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(A, m)-Isometric Unilateral Weighted Shifts in Semi-Hilbertian Spaces

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Abstract For a positive integer m, a bounded linear operator T on a Hilbert space $\mathbb{H}$ is called an (A, m)-isometry, if $\Theta_A^{(m)}(T) = \sum_{k=0}^{m}(-1)^{m-k}\binom{m}{k}T^{*k}AT^k = 0$, where A is a positive (semi-definite) operator. In this paper we give a characterization of (A, m)-isometric and strict (A, m)-isometric unilateral weighted shifts in terms of their weight sequences, respectively. Moreover, we characterize $(A, 2)$-expansive unilateral weighted shifts (i.e. operators satisfying $\Theta_A^{(2)}(T) \leq 0$).

Keywords (A, m)-isometry · (A, m)-expansive operators · Unilateral weighted shifts

Mathematics Subject Classification 46C05 · 47A05 · 47B37

1 Introduction and Preliminaries

Throughout the paper, $\mathbb{H}$ denotes a separable infinite dimensional complex Hilbert space with inner product $\langle \cdot \mid \cdot \rangle$ and $\{e_n\}_{n\geq 0}$ is an orthonormal basis of $\mathbb{H}$. A represents a nonzero ($A \neq 0$) positive operator and denote I the identity operator on $\mathbb{H}$. By $\mathcal{L}(\mathbb{H})$ we denote the Banach algebra of all linear operators on $\mathbb{H}$. For every $T \in \mathcal{L}(\mathbb{H})$ its range is denoted by $R(T)$, its null space by $N(T)$ and its adjoint by T^*. The cone

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of positive (semi-definite) operators and the set of all $T \in \mathcal{L}(\mathbb{H})$ which admit an A-adjoint are given, respectively, by

$$\mathcal{L}(\mathbb{H})^+ := \{A \in \mathcal{L}(\mathbb{H}) : \langle Au \mid u\rangle \geq 0,\ \forall\, u \in \mathbb{H}\},$$
$$\mathcal{L}_A(\mathbb{H}) := \{T \in \mathcal{L}(\mathbb{H}) : R(T^*A) \subset R(A)\}.$$

Any $A \in \mathcal{L}(\mathbb{H})^+$ defines a positive semi-definite sesquilinear form:

$$\langle \cdot \mid \cdot \rangle_A : \mathbb{H} \times \mathbb{H} \longrightarrow \mathbb{C}, \quad \langle u \mid v\rangle_A := \langle Au \mid v\rangle.$$

By $\|\cdot\|_A$ we denote the semi-norm induced by $\langle \cdot \mid \cdot\rangle_A$, i.e. $\|u\|_A = \langle u \mid u\rangle_A^{\frac{1}{2}}$. Observe that $\|u\|_A = 0$ if and only if $u \in N(A)$. Then $\|\cdot\|_A$ is a norm if and only if A is an injective operator.

Definition 1.1 For a positive operator A, we say that $u,\ v \in \mathbb{H}$ are A-orthogonal if

$$\langle u \mid v\rangle_A = \langle Au \mid v\rangle = 0.$$

More general, a family of vectors $(u_i)_i$ is A-orthogonal if $\langle u_i \mid u_j\rangle_A = 0$, for all $i \neq j$.

For any $T \in \mathcal{L}(\mathbb{H})$ and $A \in \mathcal{L}(\mathbb{H})^+$, define

$$\Theta_A^{(m)}(T) = \sum_{k=0}^{m} (-1)^{m-k} \binom{m}{k} T^{*k} A T^k, \quad m \geq 1. \tag{1}$$

We say that T is (A, m)-expansive if $\Theta_A^{(m)}(T) \leq 0$ for some positive integer m. For more details on such a family, we refer the readers to [11].

A detailed study of m-isometries was developed by Agler and Stankus [1–3]. Recently, Sid Ahmed et al. [8] generalized the concept of those operators on a Hilbert space when an additional semi-inner product is considered. They introduced the (A, m)-isometries as a special case of (A, m)-expansive operators. In [9], we gave a detailed study concerning the behavior of the orbits of such a family. In fact, for $m \in \mathbb{N}$, an operator $T \in \mathcal{L}(\mathbb{H})$ is called an (A, m)-isometry if $\Theta_A^{(m)}(T) = 0$, or equivalently,

$$\sum_{k=0}^{m} (-1)^{m-k} \binom{m}{k} \|T^k u\|_A^2 = 0 \quad \text{for all } u \in \mathbb{H}. \tag{2}$$

Remark 1.1 1. An $(A, 1)$-isometry will be called an A-isometry.
2. T will be said a strict (A, m)-isometry if it is an (A, m)-isometry but not an $(A, m-1)$-isometry.

For $T \in \mathcal{L}(\mathbb{H})$ and $k = 0, 1, 2, \ldots$, we consider the operator

$$\beta_k(T) = \frac{1}{k!}\, \Theta_A^{(k)}(T).$$

For $n \geq k$, we denote

$$n^{(k)} = \begin{cases} 1, & \text{if } n = 0 \text{ or } k = 0; \\ \binom{n}{k} k! \ = n(n-1)(n-2)\ldots(n-k+1), & \text{otherwise.} \end{cases}$$

Observe that $\beta_0(T) = A$ and if T is an (A,m)-isometry, then $\beta_k(T) = 0$ for every $k \geq m$. Hence, according to [9] we have

$$\|T^n u\|_A^2 = \sum_{k=0}^{m-1} n^{(k)} \ \langle \beta_k(T)u \mid u\rangle \quad \text{for all } u \in \mathbb{H}. \tag{3}$$

The A-covariance operator $\Delta_{A,T}$ is defined by

$$\Delta_{A,T} := \beta_{m-1}(T) = \frac{1}{(m-1)!} \ \Theta_A^{(m-1)}(T). \tag{4}$$

Theorem 2.1 [4] gives a useful characterization of m-isometries on a Hilbert space. The same proof works for (A,m)-isometries. In fact, we show that if T is an (A,m)-isometry then it is possible to explain the semi-norm $\|T^n u\|_A$ of $T^n u$ in terms of the semi-norms of the vectors $u, Tu, \ldots, T^{m-1}u$.

Theorem 1.1 *An operator $T \in \mathcal{L}(\mathbb{H})$ is an (A,m)-isometry if and only if*

$$\|T^n u\|_A^2 = \sum_{k=0}^{m-1} (-1)^{m-k-1} \frac{n(n-1)\ldots\overbrace{(n-k)}\ldots(n-m+1)}{k!(m-k-1)!} \|T^k u\|_A^2 \tag{5}$$

for all $n \geq 0$ and all $u \in \mathbb{H}$, where $\overbrace{n-k}$ denotes that the factor $(n-k)$ is omitted.

Remark 1.2 For $k = 0,\ 1,\ 2, \ldots, (m-1)$, the coefficient of $\|T^k u\|_A^2$ is a polynomial in n of degree $(m-1)$.

The paper is organized as follows. In Sect. 2 we focus on unilateral weighted shifts which are (A,m)-isometries. A characterization in terms of the weight sequence is given for the forward shifts. Then, we describe the behavior of the weights for such a family. Finally, in order that the paper will be self-contained, we give a characterization for backward shifts.

Generally, an (A,m)-isometry is not an $(A,m-1)$-isometry (see [8]). Inspired from that, the aim of Sect. 3 is the study of strict (A,m)-isometric unilateral weighted forward shifts. A characterization for a particular operator A is also given. In Sect. 4 we focus on $(A,2)$-expansive (or A-concave) operators. Some properties related to unilateral weighted shifts are given.

2 (A, m)-Isometric Unilateral Weighted Shifts

The aim of this section is to give a characterization of unilateral weighted shift operators which are (A, m)-isometries in terms of their weight sequences. Before starting the study, first we recall that a unilateral weighted shift T is unitarily equivalent to a weighted shift operator with a non-negative weight sequence. So we can assume that $w_n \geq 0$. Furthermore, if T is injective, it can be assumed that $w_n > 0$ (see [6,10]).

2.1 Unilateral Weighted Forward Shifts

An operator $T \in \mathcal{L}(\mathbb{H})$ is said to be a unilateral weighted forward shift, if there exists an orthonormal basis $\{e_n\}_{n\geq 0}$ and a sequence $\{w_n\}_{n\geq 0}$ of complex numbers such that $Te_n = w_n e_{n+1}$. It is well known and it is not difficult to see that T is a bounded operator if and only if the weight sequence $\{w_n\}_{n\geq 0}$ is bounded. The iterates of T are given by $T^0 = I$, and for $k \geq 1$,

$$T^k e_n = \left(\prod_{i=0}^{k-1} w_{n+i}\right) e_{n+k}, \quad n \geq 0. \tag{6}$$

It was shown ([4], Proposition 3.1) that if T is an m-isometric unilateral weighted forward shift operator with weight sequence $\{w_n\}_{n\geq 1}$, then $w_n \neq 0$ for all $n \geq 1$. The characterization related to (A, m)-isometric unilateral weighted shifts is given in the following proposition.

Proposition 2.1 *Let T be a unilateral weighted forward shift with weight sequence $\{w_n\}_{n\geq 0}$. If T is an (A, m)-isometry and if there exists a nonnegative integer n_0 such that $e_{n_0} \notin N(A)$, then $w_{n_0} \neq 0$.*

Proof Assume that there exists a nonnegative integer n_0 such that $\|e_{n_0}\|_A \neq 0$. Equation (6) implies that

$$\|T^k e_{n_0}\|_A^2 = \left(\prod_{i=0}^{k-1} |w_{i+n_0}|^2\right) \|e_{k+n_0}\|_A^2, \quad k \geq 1.$$

Since $T^0 e_{n_0} = e_{n_0}$, we obtain

$$\begin{aligned}
(-1)^{m+1}\|e_{n_0}\|_A^2 &= \sum_{k=1}^{m}(-1)^{m-k}\binom{m}{k}\|T^k e_{n_0}\|_A^2 \\
&= \sum_{k=1}^{m}(-1)^{m-k}\binom{m}{k}\left(\prod_{i=0}^{k-1}|w_{i+n_0}|^2\right)\|e_{k+n_0}\|_A^2 \\
&= |w_{n_0}|^2\Bigg[(-1)^{m-1}\binom{m}{1}\|e_{n_0+1}\|_A^2 \\
&\quad + \sum_{k=2}^{m}(-1)^{m-k}\binom{m}{k}\left(\prod_{i=1}^{k-1}|w_{i+n_0}|^2\right)\|e_{k+n_0}\|_A^2\Bigg]
\end{aligned}$$

which gives

$$|w_{n_0}|^2 = \frac{(-1)^{m+1}\|e_{n_0}\|_A^2}{(-1)^{m-1}\binom{m}{1}\|e_{n_0+1}\|_A^2 + \sum_{k=2}^{m}(-1)^{m-k}\binom{m}{k}\left(\prod_{i=1}^{k-1}|w_{i+n_0}|^2\right)\|e_{k+n_0}\|_A^2}.$$

Thus, the proof is achieved. □

Corollary 2.1 *Let T be a unilateral weighted forward shift with weight sequence $\{w_n\}_{n\geq 0}$. The following assertions hold true.*

1. *If T is an A-isometry and if $e_0 \notin N(A)$, then $w_n \neq 0$ for all $n \geq 0$.*
2. *If T is an $(A, 2)$-isometry and $e_{2p} \notin N(A)$ for all $p \geq 0$, then $w_n \neq 0$ for all $n \geq 0$.*

Proof 1. If T is an A-isometry then $\|Tu\|_A = \|u\|_A$ for all $u \in \mathbb{H}$. For $u = e_n$, we obtain

$$|w_n|\|e_{n+1}\|_A = \|e_n\|_A$$

which implies that if $e_0 \notin N(A)$, then $w_n \neq 0$ for all $n \geq 0$.

2. To obtain the desired claim we restrict ourselves to prove that, for each n, if $e_n \notin N(A)$ then $w_n \neq 0$ and $w_{n+1} \neq 0$.
Since T is an $(A, 2)$-isometry, $\|T^2u\|_A^2 - 2\|Tu\|_A^2 + \|u\|_A^2 = 0$ for all $u \in \mathbb{H}$. For $u = e_n$, we obtain

$$|w_n|^2\left\{|w_{n+1}|^2\|e_{n+2}\|_A^2 - 2\|e_{n+1}\|_A^2\right\} = -\|e_n\|_A^2. \tag{7}$$

If $e_n \notin N(A)$, then (7) implies that $w_n \neq 0$ and

$$|w_{n+1}|\|e_{n+2}\|_A \pm \sqrt{2}\|e_{n+1}\|_A \neq 0. \tag{8}$$

Moreover, if we assume that $w_{n+1} = 0$, then the identity (8) gives $e_{n+1} \notin N(A)$, and hence $w_{n+1} \neq 0$ which is a contradiction. □

In the following proposition we establish that if a unilateral weighted forward shift T with weight sequence $\{w_n\}_{n\geq 0}$ is A-bounded then the sequence $\left\{\frac{\|e_{n+1}\|_A}{\|e_n\|_A}\ w_n,\ n \geq 0\right\}$ is bounded.

Proposition 2.2 *Let $T \in \mathcal{L}_A(\mathbb{H})$ be a unilateral weighted forward shift with weight sequence $\{w_k\}_{k\geq 0}$. Assume that $e_k \notin N(A)$ for all $k \in \mathbb{N}$. Then, for all $n \geq 1$*

$$\|T^n\|_A \geq \sup_{k\in\mathbb{N}}\left\{\left(\prod_{i=0}^{n-1}|w_{i+k}|\right)\frac{\|e_{k+n}\|_A}{\|e_k\|_A}\right\}. \tag{9}$$

Proof Note first that if $T \in \mathcal{L}_A(\mathbb{H})$, then $T^n \in \mathcal{L}_A(\mathbb{H})$ for all $n \geq 0$ and, moreover, we have

$$\|T^n u\|_A \le \|T^n\|_A \|u\|_A \quad \text{for all } u \in \mathbb{H},\ n \ge 0.$$

Let $n \ge 1$. Since (6) holds true, we have

$$\left(\prod_{i=0}^{n-1} |w_{i+k}|\right) \|e_{k+n}\|_A = \|T^n e_k\|_A \le \|T^n\|_A \|e_k\|_A.$$

Thus taking supremum over all $k \in \mathbb{N}$, we obtain

$$\sup_{k\in\mathbb{N}} \left\{ \left(\prod_{i=0}^{n-1} |w_{i+k}|\right) \frac{\|e_{k+n}\|_A}{\|e_k\|_A} \right\} \le \|T^n\|_A.$$

Remark 2.1 Equality between the two parts of (9) does not holds true for every unilateral weighted forward shift T and any positive operator A. For example, assume that $n = 1$ and $w_k = 1$ for all $k \ge 0$. Let A be the operator given in the orthonormal basis $\{e_k\}_{k\ge 0}$ by the matrix $\{a_{jk}\}_{j,k\ge 0}$ such that $a_{jj} = 1$ for all $j \ge 0$, $a_{01} = a_{10} = \frac{1}{2}$, and all other elements are equal to 0. It is easily seen that A is an invertible positive operator. Moreover, $T \in \mathcal{L}_A(\mathbb{H})$. Remark that the right part of (9) is equal to 1. Indeed,

$$\|e_k\|_A^2 = \langle Ae_k \mid e_k\rangle = a_{kk} = 1, \quad \text{for all } k \ge 0.$$

On the other hand, we have

$$\begin{aligned} \|e_0 - e_1\|_A^2 &= \langle A(e_0 - e_1) \mid (e_0 - e_1)\rangle \\ &= a_{00} - 2a_{01} + a_{11} = 1 \\ \text{and} \quad \|T(e_0 - e_1)\|_A^2 &= \|e_1 - e_2\|_A^2 \\ &= \langle A(e_1 - e_2) \mid (e_1 - e_2)\rangle \\ &= a_{11} - 2a_{12} + a_{22} = 2. \end{aligned}$$

Consequently, $\|T\|_A \ge \sqrt{2}$.

Every $u \in \mathbb{H}$ can be written as $u = \sum_{n\ge 0} \alpha_n e_n$; $\{\alpha_n\}_{n\ge 0} \subset \mathbb{C}$. Hence, we have

$$\sum_{k=0}^{m} (-1)^{m-k} \binom{m}{k} \|T^k u\|_A^2 = \sum_{i,j\ge 0} \alpha_i \overline{\alpha}_j \ S_{i,j,A}^{(m)},$$

where

$$S_{i,j,A}^{(m)} = \sum_{k=0}^{m} (-1)^{m-k} \binom{m}{k} \langle AT^k e_i \mid T^k e_j\rangle, \quad i, j \ge 0.$$

Remark 2.2 If T is a unilateral weighted forward shift with weight sequence $\{w_n\}_{n\ge 0}$, then we have

$$S_{i,j,A}^{(m)} = \begin{cases} (-1)^m \langle Ae_i \mid e_j \rangle + \sum_{k=1}^{m} (-1)^{m-k} \binom{m}{k} \left(\prod_{p=0}^{k-1} w_{i+p} \overline{w}_{j+p} \right) \langle Ae_{i+k} \mid e_{j+k} \rangle, & (i \neq j) \\ (-1)^m \|e_i\|_A^2 + \sum_{k=1}^{m} (-1)^{m-k} \binom{m}{k} \left(\prod_{p=0}^{k-1} |w_{i+p}|^2 \right) \|e_{i+k}\|_A^2 := S_{i,A}^{(m)}, & (i = j). \end{cases}$$

Let us begin with the following theorem.

Theorem 2.2 *Let T be a unilateral weighted forward shift with weight sequence $\{w_n\}_{n \geq 0}$. Then T is an (A, m)-isometry if and only if $S_{i,j,A}^{(m)} = 0$ for all $i, j \geq 0$.*

Proof Assume that T is an (A, m)-isometry. For $u = e_n$, (2) implies that

$$\begin{aligned} 0 &= \sum_{k=0}^{m} (-1)^{m-k} \binom{m}{k} \|T^k e_n\|_A^2 \\ &= (-1)^m \|e_n\|_A^2 + \sum_{k=1}^{m} (-1)^{m-k} \binom{m}{k} \left(\prod_{i=0}^{k-1} |w_{n+i}|^2 \right) \|e_{n+k}\|_A^2. \end{aligned}$$

Hence, for all $n \in \mathbb{N}$ we have $S_{n,A}^{(m)} = 0$. As a consequence, for all sequence $\{\alpha_n\}_n \subset \mathbb{C}$, it yields that $\sum_{i \neq j} \alpha_i \overline{\alpha}_j \; S_{i,j,A}^{(m)} = 0$. Indeed, to prove such a claim we argue by contradiction. If there exists $\{\alpha_n\}_n$ such that $\sum_{i \neq j} \alpha_i \overline{\alpha}_j \; S_{i,j,A}^{(m)} \neq 0$, then if we take a nonzero $v = \sum_i \; \alpha_i e_i$ we obtain

$$\sum_{k=0}^{m} (-1)^{m-k} \binom{m}{k} \|T^k v\|_A^2 \neq 0,$$

which contradicts the fact that T is an (A, m)-isometry. Moreover, assume that there exists $i_0 \neq j_0$ such that $S_{i_0,j_0,A}^{(m)} \neq 0$. If we consider the sequence $\{\beta_n\}_n$ defined by $\beta_{i_0} \neq 0$, $\beta_{j_0} \neq 0$ and $\beta_i = 0$ otherwise, then for $s = \beta_{i_0} e_{i_0} + \beta_{j_0} e_{j_0}$, we obtain

$$\sum_{k=0}^{m} (-1)^{m-k} \binom{m}{k} \|T^k s\|_A^2 = \beta_{i_0} \overline{\beta_{j_0}} \; S_{i_0,j_0,A}^{(m)} \neq 0,$$

which contradicts the hypothesis. Hence, $S_{i,j,A}^{(m)} = 0$ for all $i \neq j$.

For the converse, suppose that $S_{i,j,A}^{(m)} = 0$ for all $i, j \geq 0$. Since every vector $u \in \mathbb{H}$ can be written as $u = \sum_i \; \alpha_i e_i$, it follows that

$$\sum_{k=0}^{m} (-1)^{m-k} \binom{m}{k} \|T^k u\|_A^2 = \sum_{i,j} \alpha_i \overline{\alpha}_j \; S_{i,j,A}^{(m)} = 0.$$

□

In the following proposition we give some properties related to the coefficients $S_{i,j,A}^{(m)}$ defined in Remark 2.2.

Proposition 2.3 *Let T be an (A, m)-isometric unilateral weighted forward shift with weight sequence $\{w_n\}_{n\geq 0}$. The following claims hold true.*

1. *If $S_{0,A}^{(m-1)} = 0$, then $S_{n,A}^{(m-1)} = 0$ for all $-1 \leq n-1 < \inf\{n \geq 0/w_n = 0\}$, where $\inf\{n \geq 0/w_n = 0\} = +\infty$ if $\{n \geq 0/w_n = 0\} = \Phi$.*
2. *If $e_n \notin N(A)$ for all $n \in \mathbb{N}$ and $S_{0,A}^{(m-1)} = 0$, then $S_{n,A}^{(m-1)} = 0$ for all $n \in \mathbb{N}$.*

Proof 1. First of all, let us remark that

$$S_{n,A}^{(m-1)} = (m-1)!\, \langle \Delta_{A,T} e_n \mid e_n \rangle \quad \text{for all } n \geq 0.$$

Since T is an (A, m)-isometry, then a simple computation shows that

$$\triangle_{A,T} = T^* \triangle_{A,T} T.$$

Moreover,

$$\begin{aligned}\langle \Delta_{A,T} e_n \mid e_n \rangle &= \langle T^* \Delta_{A,T} T e_n \mid e_n \rangle \\ &= \langle \Delta_{A,T} T e_n \mid T e_n \rangle \\ &= |w_n|^2 \, \langle \Delta_{A,T} e_{n+1} \mid e_{n+1} \rangle.\end{aligned}$$

This implies that

$$S_{n,A}^{(m-1)} = |w_n|^2 \, S_{n+1,A}^{(m-1)}. \tag{10}$$

As a consequence of (10), we have

$$S_{0,A}^{(m-1)} = \prod_{i=0}^{n} |w_i|^2 \, S_{n+1,A}^{(m-1)} \quad \text{for all } n \geq 0.$$

Hence, we obtain the result.

2. The claim is a direct consequence of (10) and Proposition 2.1. □

Corollary 2.3 *Let T be a unilateral weighted forward shift with weight sequence $\{w_n\}_{n\geq 0}$. Let $H_{i,j,A}^{(m)}$ be as*

$$H_{i,j,A}^{(m)} := \left(\prod_{p=0}^{i} w_p \right) \left(\prod_{q=0}^{j} \overline{w}_q \right) S_{i,j,A}^{(m)}, \quad i, j \geq 0. \tag{11}$$

Then, we have

1. *If T is an (A, m)-isometry, then*
   (a) $H_{i,A}^{(m)} := H_{i,i,A}^{(m)} = 0$ for all $i \geq 0$.
   (b) $\sum_{i,j} H_{i,j,A}^{(m)} = 0$ for all $i, j \geq 0$.

2. *If, for all* $i, j \geq 0$ $H_{i,j,A}^{(m)} = 0$, *then either* T *is an* (A, m)*-isometry or there exists* $k \geq 0$ *such that* $w_k = 0$.

Proof 1. Assuming that T is an (A, m)-isometry, (a) and (b) follow immediately from 1. and 2. of Proposition 2.3.

2. Suppose that for all $i, j \geq 0$ $H_{i,j,A}^{(m)} = 0$, i.e

$$\left(\prod_{p=0}^{i} w_p\right)\left(\prod_{q=0}^{j} \overline{w}_q\right) S_{i,j,A}^{(m)} = 0$$

which implies that either there exists $0 \leq k \leq \max(i, j)$ such that $w_k = 0$ or $S_{i,j,A}^{(m)} = 0$ for all $i, j \geq 0$ and, hence, we can conclude. □

Remark 2.3 Let $\{f_n\}_{n \geq 1}$ be an orthonormal basis (i.e $\|f_n\| = 1 \neq 0$, $n \geq 1$) and $T f_n = w_n f_{n+1}$ for all $n \geq 1$. Assume that $A = I$. Then Proposition 2.1 holds, that is $w_n \neq 0$ for all $n \geq 1$. Moreover, we consider

$$\widetilde{H}_{i,j,A}^{(m)} := \left(\prod_{p=0}^{i-1} w_p\right)\left(\prod_{q=0}^{j-1} \overline{w}_q\right) S_{i,j,A}^{(m)}, \quad i, j \geq 1,$$

where $w_0 := 1$. If $\widetilde{H}_{n,A}^{(m)} = 0$ for all $n \geq 1$, then assertion 2. of Proposition 2.3 implies that T is an m-isometry. Hence, in that case we recover Proposition 3.2 ([4]), that is T is an m-isometry if and only if

$$\sum_{k=0}^{m} (-1)^{m-k} \binom{m}{k} \left(\prod_{i=0}^{n+k-1} |w_i|^2\right) = 0 \quad \text{for all } n \geq 1.$$

For a fixed sequence $\{w_k\}_{k \geq 0}$, $n \geq 0$ and $m \geq 1$, let us denote

$$R_{n,A}^{(m)} := (-1)^{m-1} \frac{(n-1)(n-2)\ldots(n-m+1)}{(m-1)!} \|e_0\|_A^2 + \sum_{k=1}^{m-1} (-1)^{m-k-1} \frac{n(n-1)\ldots\overbrace{(n-k)}\ldots(n-m+1)}{k!(m-k-1)!} \left(\prod_{i=0}^{k-1} |w_i|^2\right) \|e_k\|_A^2, \tag{12}$$

where $\overbrace{n-k}$ denotes that the factor $(n-k)$ is omitted. Remark that $R_{0,A}^{(m)} = \|e_0\|_A^2$ and for $j = 1, 2, \ldots, m-1$,

$$R_{j,A}^{(m)} = \left(\prod_{i=0}^{j-1} |w_i|^2\right) \|e_j\|_A^2 \ \ (\geq 0). \tag{13}$$

As it was shown in [4] for m-isometric weighted shifts, in the following result we describe the behavior of a given unilateral weighted forward shift which is (A, m)-isometric by means of his weight sequence sequence.

Theorem 2.4 *Let T be a unilateral weighted forward shift with weight sequence $\{w_n\}_{n\geq 0}$. The following claims hold true.*

1. *Assume that there exists $0 \leq n_0 \leq m-2$ such that $e_j \notin N(A),\ 0 \leq j \leq n_0+1$. If T is an (A, m)-isometry then*

$$|w_i|^2 = \frac{R^{(m)}_{i+1,A}}{R^{(m)}_{i,A}} \frac{\|e_i\|^2_A}{\|e_{i+1}\|^2_A}\ (>0) \quad \text{for } 0 \leq i \leq n_0. \tag{14}$$

2. *Assume that there exists $n_0 \geq 0$ such that $e_i \notin N(A),\ 0 \leq i \leq n_0+m$ and*

$$|w_p|^2 = \frac{R^{(m)}_{p+1,A}}{R^{(m)}_{p,A}} \frac{\|e_p\|^2_A}{\|e_{p+1}\|^2_A}\ (>0) \quad \text{for } 0 \leq p \leq n_0+m-1. \tag{15}$$

Then $H^{(m)}_{j,A} = 0$ for all $1 \leq j \leq n_0$.

Proof 1. Assume that T is an (A, m)-isometry. From Proposition 2.1 we obtain $w_i \neq 0$ for any $0 \leq i \leq n_0+1$. By Theorem 1.1 we have that, for every $u \in \mathbb{H}$ and for all $n \geq 0$,

$$\|T^n u\|^2_A = \sum_{k=0}^{m-1} (-1)^{m-k-1} \frac{n(n-1)\ldots\overbrace{(n-k)}\ldots(n-m+1)}{k!(m-k-1)!} \|T^k u\|^2_A.$$

For $u = e_0$ and for all $n \geq 1$, we obtain

$$\begin{aligned}\left(\prod_{i=0}^{n-1}|w_i|^2\right)\|e_n\|^2_A &= (-1)^{m-1}\frac{(n-1)(n-2)\ldots(n-m+1)}{(m-1)!}\ \|e_0\|^2_A \\ &\quad + \sum_{k=1}^{m-1}(-1)^{m-k-1}\frac{n(n-1)\ldots\overbrace{(n-k)}\ldots(n-m+1)}{k!(m-k-1)!} \\ &\quad \times\left(\prod_{i=0}^{k-1}|w_i|^2\right)\|e_k\|^2_A.\end{aligned}$$

The equalities (12)–(13) give $R^{(m)}_{k,A} \neq 0$ for all $0 \leq k \leq m-1$. Moreover, we have

$$R^{(m)}_{j,A}\|e_{j+1}\|^2_A|w_j|^2 = R^{(m)}_{j+1,A}\|e_j\|^2_A \quad \text{for all } 0 \leq j \leq n_0.$$

Consequently,

$$|w_j|^2 = \frac{R_{j+1,A}^{(m)}}{R_{j,A}^{(m)}} \frac{\|e_j\|_A^2}{\|e_{j+1}\|_A^2} \ (> 0) \qquad (0 \le j \le n_0).$$

2. Assume that (15) is verified. First note that

$$R_{j+k,A}^{(m)} = \left(\prod_{i=0}^{j+k-1} |w_i|^2 \right) \|e_{j+k}\|_A^2 \neq 0 \quad (j \ge 1,\ k \ge 0).$$

For $0 \le j \le n_0$, we have

$$\begin{aligned} H_{j,A}^{(m)} &= \left(\prod_{i=0}^{j} |w_i|^2 \right) S_{j,A}^{(m)} \\ &= \left(\prod_{i=0}^{j} |w_i|^2 \right) \left[(-1)^m \|e_j\|_A^2 + \sum_{k=1}^{m} (-1)^{m-k} \binom{m}{k} \left(\prod_{i=0}^{k-1} |w_{i+j}|^2 \right) \|e_{j+k}\|_A^2 \right] \\ &= (-1)^m \left(\prod_{i=0}^{j} |w_i|^2 \right) \|e_j\|_A^2 + |w_j|^2 \sum_{k=1}^{m} (-1)^{m-k} \binom{m}{k} \left(\prod_{i=0}^{j+k-1} |w_i|^2 \right) \\ &\quad \times \|e_{j+k}\|_A^2. \end{aligned}$$

Taking into account (13) and for $1 \le j \le n_0$, we obtain

$$\begin{aligned} H_{j,A}^{(m)} &= |w_j|^2 \Big[\sum_{k=0}^{m} (-1)^{m-k} \binom{m}{k} R_{j+k,A}^{(m)} \Big] \\ &= |w_j|^2 \Big\{ \sum_{k=0}^{m} (-1)^{m-k} \binom{m}{k} (-1)^{m-1} \\ &\quad \times \frac{(j+k-1)(j+k-2)\ldots(j+k-m+1)}{(m-1)!} \|e_0\|_A^2 \\ &\quad + \sum_{k=0}^{m} (-1)^{m-k} \binom{m}{k} \\ &\quad \times \Big[\sum_{h=1}^{m-1} (-1)^{m-h-1} \frac{(j+k)\ldots\overbrace{(j+k-h)}\ldots(j+k-m+1)}{h!(m-h-1)!} \\ &\quad \times \prod_{i=0}^{h-1} |w_i|^2 \|e_h\|_A^2 \Big] \Big\} \\ &= |w_j|^2 (A+B), \end{aligned}$$

where,

$$
\begin{aligned}
A &= \sum_{k=0}^{m}(-1)^{m-k}\binom{m}{k}(-1)^{m-1} \\
&\quad \times \frac{(j+k-1)(j+k-2)\dots(j+k-m+1)}{(m-1)!}\|e_0\|_A^2 \\
&= (-1)^{m-1}m\|e_0\|_A^2 \\
&\quad \times \sum_{k=0}^{m}(-1)^{m-k}\frac{(j+k-1)(j+k-2)\dots(j+k-m+1)\overbrace{(j+k-m)}}{k!(m-k)!},\\
B &= \sum_{k=0}^{m}(-1)^{m-k}\binom{m}{k} \\
&\quad \times \Big[\sum_{h=1}^{m-1}(-1)^{m-h-1}\frac{(j+k)\dots\overbrace{(j+k-h)}\dots(j+k-m+1)}{h!(m-h-1)!} \\
&\quad \times \prod_{i=0}^{h-1}|w_i|^2\|e_h\|_A^2\Big] \\
&= \sum_{h=1}^{m-1}(-1)^{m-h-1}\frac{m!}{h!(m-h-1)!}\prod_{i=0}^{h-1}|w_i|^2\|e_h\|_A^2 \\
&\quad \times \Big[\sum_{k=0}^{m}(-1)^{m-k}\frac{(j+k)\dots\overbrace{(j+k-h)}\dots(j+k-m+1)}{k!(m-k)!}\Big].
\end{aligned}
$$

Taking into account equality (3.2) [4], Lemma 3.3 we obtain $A=0$ and $B=0$. Hence, $H_{j,A}^{(m)}=0$ for $1\leq j\leq n_0$. □

Remark 2.4 Let T be the unilateral weighted forward shift with weight sequence $\{w_n\}_{n\geq 0}$. From Theorem 2.4 we obtain the following characterizations:

1. Assume that $(e_n)_{n\geq 0}$ is A-orthogonal, $e_n\notin N(A)$ for all $n\geq 0$ and $S_{0,A}^{(m)}=0$. Then, T is an (A,m)-isometry if and only if

$$
R_{n,A}^{(m)}=\left(\prod_{i=0}^{n-1}|w_i|^2\right)\|e_n\|_A^2>0 \quad \text{for every } n\geq 1.
$$

   If $A=I$, then we obtain a conclusion similar to that of Remark 3.5 ([4]).
2. Assume that there exists $0\leq n_0\leq m-2$ such that $e_j\notin N(A),\ 0\leq j\leq n_0+1$. We have:
   (a) If T is an A-isometry, then

$$
\|e_i\|_A=|w_i|\,\|e_{i+1}\|_A \quad \text{for every } i\geq 0.
$$

(b) If T is an $(A, 2)$-isometry, then

$$|w_i|^2 = \frac{(i+1)|w_0|^2 - i\frac{\|e_0\|_A^2}{\|e_1\|_A^2}}{i|w_0|^2 - (i-1)\frac{\|e_0\|_A^2}{\|e_1\|_A^2}} \ \frac{\|e_i\|_A^2}{\|e_{i+1}\|_A^2} \quad (0 \le i \le n_0)$$

$$= \frac{i\Big(|w_0|^2 - \frac{\|e_0\|_A^2}{\|e_1\|_A^2}\Big) + |w_0|^2}{(i-1)\Big(|w_0|^2 - \frac{\|e_0\|_A^2}{\|e_1\|_A^2}\Big) + |w_0|^2} \ \frac{\|e_i\|_A^2}{\|e_{i+1}\|_A^2}.$$

Moreover, observe that

$$|w_0| \ \ge \ \frac{\|e_0\|_A}{\|e_1\|_A} \Longleftrightarrow |w_i| > 0 \quad (0 \le i \le n_0).$$

In particular, if $|w_0| > \frac{\|e_0\|_A}{\|e_1\|_A}$, then T is a strict $(A, 2)$-isometry.

2.2 Unilateral Weighted Backward Shifts

A unilateral weighted backward shift B_w with weight sequence $\{w_n\}_{n\ge 1}$ is defined by $B_w e_n = w_n e_{n-1}$ if $n \ge 1$ and by $B_w e_o = 0$. The iterates of B_w are given by

$$B_w^k e_n = \begin{cases} \Big(\prod_{i=0}^{k-1} w_{n-i}\Big) e_{n-k} & \text{if } 1 \le k \le n; \\ 0 & \text{otherwise.} \end{cases} \tag{16}$$

Let $u = \sum_{n=0}^{\infty} \alpha_n e_n \in \mathbb{H}$. We have

$$\begin{aligned} &\sum_{k=0}^{m} (-1)^{m-k} \binom{m}{k} \|B_w^k u\|_A^2 \\ &= \sum_{n=0}^{\infty} |\alpha_n|^2 \Big\{(-1)^m \ \|e_n\|_A^2 + \sum_{k=1}^{m} (-1)^{m-k} \binom{m}{k} \|B_w^k e_n\|_A^2\Big\} \\ &\quad + \sum_{i\ne j} \alpha_i \overline{\alpha}_j \Big\{(-1)^m \langle A e_i \mid e_j\rangle + \sum_{k=1}^{m} (-1)^{m-k} \binom{m}{k} \langle A B_w^k e_i \mid B_w^k e_j\rangle\Big\}, \end{aligned}$$

where

$$\langle A B_w^k e_i \mid B_w^k e_j\rangle = \begin{cases} \left(\prod_{p=0}^{k-1} w_{i-p}\overline{w}_{j-p}\right) \langle A e_{i-k} \mid e_{j-k}\rangle & \text{if } 1 \le k \le i < j; \\ 0 & \text{if } k > \min(i, j). \end{cases} \tag{17}$$

Unilateral weighted backward shifts cannot be m-isometric for any positive integer number m, since B_w does not satisfies Eq. (2) for the vector e_0. We prove in the following result that they cannot also be (A, m)-isomeric.

Theorem 2.5 *A unilateral weighted backward shift can not be an (A, m)-isometry for any positive integer m.*

Proof Let B_w be a unilateral weighted backward shift with weight sequence $\{w_n\}_{n\geq 1}$. Assume that B_w is an (A, m)-isometry. Then $e_n \in N(A)$ for all $n \geq 0$. Indeed, we use an induction argument to prove such a claim. Since B_w is an (A, m)-isometry, Eq. (2) gives

$$(-1)^m \, \|e_n\|_A^2 + \sum_{k=1}^{m}(-1)^{m-k}\binom{m}{k}\|B_w^k e_n\|_A^2 = 0 \quad \text{for all } n \geq 0. \tag{18}$$

Since $B_w^k e_0 = 0$ for all $k \geq 1$, we obtain

$$0 = \sum_{k=1}^{m}(-1)^{m-k}\binom{m}{k}\|B_w^k e_0\|_A^2 = (-1)^{m+1}\,\|e_0\|_A^2, \tag{19}$$

which implies that $e_0 \in N(A)$ and, hence, the property is satisfied for $n = 0$. Assume that for $p = 0, 1, 2, \ldots, n-1$ $(n \geq 1)$, $e_p \in N(A)$ and let us show the property for the step n. Taking (17) and (18) into account, we deduce

$$\begin{aligned}
&(-1)^m \, \|e_n\|_A^2 + \sum_{k=1}^{n}(-1)^{m-k}\binom{m}{k}\left(\prod_{i=0}^{k-1}|w_{n-i}|^2\right)\|e_{n-k}\|_A^2 = 0 \quad \text{if } 1 \leq n \leq m,\\
&(-1)^m \, \|e_n\|_A^2 + \sum_{k=1}^{m}(-1)^{m-k}\binom{m}{k}\left(\prod_{i=0}^{k-1}|w_{n-i}|^2\right)\|e_{n-k}\|_A^2 = 0 \quad \text{if } n \geq m+1,
\end{aligned}$$

and the claim follows from that. On the other hand, every $u \in \mathbb{H}$ can be written as $u = \sum_{i=0}^{\infty}\alpha_i e_i$. Hence,

$$\langle Au \mid u\rangle = \sum_{i=0}^{\infty}|\alpha_i|^2\|e_i\|_A^2 + \sum_{0\leq i\neq j}\alpha_i\overline{\alpha_j}\langle Ae_i \mid e_j\rangle = 0.$$

According to Proposition 2.15, [5], we obtain $A = 0$ which is impossible. □

3 Strict (A, m)-Isometric Unilateral Weighted Forward Shifts

Generally, an (A, m)-isometry is not an $(A, m-1)$-isometry. Sid Ahmed et al. (Theorem 2.1 [8]) proved that if T is an (A, m)-isometry satisfying $N(\triangle_{A,T})$ is invariant for A, then $T|_{N(\triangle_{A,T})}$ is an $(A|_{N(\triangle_{A,T})}, m-1)$-isometry. Moreover we can see that if

T is an invertible (A, m)-isometry and m is even, then it is an $(A, m-1)$-isometry. In the following we describe when a unilateral weighted shift operator is a strict (A, m)-isometry in terms of the weights sequence. To be precise we define

$$Z_n := |w_n|^2 S^{(m-1)}_{n+1,A} = \sum_{k=0}^{m-1} (-1)^{m-1-k} \binom{m-1}{k} \left(\prod_{i=0}^{k} |w_{n+i}|^2 \right) \|e_{n+k+1}\|^2_A, \text{ for all } n \geq 0.$$

Theorem 3.1 *Let T be an (A, m)-isometric unilateral weighted forward shift with weight sequence $\{w_n\}_{n\geq 0}$. If there exists $n_0 \in \mathbb{N}$ such that $Z_{n_0} \neq 0$ then $w_{n_0} \neq 0$ and T is a strict (A, m)-isometry.*

Proof Two proofs for this theorem will be given.
First proof Since there exists $n_0 \in \mathbb{N}$ such that $Z_{n_0} \neq 0$, we have $w_{n_0} \neq 0$ and $S^{(m-1)}_{n_0+1,A} \neq 0$. Hence, Theorem 2.2 allows to conclude.
Second proof Let us prove the converse. Assume that T is an (A, m)-isometry and an $(A, m-1)$-isometry. Then, for $u = e_n$, the identity (2) gives

$$0 = (-1)^m \|e_n\|^2_A + \sum_{k=1}^{m} (-1)^{m-k} \binom{m}{k} \left(\prod_{i=0}^{k-1} |w_{n+i}|^2 \right) \|e_{n+k}\|^2_A, \tag{20}$$

$$0 = (-1)^m \|e_n\|^2_A + \sum_{k=1}^{m-1} (-1)^{m-k} \binom{m-1}{k} \left(\prod_{i=0}^{k-1} |w_{n+i}|^2 \right) \|e_{n+k}\|^2_A. \tag{21}$$

Remarking that $\binom{m}{k} - \binom{m-1}{k} = \binom{m-1}{k-1}$, the identities (20) and (21) give $Z_n = 0$ for all $n \geq 0$. As a consequence, if there exists $n_0 \in \mathbb{N}$ such that $Z_{n_0} \neq 0$ then either T is an (A, m)-isometry and not an $(A, m-1)$-isometry or T is not an (A, m)-isometry and it is an $(A, m-1)$-isometry. The second conclusion is impossible since it is well known that every $(A, m-1)$-isometry is an (A, m)-isometry. Hence T is a strict (A, m)-isometry. □

We have the following result.

Theorem 3.2 *Let T be a unilateral weighted forward shift with weight sequence $\{w_n\}_{n\geq 0}$ ($w_n \neq 0$ for all $n \in \mathbb{N}$). Assume that $(e_n)_{n\geq 0}$ is A-orthogonal. If T is a strict (A, m)-isometry then for every nonnegative integer n, $S^{(m)}_{n,A} = 0$ and $S^{(m-1)}_{n,A} \neq 0$.*

Proof Note that if $(e_n)_{n\geq 0}$ is A-orthogonal, then $S^{(m-1)}_{i,j,A} = 0$ for all $i \neq j \geq 0$. Suppose that T is a strict (A, m)-isometry. Then, Eq. (2) gives

$$S^{(m)}_{n,A} = (-1)^m \|e_n\|^2_A + \sum_{k=1}^{m} (-1)^{m-k} \binom{m}{k} \left(\prod_{i=0}^{k-1} |w_{n+i}|^2 \right) \|e_{n+k}\|^2_A = 0 \quad \text{for all } n \geq 0.$$

Let us prove now that $S^{(m-1)}_{n,A} \neq 0$ for all $n \geq 0$. Assume the contrary, that n_0 is the smallest non-negative integer such that $S^{(m-1)}_{n_0,A} = 0$. Our aim is to prove that $n_0 = 0$.

If $n_0 \geq 1$ we obtain $S_{n_0-1,A}^{(m-1)} \neq 0$, which is impossible from (10). Furthermore, Proposition 2.3 yields T is an $(A, m-1)$-isometry. Hence we obtain the desired result. □

As an immediate corollary to Theorem 3.2, we characterize unilateral weighted forward shifts that are strictly (A, m)-isometric.

Corollary 3.3 *Let T be a unilateral weighted forward shift with weight sequence $\{w_n\}_{n\geq 0}$. Assume that $(e_n)_{n\geq 0}$ is A-orthogonal. Then, the following assertions are equivalent:*

1. *T is a strict (A, m)-isometry.*
2. *For every $n \in \mathbb{N}$, we have*
   (a) $S_{n,A}^{(m)} = 0$.
   (b) $S_{n,A}^{(m-1)} \neq 0$.

Proof Suppose that T is a strict (A, m)-isometry. Then Theorem 3.2 shows that 2. holds. Suppose, now, that (a) and (b) hold true. Since $(e_n)_{n\geq 0}$ is an orthonormal basis, then every $u \in \mathbb{H}$ can be written as $u = \sum_{i=0}^{\infty} \alpha_i e_i$. According to (a), we have

$$\sum_{k=0}^{m}(-1)^{m-k}\binom{m}{k}\|T^k u\|_A^2 = \sum_{i=0}^{\infty}|\alpha_i|^2 S_{i,A}^{(m)} = 0.$$

This implies that (2) holds for every $u \in \mathbb{H}$ and hence T is an (A, m)-isometry. On the other hand, for all $n \geq 0$,

$$\sum_{k=0}^{m-1}(-1)^{m-k}\binom{m-1}{k}\|T^k e_n\|_A^2 = S_{n,A}^{(m-1)} \neq 0.$$

Thus, T is not an $(A, m-1)$-isometry. □

We now apply Corollary 3.3 to investigate the following example.

Example 3.1 Let $\mathbb{H}$ be a separable Hilbert space with an orthonormal basis $\{e_n\}_{n\geq 0}$. Let $T, A \in \mathcal{L}(\mathbb{H})$ where T is the unilateral weighted shift defined by

$$Te_n = \sqrt{\frac{n+3}{n+1}}\, e_{n+1}, \quad n \geq 0$$

and A is the positive operator given by $Ae_n = \frac{n+1}{n+2}\, e_n$, $n \geq 0$. It is not difficult to verify that

$$S_{n,A}^{(3)} = 0, \qquad S_{n,A}^{(2)} = \frac{2}{(n+1)(n+2)}, \quad n \geq 0$$

and hence T is an $(A, 3)$-isometry which is not an $(A, 2)$-isometry.

In general, (A, m)-isometries are not A-isometries. Theorem 2.3 [9] gives a first characterization of (A, m)-isometric operators which are A-isometries. In the same way, we will describe in the following result the related characterization for a family of unilateral weighted shifts.

Proposition 3.1 *Let T be a unilateral weighted forward shift with weight sequence $\{w_n\}_{n\geq 0}$ ($w_n \neq 0$ for all $n \in \mathbb{N}$). Assume that $(e_n)_{n\geq 0}$ is A-orthogonal. If T is an (A, m)-isometry and if, for some nonzero $u \in \mathbb{H}$, we have*

$$\|u\|_A = \|T^k u\|_A, \quad 1 \leq k \leq m-1, \tag{22}$$

then T is an A-isometry.

Proof Suppose that $\{e_n\}_{n\geq 0}$ is an orthonormal basis for $\mathbb{H}$ and $Te_n = w_n e_{n+1}$ for all $n \geq 0$. Put $u = \sum_{n=0}^{\infty} \alpha_n e_n$. Then, (22) gives

$$\begin{aligned}
\sum_{k=0}^{m-1} (-1)^{m-k-1} \binom{m-1}{k} \|T^k u\|_A^2 &= \sum_{k=0}^{m-1} (-1)^{m-k-1} \binom{m-1}{k} \|u\|_A^2 \\
&= \|u\|_A^2 \left(\sum_{k=0}^{m-1} \binom{m-1}{k} (-1)^{m-k-1} \right) \\
&= \|u\|_A^2 \Big(1 + (-1)\Big)^{m-1} \\
&= 0.
\end{aligned}$$

Moreover, since $(e_n)_{n\geq 0}$ is A-orthogonal, (4) implies that

$$\begin{aligned}
\sum_{k=0}^{m-1} (-1)^{m-k-1} \binom{m-1}{k} \|T^k u\|_A^2 &= \sum_{n=0}^{\infty} |\alpha_n|^2 \Bigg[(-1)^{m-1} \|e_n\|_A^2 \\
&\quad + \sum_{k=1}^{m-1} (-1)^{m-1-k} \binom{m-1}{k} \\
&\quad \times \left(\prod_{i=0}^{k-1} |w_{n+i}|^2 \right) \|e_{n+k}\|_A^2 \Bigg] \\
&= \sum_{n=0}^{\infty} |\alpha_n|^2 S_{n,A}^{(m-1)} \\
&= \sum_{n=0}^{\infty} |\alpha_n|^2 \Big((m-1)! \langle \Delta_{A,T} e_n \mid e_n \rangle \Big).
\end{aligned}$$

According to Theorem 2.1 [8], $\Delta_{A,T}$ is a positive operator (i.e $\langle \Delta_{A,T} e_n \mid e_n \rangle \geq 0$, for all $n \geq 0$). Since u is non-zero, $\alpha_{n_0} \neq 0$ for some n_0, then

$$S_{n_0,A}^{(m-1)} = (m-1)!\langle \Delta_{A,T} e_{n_0} \mid e_{n_0} \rangle = 0.$$

Thus, condition (b) in Corollary 3.3 does not occur and so T must be an $(A, m-1)$-isometry. Now, applying an argument similar to the above and using Corollary 3.3, $(m-1)$ times, we conclude that T must be an A-isometry. □

Remark 3.1 The conclusion of Proposition 3.1 is not valid, in general, for any operator T and any operator A. Indeed, let $\mathbb{K} = \mathbb{C}^2$ be equipped with the norm $\|(x, y)\|^2 = |x|^2 + |y|^2$ and consider the operators

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}, \qquad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

Note that $A \in \mathcal{L}(\mathbb{H})^+$ and $T \in \mathcal{L}(\mathbb{H})$. Moreover, by direct computation, we see that

$$\begin{aligned} \|(x, y)\|_A^2 &= x^2 + 2xy + 2y^2, \\ \|T(x, y)\|_A^2 &= x^2 + 4xy + 5y^2, \\ \|T^2(x, y)\|_A^2 &= x^2 + 6xy + 10y^2, \\ \|T^3(x, y)\|_A^2 &= x^2 + 8xy + 17y^2. \end{aligned}$$

Consequently,

$$\begin{aligned} &\|T^3(x, y)\|_A^2 - 3\|T^2(x, y)\|_A^2 + 3\|T(x, y)\|_A^2 - \|(x, y)\|_A^2 = 0, \\ &\|T^2(x, y)\|_A^2 - 2\|T(x, y)\|_A^2 + \|(x, y)\|_A^2 = 2y^2 \neq 0 \\ \text{and} \quad &\|T(x, y)\|_A^2 - \|(x, y)\|_A^2 = 2xy + 3y^2 \neq 0. \end{aligned}$$

Thus, T is an $(A, 3)$-isometry but is nor an $(A, 2)$-isometry neither an A-isometry. Furthermore, if $u = (1, 0)$ then $\|u\|_A = \|Tu\|_A = \|T^2u\|_A = 1$.

4 $(A, 2)$-Expansive Weighted Shift Operators

An operator $T \in \mathcal{L}(\mathbb{H})$ is said to be $(A, 2)$-expansive (also A-concave), if it satisfies the inequality

$$\Theta_A^{(2)}(T) = T^{*2}AT^2 - 2T^*AT + A \leq 0$$

or, equivalently,

$$\|T^2u\|_A^2 - 2\|Tu\|_A^2 + \|u\|_A^2 \leq 0 \quad \text{for all } u \in \mathbb{H}. \tag{23}$$

As a matter of fact, $(A, 2)$-isometries and A-isometries are $(A, 2)$-expansive operators. In addition, it was shown in [11], Proposition 3.9 that all powers of an $(A, 2)$-expansive operator is also $(A, 2)$-expansive.

In this section we give some properties of $(A, 2)$-expansive operators that generalizes those described in [7].

The A-covariance operator for an $(A, 2)$-expansive operator is given by $\Delta_{A,T} = T^*AT - A$. We begin with the following preliminary result.

Lemma 4.1 *Let* $T \in \mathcal{L}(\mathbb{H})$. *Then,*
T *is* $(A, 2)$*-expansive* *if and only if* $T^*\Delta_{A,T}T \le \Delta_{A,T}$.

Proof Let $u \in \mathbb{H}$. We have

$$\langle (T^*\Delta_{A,T}T - \Delta_{A,T})u, u\rangle = \|T^2u\|_A^2 - 2\|Tu\|_A^2 + \|u\|_A^2$$

thus the claim follows from that. □

Jung et al. proved in [11], Theorem 3.10 that the A-covariance operator for an $(A, 2)$-expansive operator is positive. In the following theorem we show the same claim using different approach.

Theorem 4.1 *Let* $T \in \mathcal{L}(\mathbb{H})$. *If* T *is* $(A, 2)$*-expansive, then:*

1. $\Delta_{A,T}$ *is a positive operator.*
2. *If* A *is injective then* T *is also injective.*
3. *If* T *is invertible, then* T^{-1} *is* $(A, 2)$*-expansive.*

Proof 1. Let $u \in \mathbb{H}$. We have

$$\langle \Delta_{A,T}u, u\rangle = \langle (T^*AT - A)u, u\rangle = \|Tu\|_A^2 - \|u\|_A^2.$$

To obtain the claim let us suppose on the contrary that $\|Tu_0\|_A < \|u_0\|_A$ for some $u_0 \in \mathbb{H}$. By using on induction argument we obtain

$$\|T^nu_0\|_A^2 < \|T^{n-1}u_0\|_A^2 < \cdots < \|u_0\|_A^2$$

for each positive integer number n. We deduce that the sequence $\{\|T^nu_0\|_A^2\}_{n\ge 0}$ is strictly decreasing, bounded and hence it is convergent. Moreover we have

$$0 = \lim_{n \longrightarrow +\infty} (\|T^{n+1}u_0\|_A^2 - \|T^nu_0\|_A^2) < 0$$

which is a contradiction. Thus $\langle \Delta_{A,T}u, u\rangle \ge 0$ for all $u \in \mathbb{H}$.

2. If $T \in \mathcal{L}(\mathbb{H})$ is $(A, 2)$ -expansive and $u \in N(T)$ thus $Tu = 0$. Moreover, $\|Tu\|_A^2 = \|T^2u\|_A^2 = 0$. Since T is A-concave, $\|u\|_A = 0$ which is equivalent to the fact that $u \in N(A)$.
3. By hypothesis, T is $(A, 2)$-expansive. Then, we have

$$\sum_{k=0}^{2}(-1)^k \binom{2}{k} \|T^{2-k}u\|_A^2 \le 0 \quad \text{for all } u \in \mathbb{H}. \tag{24}$$

Replacing u by $(T^{-1})^2u$ in (24), we deduce that

$$\begin{aligned}\sum_{k=0}^{2}(-1)^k\binom{2}{k}\|T^{2-k}((T^{-1})^2u)\|_A^2 &= \sum_{k=0}^{2}(-1)^k\binom{2}{2-k}\|T^{-k}u\|_A^2\\ &= \sum_{k=0}^{2}(-1)^{2-k}\binom{2}{k}\|(T^{-1})^{2-k}u\|_A^2\\ &= \sum_{k=0}^{2}(-1)^k\binom{2}{k}\|(T^{-1})^{2-k}u\|_A^2.\end{aligned}$$

□

Proposition 4.1 *Let $T\in\mathcal{L}(\mathbb{H})$ be an A-isometry and $S\in\mathcal{L}(\mathbb{H})$ with $ST=TS$, then ST is $(A,2)$-expansive if and only if S is $(A,2)$-expansive.*

Proof Since T is an A-isometry, we have

$$\|T^kS^ku\|_A=\|S^ku\|_A,\quad k=0,\ 1,\ 2.$$

On the other hand, we have

$$\begin{aligned}\sum_{k=0}^{2}(-1)^k\binom{2}{k}\|(ST)^{2-k}u\|_A^2 &= \sum_{k=0}^{2}(-1)^k\binom{2}{k}\|(TS)^{2-k}u\|_A^2\\ &= \sum_{k=0}^{2}(-1)^k\binom{2}{k}\|T^{2-k}S^{2-k}u\|_A^2\\ &= \sum_{k=0}^{2}(-1)^k\binom{2}{k}\|S^{2-k}u\|_A^2\end{aligned}$$

which allows us to conclude. □

Corollary 4.2 *Let $T\in\mathcal{L}(\mathbb{H})$ be an inversible $(A,2)$-expansive operator and $S\in\mathcal{L}(\mathbb{H})$ with $ST=TS$, then ST is $(A,2)$-expansive if and only if S is $(A,2)$-expansive.*

Proof It suffises to prove that T is an A-isometry. If T is an invertible $(A,2)$-expansive operator, then T^{-1} is also $(A,2)$-expansive. Hence, by (1.)-Theorem 4.1 $\Delta_{A,T}\geq 0$ and

$$\Delta_{A,T^{-1}}=T^{-1*}AT^{-1}-A\geq 0.$$

On the other hand,

$$\Delta_{A,T}=T^*AT-A=-T^*(T^{-1*}AT^{-1}-A)T\leq 0,$$

which implies that T is an A-isometry. Thus we complete the proof by involving Proposition 4.1. □

Now, we specify the study to unilateral weighted shifts. We give results generalizing those described in [7].

Theorem 4.3 *Let T be a unilateral weighted forward shift with weights $\{w_n\}_{n\geq 0}$. Assume that for all $n \in \mathbb{N}$, $e_n \notin N(A)$. If T is $(A, 2)$-expansive, then the following assertions holds.*

1. $S^{(2)}_{n,A} = |w_n|^2|w_{n+1}|^2\|e_{n+2}\|^2_A - 2|w_n|^2\|e_{n+1}\|^2_A + \|e_n\|^2_A \leq 0$ *for each n;*
2. $\{V_n\}_n = \left\{|w_n|\frac{\|e_{n+1}\|_A}{\|e_n\|_A}\right\}_n$ *is a decreasing sequence of real numbers converging to 1;*
3. $\frac{\|e_n\|_A}{\|e_{n+1}\|_A} \leq |w_n| < \sqrt{2}\frac{\|e_n\|_A}{\|e_{n+1}\|_A}$, *for all $n \geq 0$.*

Proof 1. Applying (23) for $u = e_n$ we obtain the claim.

2. Let $V_n = |w_n|\frac{\|e_{n+1}\|_A}{\|e_n\|_A}$. To prove the assertion let us assume the contrary that $V_n < V_{n+1}$ for some non-negative integer n. Therefore,

$$0 \leq \left(|w_n|^2\frac{\|e_{n+1}\|^2_A}{\|e_n\|_A} - \|e_n\|_A\right)^2 < S^{(2)}_{n,A} \leq 0,$$

which is a contradiction. Hence $\{V_n\}_n$ is a decreasing sequence of non-negative numbers. On the other hand, by Theorem 4.1 the operator $\Delta_{A,T}$ is positive, that is

$$\langle \Delta_{A,T}u, u\rangle = \|Tu\|^2_A - \|u\|^2_A \geq 0 \quad \text{for all } u \in \mathbb{H}. \tag{25}$$

Thus for $u = e_n$ the identity (25) gives

$$\|Te_n\|_A = |w_n|\|e_{n+1}\|_A \geq \|e_n\|_A,$$

which implies that $V_n \geq 1$ for all $n \geq 0$. Since the sequence $\{V_n\}_n$ is decreasing, it must be convergent. Let $l = \lim_{n\longrightarrow+\infty} V_n$. Our aim now is to prove that $l = 1$. Taking into account that $S^{(2)}_{n,A} \leq 0$, it is easily seen that V_n satisfies

$$V_n^2V_{n+1}^2 - 2V_n^2 + 1 \leq 0 \quad \text{for all } n \geq 0. \tag{26}$$

It holds that

$$\lim_{n\longrightarrow+\infty}(V_n^2V_{n+1}^2 - 2V_n^2 + 1) = (l^2 - 1)^2 \leq 0$$

and hence $l = 1$.

3. The identity (26) implies

$$V_{n+1}^2 - 2 = (V_{n+1} - \sqrt{2})(V_{n+1} + \sqrt{2}) \leq -\frac{1}{V_n^2} < 0,$$

so $1 \leq V_n < \sqrt{2}$, for each $n \geq 0$, which allows us to conclude. □

Theorem 4.4 *A unilateral weighted backward shift cannot be $(A, 2)$-expansive.*

Proof We argue by contradiction. Assume that B_w is $(A, 2)$-expansive. Since B_w is a unilateral weighted backward shift, $B_w e_n = w_n e_{n-1}$ for all $n \geq 1$ and $B_w e_0 = 0$. On the other hand, (23) gives

$$\|B_w^2 e_n\|_A^2 - 2\|B_w e_n\|_A^2 + \|e_n\|_A^2 \leq 0 \quad \text{for all } n \geq 0. \tag{27}$$

It is easily seen that for $i = 0, 1$, $e_i \in N(A)$. Using (27) we prove by an induction argument that $e_n \in N(A)$ for all $n \geq 2$. Hence $A = 0$ which is impossible. □

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