed and convex subset of a complete CAT(0) space X. Let T_1 and T_2 be two nonexpansive mappings of C. Let $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$ be in $[\delta, 1 - \delta]$ for all $n \in N$ and for some $\delta \in \left(0, \frac{1}{2}\right)$ with $a_n + b_n + c_n = 1$. If $F = F(T_1) \cap F(T_2) \neq \emptyset$, then $\{x_n\}$ defined by the iteration process*

$$x_{n+1} = a_n x_n \oplus (1 - a_n)\left[\frac{b_n}{1 - a_n} T_1 x_n \oplus \frac{c_n}{1 - a_n} T_2 x_n\right], \tag{4}$$

$\triangle$-converges to a common fixed point of T_1 and T_2.

Proof Set $a_{n,1} = a_n$, $a_{n,2} = b_n$, $a_{n,3} = c_n$ and $m = 3$ in Theorem 2. □

Corollary 2 [1] *Let C be a nonempty, closed and convex subset of a complete CAT(0) space X. Let T_1 and T_2 be two nonexpansive mappings of C. Let $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$ be in $[\delta, 1 - \delta]$ with $a_n + b_n + c_n = 1$ for all $n \in N$ and for some $\delta \in \left(0, \frac{1}{2}\right)$. If $F = F(T_1) \cap F(T_2) \neq \emptyset$, then $\{x_n\}$ defined by the iteration process (4) converges strongly to a common fixed point of T_1 and T_2 if and only if $\liminf_{n\to\infty} d(x_n, F) = 0$.*

Corollary 3 [1] *Let C be a nonempty, closed and convex subset of a complete CAT(0) space X. Let T_1 and T_2 be two nonexpansive mappings of C satisfying condition (AR). Let $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$ be in $[\delta, 1 - \delta]$ for all $n \in N$ and for some $\delta \in \left(0, \frac{1}{2}\right)$ with $a_n + b_n + c_n = 1$. If $F = F(T_1) \cap F(T_2) \neq \emptyset$, then $\{x_n\}$ defined by the iteration process (4) converges strongly to a common fixed point of T_1 and T_2.*

Remark 1 Our results are new in literature and improve the theorems of Kuhfittig [17], Rhoades[22] and many others due to the following reasons:

(1) Domain is nonlinear;
(2) Mappings are generalized nonexpansive;
(3) iterative scheme is one-step instead of multi-step.

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