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Ore Extensions Over Right Strongly Hopfian Rings

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Abstract An associative ring is said to be right strongly Hopfian if the chain of right annihilators $r_R(a) \subseteq r_R(a^2) \subseteq \cdots$ stabilizes for each $a \in R$. In this article, we are interested in the class of right strongly Hopfian rings and the transfer of this property from an associative ring R to the Ore extension $R[x;\alpha,\delta]$ and the monoid ring $R[M]$. It is proved that if R is (α,δ)-compatible and $R[x;\alpha,\delta]$ is reversible, then the Ore extension $R[x;\alpha,\delta]$ is right strongly Hopfian if and only if R is right strongly Hopfian, and it is also shown that if M is a strictly totally ordered monoid and $R[M]$ is a reversible ring, then the monoid ring $R[M]$ is right strongly Hopfian if and only if R is right strongly Hopfian. Consequently, several known results regarding strongly Hopfian rings are extended to a more general setting.

Keywords Strongly Hopfian ring · Ore extension · Monoid ring

Mathematics Subject Classification 16S99

1 Introduction

Throughout this paper, all rings are associative with identity. For a nonempty subset X of a ring R, $l_R(X) = \{a \in R \mid aX = 0\}$ and $r_R(X) = \{a \in R \mid Xa = 0\}$ denote

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the left and the right annihilators of X in R, respectively. Following A. Hmaimou et al [5], a ring R is left strongly Hopfian if for every endomorphism f of R, the chain $\ker f \subseteq \ker f^2 \subseteq \cdots$ stabilizes. Equivalently, R is left strongly Hopfian if the chain of left annihilators $l_R(a) \subseteq l_R(a^2) \subseteq \cdots$ stabilizes for each $a \in R$. The class of left strongly Hopfian rings is very large. It contains Noetherian rings, Laskerian rings, rings satisfying acc on d-annihilators, those satisfying acc on d-colons, and so on [4]. If R is a commutative ring, then a left strongly Hopfian ring is also called a strongly Hopfian ring. A. Hmaimou et al [5] showed that for a commutative ring R, the ring R is strongly Hopfian if and only if the polynomial ring $R[x]$ is strongly Hopfian if and only if the Laurent polynomial ring $R[x; x^{-1}]$ is strongly Hopfian. Let R be a commutative ring. In [4], S. Hizem provided an example of a strongly Hopfian ring R such that the power series ring $R[[x]]$ is not necessarily strongly Hopfian, and also gave some necessary and sufficient conditions for $R[[x]]$ to be strongly Hopfian. For more details and properties of left strongly Hopfian rings, see [2,4,5,7,8].

Let α be an endomorphism, and δ an α-derivation of R, that is, δ is an additive map such that $\delta(ab) = \delta(a)b + \alpha(a)\delta(b)$, for $a, b \in R$. According to Annin [1], a ring R is said to be α-compatible if for each $a, b \in R, ab = 0 \Leftrightarrow a\alpha(b) = 0$. Clearly, this may only happen when the endomorphism α is injective. Moreover, R is said to be δ-compatible if for each $a, b \in R, ab = 0 \Rightarrow a\delta(b) = 0$. A ring R is (α, δ)-compatible if it is both α-compatible and δ-compatible. Recall that a ring R is reversible if $ab = 0 \Rightarrow ba = 0$ for all $a, b \in R$, and a ring R is semicommutative if $ab = 0$ implies $aRb = 0$ for any $a, b \in R$. Clearly, any subring of a reversible ring is also reversible, and if R is a reversible ring, then for any $n \in \mathbb{N}$ and any permutation $\sigma \in S_n$, $x_1x_2 \cdots x_n = 0$ implies $x_{\sigma(1)}Rx_{\sigma(2)}R \cdots x_{\sigma(n)}R = 0$ for any $x_i \in R$, $1 \leq i \leq n$. Reversible rings are semicommutative, but the reverse is not true in general [6, Example 1.5]. Let $f(x) = a_0 + a_1x + \cdots + a_nx^n \in R[x; \alpha, \delta]$, $\{a_0, a_1, \ldots, a_n\}$ denote the subset of R that comprised the coefficients of $f(x)$.

In this article, we are interested in the class of right strongly Hopfian rings and the transfer of this property from an associative ring R to the Ore extension $R[x; \alpha, \delta]$ and the monoid ring $R[M]$. We first provide some examples of right strongly Hopfian rings. We next show that (1) if R is (α, δ)-compatible and $R[x; \alpha, \delta]$ is reversible, then the Ore extension $R[x; \alpha, \delta]$ is right strongly Hopfian if and only if R is right strongly Hopfian; (2) if M is a strictly totally ordered monoid and $R[M]$ a reversible ring, then the monoid ring $R[M]$ is right strongly Hopfian if and only if R is right strongly Hopfian.

2 Extensions of Right Strongly Hopfian Rings

Definition 2.1 A ring R is right strongly Hopfian if the chain of right annihilators $r_R(a) \subseteq r_R(a^2) \subseteq \cdots$ stabilizes for each $a \in R$.

The next Lemma is known and very useful. We leave the proof for the reader.

Lemma 2.2 *Let $a \in R$. Then the chain $r_R(a) \subseteq r_R(a^2) \subseteq \cdots$ stabilizes if and only if there exists $n > m$ such that $r_R(a^n) = r_R(a^m)$.*

Lemma 2.3 *Let $A \subset B$ be an extension of rings. If B is right strongly Hopfian, then so is A.*

Proof Let $a \in A$. Then $r_A(a) = r_B(a) \cap A$.

Proposition 2.4 *Let $T_n(R)$ denote the $n \times n$ upper triangular matrix ring over a ring R. Then the following conditions are equivalent:*

(1) R is right strongly Hopfian;
(2) $T_n(R)$ is right strongly Hopfian.

Proof (1) $\Rightarrow$ (2). Suppose R is right strongly Hopfian and let

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & a_{nn} \end{pmatrix} \in T_n(R).$$

We proceed by induction on n to show that $T_n(R)$ is right strongly Hopfian. Let $n = 2$. Put $\alpha = \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \in T_2(R)$. Since R is right strongly Hopfian, there exists $m \in \mathbb{N}$ such that for any $n > m$, $r_R(a^n) = r_R(a^m)$ and $r_R(c^n) = r_R(c^m)$. Now we show that $r_{T_2(R)}(\alpha^{2m+1}) = r_{T_2(R)}(\alpha^{2m})$. If $\beta = \begin{pmatrix} x & y \\ 0 & z \end{pmatrix} \in r_{T_2(R)}(\alpha^{2m+1})$, then

$$\begin{aligned} &\alpha^{2m+1}\beta \\ &\quad = \begin{pmatrix} a^{2m+1} & a^{2m}b + a^{2m-1}bc + \cdots + a^m bc^m + \cdots + bc^{2m} \\ 0 & c^{2m+1} \end{pmatrix} \begin{pmatrix} x & y \\ 0 & z \end{pmatrix} \\ &\quad = \begin{pmatrix} a^{2m+1}x & a^{2m+1}y + (a^{2m}b + a^{2m-1}bc + \cdots + a^m bc^m + \cdots + bc^{2m})z \\ 0 & c^{2m+1}z \end{pmatrix} = 0. \end{aligned}$$

Thus $x \in r_R(a^{2m+1}) = r_R(a^{2m})$ and $z \in r_R(c^{2m+1}) = r_R(c^{2m}) = \cdots = r_R(c^m)$. Hence the equation

$$a^{2m+1}y + (a^{2m}b + a^{2m-1}bc + \cdots + a^m bc^m + \cdots + bc^{2m})z = 0$$

becomes

$$\begin{aligned} &a^{2m+1}y + (a^{2m}b + a^{2m-1}bc + \cdots + a^{m+1}bc^{m-1})z \\ &\quad = a^{m+1}(a^m y + (a^{m-1}b + a^{m-2}bc + \cdots + bc^{m-1})z) = 0. \end{aligned}$$

Then

$$a^m y + (a^{m-1}b + a^{m-2}bc + \cdots + bc^{m-1}) \quad z \in r_R(a^{m+1}),$$

and so

$$a^m y + (a^{m-1}b + a^{m-2}bc + \cdots + bc^{m-1})z \in r_R(a^m).$$

Hence

$$\begin{aligned} &a^m(a^m y + (a^{m-1}b + a^{m-2}bc + \cdots + bc^{m-1})z) \\ &\quad = a^{2m} y + (a^{2m-1}b + a^{2m-2}bc + \cdots + a^m bc^{m-1})z = 0. \end{aligned}$$

Then

$$\begin{aligned} &a^{2m}\beta \\ &\quad = \begin{pmatrix} a^{2m} & a^{2m-1}b + a^{2m-2}bc + \cdots + abc^{2m-2} + bc^{2m-1} \\ 0 & c^{2m} \end{pmatrix} \begin{pmatrix} x & y \\ 0 & z \end{pmatrix} \\ &\quad = \begin{pmatrix} a^{2m}x & a^{2m}y + (a^{2m-1}b + a^{2m-2}bc + \cdots + a^m bc^{m-1} + \cdots + bc^{2m-1})z \\ 0 & c^{2m}z \end{pmatrix} \\ &\quad = \begin{pmatrix} 0 & a^{2m}y + (a^{2m-1}b + a^{2m-2}bc + \cdots + a^m bc^{m-1})z \\ 0 & 0 \end{pmatrix} = 0. \end{aligned}$$

Hence $r_{T_2(R)}(\alpha^{2m+1}) \subseteq r_{T_2(R)}(\alpha^{2m})$ and so $r_{T_2(R)}(\alpha^{2m+1}) = r_{T_2(R)}(\alpha^{2m})$. Therefore $T_2(R)$ is right strongly Hopfian.

Next, we assume that the result is true for $n-1$, $n > 2$, and let

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & a_{nn} \end{pmatrix} \in T_n(R).$$

We show that $r_{T_n(R)}(A) \subseteq r_{T_n(R)}(A^2) \subseteq \cdots$ stabilizes. Put

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & a_{nn} \end{pmatrix} = \begin{pmatrix} A_{n-1} & B \\ 0 & a_{nn} \end{pmatrix}.$$

By the induction hypothesis, we can find $m \in \mathbb{N}$ such that for any $s > m$, $r_{T_{n-1}(R)}(A^s_{n-1}) = r_{T_{n-1}(R)}(A^m_{n-1})$ and $r_R(a^s_{nn}) = r_R(a^m_{nn})$. Then using the same way as above, we can show that $r_{T_n(R)}(A^{2m+1}) = r_{T_n(R)}(A^{2m})$ and so $T_n(R)$ is right strongly Hopfian by induction.

(2) ⇒ (1) This follows easily from Lemma 2.3.

Corollary 2.5 *Let $L_n(R)$ denote the lower triangular matrix ring over R. Then the following conditions are equivalent:*

(1) R is right strongly Hopfian;

(2) $L_n(R)$ is right strongly Hopfian.

Let

$$S_n(R) = \left\{ \left. \begin{pmatrix} a & a_{12} & \cdots & a_{1n} \\ 0 & a & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & a \end{pmatrix} \right| a, a_{ij} \in R \right\},$$

$$G_n(R) = \left\{ \left. \begin{pmatrix} a_1 & a_2 & \cdots & a_n \\ 0 & a_1 & \cdots & a_{n-1} \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & a_1 \end{pmatrix} \right| a_i \in R, 1 \le i \le n \right\},$$

and let $R \bowtie R$ denote the trivial extension of R by R.

Corollary 2.6 *The following conditions are equivalent:*

(1) R is right strongly Hopfian;
(2) $S_n(R)$ is right strongly Hopfian;
(3) $G_n(R)$ is right strongly Hopfian;
(4) $R[x]/(x^n)$ is right strongly Hopfian;
(5) $R \bowtie R$ is right strongly Hopfian.

Proof Note that $R[x]/(x^n) \cong G_n(R)$ and $R \bowtie R \cong G_2(R)$.

Let R be a ring. Immediately, we deduce that the lower triangular matrix ring over R is right strongly Hopfian if and only if the upper triangular matrix ring is right strongly Hopfian. Let R be a ring, and let $W(R) = \left\{ \left. \begin{pmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{pmatrix} \right| a_{ij} \in R \right\}$. Then $W(R)$ is a 3×3 subring of $M_3(R)$ under usual matrix addition and multiplication. A natural problem asks if the right strongly Hopfian property of such a ring coincides with that of R. This inspires us to consider the right strongly Hopfian property of $W(R)$.

Proposition 2.7 *Let R be a ring. Then $W(R)$ is right strongly Hopfian if and only if R is right strongly Hopfian.*

Proof Suppose R is right strongly Hopfian and let

$$\alpha = \begin{pmatrix} a & 0 & 0 \\ x & b & y \\ 0 & 0 & c \end{pmatrix} \in W(R).$$

Then there exists $m \in \mathbb{N}$ such that for any $n > m, r_R(a^n) = r_R(a^m), r_R(b^n) = r_R(b^m)$, and $r_R(c^n) = r_R(c^m)$. Now we show that $r_{W(R)}(\alpha^{2m+1}) = r_{W(R)}(\alpha^{2m})$. If

$$\beta = \begin{pmatrix} d & 0 & 0 \\ s & e & t \\ 0 & 0 & f \end{pmatrix} \in r_{W(R)}(\alpha^{2m+1}),$$

then

$$\alpha^{2m+1}\beta = \begin{pmatrix} a^{2m+1}d & 0 & 0 \\ ud + b^{2m+1}s & b^{2m+1}e & b^{2m+1}t + vf \\ 0 & 0 & c^{2m+1}f \end{pmatrix} = 0,$$

where

$$u = xa^{2m} + bxa^{2m-1} + \cdots + b^m xa^m + b^{m+1}xa^{m-1} + \cdots + b^{2m-1}xa + b^{2m}x,$$

and

$$v = b^{2m}y + b^{2m-1}yc + b^{2m-2}yc^2 + \cdots + byc^{2m-1} + yc^{2m}.$$

Hence

$$\begin{aligned} d &\in r_R(a^{2m+1}) = r_R(a^{2m}) = \cdots = r_R(a^m), \\ e &\in r_R(b^{2m+1}) = r_R(b^{2m}) = \cdots = r_R(b^m), \end{aligned}$$

and

$$f \in r_R(c^{2m+1}) = r_R(c^{2m}) = \cdots = r_R(c^m).$$

Then

$$\begin{aligned} 0 &= ud + b^{2m+1}s \\ &= (xa^{2m} + bxa^{2m-1} + \cdots + b^m xa^m + b^{m+1}xa^{m-1} + \cdots + b^{2m}x)d + b^{2m+1}s \\ &= (b^{m+1}xa^{m-1} + b^{m+2}xa^{m-2} + \cdots + b^{2m}x)d + b^{2m+1}s \\ &= b^{m+1}((xa^{m-1} + bxa^{m-2} + \cdots + b^{m-1}x)d + b^m s), \end{aligned}$$

and

$$\begin{aligned} 0 &= b^{2m+1}t + vf \\ &= b^{2m+1}t + (b^{2m}y + b^{2m-1}yc + \cdots + b^{m+1}yc^{m-1} + b^m yc^m + \cdots + yc^{2m})f \\ &= b^{2m+1}t + (b^{2m}y + b^{2m-1}yc + \cdots + b^{m+1}yc^{m-1})f \\ &= b^{m+1}(b^m t + (b^{m-1}y + b^{m-2}yc + \cdots + yc^{m-1})f). \end{aligned}$$

Hence

$$(xa^{m-1} + bxa^{m-2} + \cdots + b^{m-1}x)d + b^m s \in r_R(b^{m+1}) = r_R(b^m)$$

and

$$b^m t + (b^{m-1}y + b^{m-2}yc + \cdots + yc^{m-1})f \in r_R((b^{m+1}) = r_R(b^m).$$

So

$$b^m((xa^{m-1}+bxa^{m-2}+\cdots+b^{m-1}x)d+b^m s)$$
$$=b^m xa^{m-1}+b^{m+1}xa^{m-2}+\cdots+b^{2m-1}x)d+b^{2m}s=0,$$

and

$$b^m(b^m t+(b^{m-1}y+b^{m-2}yc+\cdots+yc^{m-1})f)$$
$$=b^{2m}t+(b^{2m-1}y+b^{2m-2}yc+\cdots+b^m yc^{m-1})f=0.$$

Then by routine computations, we can show that $\alpha^{2m}\beta=0$ and so $\beta\in r_{W(R)}(\alpha^{2m})$. Hence $r_{W(R)}(\alpha^{2m+1})=r_{W(R)}(\alpha^{2m})$. Therefore $W(R)$ is right strongly Hopfian.

Conversely, if $W(R)$ is right strongly Hopfian, then by Lemma 2.3 R is right strongly Hopfian.

Let α be an endomorphism and δ an α-derivation of R. We denote by $R[x;\alpha,\delta]$ the Ore extension whose elements are the polynomials over R, the addition is defined as usual, and the multiplication is subject to the relation $xa=\alpha(a)x+\delta(a)$ for any $a\in R$. From this rule, an inductive argument can be made in order to calculate an expression for $x^j a$, for all $j\in\mathbb{N}$ and $a\in R$. To recall this result, we shall use some convenient notation introduced in [9].

Notation 2.8 *Let δ be an α-derivation of R. For integers i, j with $0\le i\le j$, $f_i^j\in End(R,+)$ will denote the map which is the sum of all possible words in α,δ built with i letters α and $j-i$ letters δ. For instance, $f_0^0=1$, $f_j^j=\alpha^j$, $f_0^j=\delta^j$, and $f_{j-1}^j=\alpha^{j-1}\delta+\alpha^{j-2}\delta\alpha+\cdots+\delta\alpha^{j-1}$. Using recursive formulas for the f_i^j's and induction, as done in [9], one can show with a routine computation that*

$$x^j a=\sum_{i=0}^{j} f_i^j(a)x^i.$$

The following Lemma is well known and we omit the proof (see [3, Lemma 2.1]).

Lemma 2.9 *Let R be an (α,δ)-compatible ring. Then we have the following:*

(1) *If $ab=0$, then $a\alpha^n(b)=\alpha^n(a)b=0$ for all positive integers n.*
(2) *If $\alpha^k(a)b=0$ for some positive integer k, then $ab=0$.*
(3) *If $ab=0$, then $\alpha^n(a)\delta^m(b)=0=\delta^m(a)\alpha^n(b)$ for all positive integers m,n.*
(4) *If $ab=0$, then $af_i^j(b)=0$ and $f_i^j(a)b=0$ for all i, j.*

Lemma 2.10 *Let R be an α-compatible ring. If $\alpha^{k_1}(a_1)\alpha^{k_2}(a_2)\cdots\alpha^{k_n}(a_n)=0$ for some positive integers, then $a_1a_2\cdots a_n=0$.*

Proof Using induction, for $n=1$, the result is true by the injectivity of α. Now suppose $\alpha^{k_1}(a_1)\alpha^{k_2}(a_2)\cdots\alpha^{k_n}(a_n)=0$. Then $\alpha^{k_1}(a_1)\alpha^{k_2}(a_2)\cdots\alpha^{k_{n-1}}(a_{n-1})a_n=0$, and so $\alpha^{k_1}(a_1)\alpha^{k_2}(a_2)\cdots\alpha^{k_{n-1}}(a_{n-1}a_n)=0$. Then $a_1a_2\cdots a_n=0$.

Lemma 2.11 *Let R be an (α,δ)-compatible ring, $f(x)=a_0+a_1x+\cdots+a_nx^n$ and $g(x)=b_0+b_1x+\cdots+b_mx^m$ be two polynomials in $R[x;\alpha,\delta]$. Then we have the following:*

(1) *If for all $0\le i\le n$ and $0\le j\le m$, $a_ib_j=0$, then $f(x)g(x)=0$.*
(2) *If R is semicommutative and $c\in R$ is such that for all $0\le j\le m$, $cb_j=0$, then $cf(x)g(x)=0$.*

Proof (1) We have

$$\begin{aligned}f(x)g(x)&=(a_0+a_1x+\cdots+a_nx^n)(b_0+b_1x+\cdots+b_mx^m)\\&=\sum_{l=0}^{m+n}\left(\sum_{s+t=l}\left(\sum_{i=s}^{n}a_if_s^i(b_t)\right)\right)x^l.\end{aligned}$$

By Lemma 2.9, $a_ib_t=0$ implies $a_if_s^i(b_t)=0$. Thus it is easy to see that $f(x)g(x)=0$.

(2) Since R is semicommutative, for all $0\le i\le n$ and $0\le j\le m$, $cb_j=0$ implies $ca_ib_j=0$. Thus by (1) we complete the proof.

For two polynomials $f(x)$ and $g(x)$ in $R[x;\alpha,\delta]$, in order to calculate an expression for $(f(x)+g(x))^n$, for all $n\in\mathbb{N}$, we denote by $[Q_i^nf(x)g(x)]$ the polynomial which is the sum of all possible terms, with each term being a product of i polynomials $f(x)$ and $n-i$ polynomials $g(x)$. Using this convenient notation, we have $(f(x)+g(x))^n=f(x)^n+[Q_{n-1}^nf(x)g(x)]+[Q_{n-2}^nf(x)g(x)]+\cdots+[Q_1^nf(x)g(x)]+g(x)^n$.

Lemma 2.12 *Let R be an (α,δ)-compatible semicommutative ring, ax^r, $f(x)=b_0+b_1x+\cdots+b_mx^m$, $g(x)=c_0+c_1x+\cdots+c_qx^q$ be three polynomials in $R[x;\alpha,\delta]$. If $c_j\in r_R(a^n)$ for all $0\le j\le q$, then for any $p>n$, $[Q_n^p(ax^r)f(x)]g(x)=0$.*

Proof It is easy to check that the coefficients of $[Q_n^p(ax^r)f(x)]$ can be written as sums of monomials of length p in $f_s^t(a)$ and $f_u^v(b_j)$, where $b_j\in\{b_0,b_1,\dots,b_m\}$ and $t\ge s\ge 0$, $v\ge u\ge 0$ are nonnegative positive integers. Consider each monomial $f_{s_1}^{t_1}(v_1)f_{s_2}^{t_2}(v_2)\cdots f_{s_p}^{t_p}(v_p)$ where $v_1,v_2,\dots,v_p\in\{a,b_0,b_1,\dots,b_m\}$. It would contain n letters a. Suppose $v_{r_1}=v_{r_2}=\cdots=v_{r_n}=a$ for some $1\le r_1<r_2<\cdots<r_n\le p$. Then we write the monomial $f_{s_1}^{t_1}(v_1)f_{s_2}^{t_2}(v_2)\cdots f_{s_p}^{t_p}(v_p)$ as

$$f_{s_1}^{t_1}(v_1)\cdots f_{s_{r_1}}^{t_{r_1}}(a)f_{s_{r_1+1}}^{t_{r_1+1}}(v_{r_1+1})\cdots f_{s_{r_n-1}}^{t_{r_n-1}}(v_{r_n-1})f_{s_{r_n}}^{t_{r_n}}(a)f_{s_{r_n+1}}^{t_{r_n+1}}(v_{r_n+1})\cdots f_{s_p}^{t_p}(v_p),$$

where $v_s\in\{b_0,b_1,\dots,b_m\}$ if $s\notin\{r_1,r_2,\dots,r_n\}$. For each $0\le j\le q$, since R is (α,δ)-compatible and semicommutative, $a^nc_j=aa\cdots ac_j=0$ implies

$$f_{s_{r_1}}^{t_{r_1}}(a)f_{s_{r_2}}^{t_{r_2}}(a)\cdots f_{s_{r_n}}^{t_{r_n}}(a)c_j=0,$$

and so

$$\begin{aligned}&f_{s_1}^{t_1}(v_1)\cdots f_{s_{r_1}}^{t_{r_1}}(a)f_{s_{r_1+1}}^{t_{r_1+1}}(v_{r_1+1})\cdots\\&f_{s_{r_n-1}}^{t_{r_n-1}}(v_{r_n-1})f_{s_{r_n}}^{t_{r_n}}(a)f_{s_{r_n+1}}^{t_{r_n+1}}(v_{r_n+1})\cdots f_{s_p}^{t_p}(v_p)c_j=0.\end{aligned}$$

Thus by Lemma 2.11 we complete the proof.

The same idea can be used to prove the following.

Corollary 2.13 *Let R be an (α, δ)-compatible semicommutative ring, ax^r, $f(x) = b_0 + b_1x + \cdots + b_mx^m$, $g(x) = c_0 + c_1x + \cdots + c_qx^q$ be three polynomials in $R[x; \alpha, \delta]$. If $c_j \in r_R(a^n)$ for all $0 \le j \le q$, Then we have the following:*

(1) For any $p > n + l$, $[Q^p_{n+l}(ax^r)f(x)]g(x) = 0$.
(2) $R[x; \alpha, \delta](a^{i_1}x^{n_1})R[x; \alpha, \delta](a^{i_2}x^{n_2})R[x; \alpha, \delta] \cdots (a^{i_k}x^{n_k})R[x; \alpha, \delta]g(x) = 0$ if $i_1 + i_2 + \cdots + i_k \ge n$.

Proposition 2.14 *Let R be (α, δ)-compatible and $R[x; \alpha, \delta]$ be reversible. Then the following conditions are equivalent:*

(1) R is right strongly Hopfian;
(2) $R[x; \alpha, \delta]$ is right strongly Hopfian.

Proof (1) $\Rightarrow$ (2) Let $f(x) = a_0 + a_1x + \cdots + a_nx^n \in R[x; \alpha, \delta]$. Since R is right strongly Hopfian, there exists $k \in \mathbb{N}$ such that for all $l > k$ and all $0 \le i \le n$, $r_R(a_i^l) = r_R(a_i^k)$. Now we show that $r_{R[x;\alpha,\delta]}(f(x)^{(n+1)k+1}) = r_{R[x;\alpha,\delta]}(f(x)^{(n+1)k})$. If

$$g(x) = b_0 + b_1x + \cdots + b_mx^m \in r_{R[x;\alpha,\delta]}(f(x)^{(n+1)k+1}),$$

then

$$\begin{aligned}0 &= f(x)^{(n+1)k+1}g(x) = (a_0 + a_1x + \cdots + a_nx^n)^{(n+1)k+1}(b_0 + b_1x + \cdots + b_mx^m)\\ &= a_n\alpha^n(a_n)\alpha^{2n}(a_n)\cdots\alpha^{(n+1)kn}(a_n)\alpha^{[(n+1)k+1]n}(b_m)x^{[(n+1)k+1]n+m} + \text{lower terms}.\end{aligned}$$

Hence

$$a_n\alpha^n(a_n)\alpha^{2n}(a_n)\cdots\alpha^{(n+1)kn}(a_n)\alpha^{[(n+1)k+1]n}(b_m) = 0.$$

By Lemma 2.10, we obtain $a_n^{(n+1)k+1}b_m = 0$. Hence

$$b_m \in r_R\left(a_n^{(n+1)k+1}\right) = r_R\left(a_n^k\right).$$

From $f(x)^{(n+1)k+1}g(x) = 0$, we have $a_n^kf(x)^{(n+1)k+1}g(x) = 0$. Then by Lemma 2.11 we obtain

$$\begin{aligned}0 &= a_n^kf(x)^{(n+1)k+1}g(x)\\ &= a_n^k(a_0 + a_1x + \cdots + a_nx^n)^{(n+1)k+1}(b_0 + b_1x + \cdots + b_mx^m)\\ &= a_n^k(a_0 + a_1x + \cdots + a_nx^n)^{(n+1)k+1}(b_0 + b_1x + \cdots + b_{m-1}x^{m-1})\\ &\quad + a_n^k(a_0 + a_1x + \cdots + a_nx^n)^{(n+1)k+1}b_mx^m\\ &= a_n^k(a_0 + a_1x + \cdots + a_nx^n)^{(n+1)k+1}(b_0 + b_1x + \cdots + b_{m-1}x^{m-1})\\ &= a_n^ka_n\alpha^n(a_n)\cdots\alpha^{(n+1)kn}(a_n)\alpha^{[(n+1)k+1]n}(b_{m-1})x^{[(n+1)k+1]n+m-1} + \text{lower terms}.\end{aligned}$$

Hence

$$a_n^{k+1}\alpha^n(a_n)\alpha^{2n}(a_n)\cdots\alpha^{(n+1)kn}(a_n)\alpha^{[(n+1)k+1]n}(b_{m-1})=0$$

and so

$$b_{m-1}\in r_R\left(a_n^{(n+2)k+1}\right)=r_R\left(a_n^k\right).$$

Using the same method repeatedly, we obtain

$$b_j\in r_R(a_n^{(n+1)k+1})=r_R(a_n^k)\text{ for all }0\le j\le m.$$

Consider the polynomial $f(x)$ as the sum of two polynomials a_nx^n and $h(x)=a_{n-1}x^{n-1}+a_{n-2}x^{n-2}+\cdots+a_0$. Then by Corollary 2.13 we obtain

$$\begin{aligned}0&=f(x)^{(n+1)k+1}g(x)=\left(a_nx^n+h(x)\right)^{(n+1)k+1}g(x)\\&=(a_nx^n)^{(n+1)k+1}g(x)+\left[Q_{(n+1)k}^{(n+1)k+1}(a_nx^n)h(x)\right]g(x)+\cdots\\&\quad+\left[Q_k^{(n+1)k+1}(a_nx^n)h(x)\right]g(x)+\left[Q_{k-1}^{(n+1)k+1}(a_nx^n)h(x)\right]g(x)\\&\quad+\cdots+\left[Q_1^{(n+1)k+1}(a_nx^n)h(x)\right]g(x)+h(x)^{(n+1)k+1}g(x)\\&=\left[Q_{k-1}^{(n+1)k+1}(a_nx^n)h(x)\right]g(x)+\left[Q_{k-2}^{(n+1)k+1}(a_nx^n)h(x)\right]g(x)+\cdots\\&\quad+\left[Q_1^{(n+1)k+1}(a_nx^n)h(x)\right]g(x)+h(x)^{(n+1)k+1}g(x).\end{aligned}\tag{1}$$

Multiplying Eq. (1) on the left side by $(a_nx^n)^{k-1}$, then by Lemma 2.12 and Corollary 2.13, we obtain

$$(a_nx^n)^{k-1}h(x)^{(n+1)k+1}g(x)=0.$$

Since $R[x;\alpha,\delta]$ is reversible, this implies

$$\left[Q_{k-1}^{(n+1)k+1}(a_nx^n)h(x)\right]g(x)h(x)^{k-1}=0.\tag{2}$$

Multiplying Eq. (1) on the right side by $h(x)^{k-1}$, we obtain

$$\begin{aligned}&\left[Q_{k-2}^{(n+1)k+1}(a_nx^n)h(x)\right]g(x)h(x)^{k-1}+\left[Q_{k-3}^{(n+1)k+1}(a_nx^n)h(x)\right]g(x)h(x)^{k-1}\\&\quad+\cdots+h(x)^{(n+1)k+1}g(x)h(x)^{k-1}=0.\end{aligned}\tag{3}$$

Multiplying Eq. (3) on the left side by $(a_nx^n)^{k-2}$, we obtain

$$\begin{aligned}&(a_nx^n)^{k-2}\left[Q_1^{(n+1)k+1}(a_nx^n)h(x)\right]g(x)h(x)^{k-1}\\&\quad+(a_nx^n)^{k-2}h(x)^{(n+1)k+1}g(x)h(x)^{k-1}=0.\end{aligned}\tag{4}$$

By equation $(a_nx^n)^{k-1}h(x)^{(n+1)k+1}g(x) = 0$ and because $R[x;\alpha,\delta]$ is reversible, it is easy to see that $(a_nx^n)^{k-2}\left[Q_1^{(n+1)k+1}(a_nx^n)h(x)\right]g(x)h(x)^{k-1} = 0$. Hence Eq. (4) becomes

$$(a_nx^n)^{k-2}h(x)^{(n+1)k+1}g(x)h(x)^{k-1} = 0.$$

Since $R[x;\alpha,\delta]$ is reversible, this implies

$$\left[Q_{k-2}^{(n+1)k+1}(a_nx^n)h(x)\right]g(x)h(x)^{k-1}h(x)^{k-2} = 0.$$

Multiplying Eq. (3) on the right side by $h(x)^{k-2}$, we obtain

$$\left[Q_{k-3}^{(n+1)k+1}(a_nx^n)h(x)\right]g(x)h(x)^{k-1}h(x)^{k-2} + \cdots + h(x)^{(n+1)k+1}g(x)h(x)^{k-1}h(x)^{k-2} = 0.$$

Continuing this process yields that

$$h(x)^{(n+1)k+1}g(x)h(x)^{k-1}h(x)^{k-2}\cdots h(x) = 0,$$

and so

$$\begin{aligned} &h(x)^{(n+1)k+1+\frac{k(k-1)}{2}}g(x) \\ &= (a_0 + a_1x + \cdots + a_{n-1}x^{n-1})^{(n+1)k+1+\frac{k(k-1)}{2}}(b_0 + b_1x + \cdots b_mx^m) = 0, \end{aligned}$$

since $R[x;\alpha,\delta]$ is reversible. Now by the same way as above, we obtain

$$b_j \in r_R\left(a_{n-1}^{(n+1)k+1+\frac{k(k-1)}{2}}\right) = r_R\left(a_{n-1}^k\right)$$

for all $0 \le j \le m$. Using induction on n, we obtain

$$b_j \in r_R\left(a_i^{(n+1)k+1}\right) = r_R\left(a_i^k\right)$$

for all $0 \le j \le m$ and $0 \le i \le n$. It is now easy to check that $f(x)^{(n+1)k}g(x) = 0$. Hence $r_{R[x;\alpha,\delta]}(f(x)^{(n+1)k+1}) = r_{R[x;\alpha,\delta]}(f(x)^{(n+1)k})$. Therefore $R[x;\alpha,\delta]$ is right strongly Hopfian.

(2) $\Rightarrow$ (1) It is trivial.

Corollary 2.15 *We have the following:*

(1) *If R is α-compatible and the skew polynomial ring $R[x;\alpha]$ is reversible, then the skew polynomial ring $R[x;\alpha]$ is right strongly Hopfian if and only if R is right strongly Hopfian.*

(2) *If R is δ-compatible and the differential polynomial ring $R[x;\delta]$ is reversible, then the differential polynomial ring $R[x;\delta]$ is right strongly Hopfian if and only if R is right strongly Hopfian.*

Proof It is an immediate consequence of Proposition 2.14.

Corollary 2.16 ([5, Theorem 5.1]). *Let R be a commutative strongly Hopfian ring, then the polynomial ring $R[x]$ is strongly Hopfian.*

Let M be a multiplicative monoid. In the following, e will always stand for the identity of M. Then $R[M]$ will denote the monoid ring over R consisting of all elements of the form $\sum_{i=1}^{n} r_i g_i$ with $r_i \in R$, $g_i \in M$, $i = 1, 2, \ldots, n$, where the addition is given naturally and the multiplication is given by

$$\left(\sum_{i=1}^{n} r_i g_i\right)\left(\sum_{j=1}^{m} s_j h_j\right) = \sum_{i=1}^{n}\sum_{j=1}^{m}(r_i s_j)(g_i h_j).$$

Recall that the ordered monoid $(M, \leq)$ is a strictly ordered monoid if for any g, g', $h \in M$, $g < g'$ implies that $gh < g'h$ and $hg < hg'$.

For two elements α and β in $R[M]$, in order to calculate an expression for $(\alpha+\beta)^n$, for all $n \in \mathbb{N}$, we denote by $[Q_i^n \alpha\beta]$ the sum of all possible terms with each term being a product of i elements α and $n-i$ elements β. Using this convenient notation, we have $(\alpha+\beta)^n = \alpha^n + [Q_{n-1}^n \alpha\beta] + [Q_{n-2}^n \alpha\beta] + \cdots + [Q_1^n \alpha\beta] + \beta^n$.

Lemma 2.17 *Let $(M, \leq)$ be a strictly totally ordered monoid and R a semicommutative ring, $\alpha = ag$, $\beta = b_1h_1 + b_2h_2 + \cdots + b_nh_n$ and $\gamma = c_1v_1 + c_2v_2 + \cdots + c_mv_m$ be three elements in $R[M]$. If there exists a positive integer $n \in \mathbb{Z}$ such that $c_j \in r_R(a^n)$ for all $1 \leq j \leq m$, then for any $p > n$, $[Q_n^p \alpha\beta]\gamma = 0$.*

Proof The coefficients of $[Q_n^p \alpha\beta]$ can be written as sums of monomials of length p in a and b_j, where $b_j \in \{b_1, b_2, \ldots, b_n\}$. Consider one of such monomials, $d_1 d_2 \cdots d_p$, where $d_i \in \{a, b_1, b_2, \ldots, b_n\}$, $0 \leq i \leq p$. It would contain n letters a. Suppose $d_{r_1} = d_{r_2} = d_{r_n} = a$ for some $1 \leq r_1 < r_2 < \cdots < r_n \leq p$. Then we can write the monomial as $d_1 d_2 \cdots d_{r_1-1} a d_{r_1+1} \cdots d_{r_n-1} a d_{r_n+1} \cdots d_p$. Since R is semicommutative and $c_j \in r_R(a^n)$ for all $1 \leq j \leq m$, $a^n c_j = aa \cdots ac_j = 0$ implies

$$d_1 d_2 \cdots d_{r_1-1} a d_{r_1+1} \cdots d_{r_n-1} a d_{r_n+1} \cdots d_p c_j = 0$$

for all $1 \leq j \leq m$. Hence each monomial appearing in $[Q_n^p \alpha\beta]\gamma$ is equal to 0. Therefore $[Q_n^p \alpha\beta]\gamma = 0$.

The same idea can be used to prove the following.

Corollary 2.18 *Let $(M, \leq)$ be a strictly totally ordered monoid and R a semicommutative ring, $\alpha = ag$, $\beta = b_1h_1 + b_2h_2 + \cdots + b_nh_n$ and $\gamma = c_1v_1 + c_2v_2 + \cdots + c_mv_m$ be three elements in $R[M]$. If there exists a positive integer $n \in \mathbb{Z}$ such that $c_j \in r_R(a^n)$ for all $1 \leq j \leq m$, then for any $p > n + l$, we have the following:*

(1) $[Q^{p}_{n+l}\alpha\beta]\gamma = 0$.
(2) $R[M](ag)^{n_1}R[M](ag)^{n_2}R[M]\cdots(ag)^{n_k}R[M]\gamma R[M] = 0$ *if* $n_1 + n_2 + \cdots + n_k \geq n$.

Proposition 2.19 *Let M be a strictly totally ordered monoid and $R[M]$ a reversible ring. Then the following conditions are equivalent:*

(1) *R is a right strongly Hopfian ring;*
(2) *$R[M]$ is a right strongly Hopfian ring.*

Proof (1) $\Rightarrow$ (2). Let $\alpha = a_1g_1 + a_2g_2 + \cdots + a_ng_n \in R[M]$ with $g_i < g_j$ if $i < j$. Since R is right strongly Hopfian, there exists $k \in \mathbb{N}$ such that for all $l > k$ and all $1 \leq i \leq n$, $r_R(a_i^l) = r_R(a_i^k)$. Now we show that $r_{R[M]}(\alpha^{nk+1}) = r_{R[M]}(\alpha^{nk})$. If $\beta = b_1h_1 + b_2h_2 + \cdots b_mh_m \in r_{R[M]}(\alpha^{nk+1})$ with $h_s < h_t$ if $s < t$, then

$$0 = \alpha^{nk+1}\beta = (a_1g_1 + a_2g_2 + \cdots + a_ng_n)^{nk+1}(b_1h_1 + b_2h_2 + \cdots b_mh_m).$$

Considering the coefficient of the largest element $g_n^{nk+1}h_m$ in $\alpha^{nk+1}\beta$, we obtain $a_n^{nk+1}b_m = 0$. Hence $b_m \in r_R(a_n^{nk+1}) = r_R(a_n^k)$. From $\alpha^{nk+1}\beta = 0$, we have

$$\begin{aligned}
0 &= (a_n^k e)\alpha^{nk+1}\beta = (a_n^k e)(a_1g_1 + a_2g_2 + \cdots + a_ng_n)^{nk+1}(b_1h_1 + b_2h_2 + \cdots b_mh_m)\\
&= (a_n^k e)(a_1g_1 + a_2g_2 + \cdots + a_ng_n)^{nk+1}(b_1h_1 + b_2h_2 + \cdots b_{m-1}h_{m-1})\\
&\quad + (a_n^k e)(a_1g_1 + a_2g_2 + \cdots + a_ng_n)^{nk+1}b_mh_m\\
&= (a_n^k e)(a_1g_1 + a_2g_2 + \cdots + a_ng_n)^{nk+1}(b_1h_1 + b_2h_2 + \cdots b_{m-1}h_{m-1})\\
&= (a_n^k e)\alpha^{nk+1}(\beta - b_mh_m).
\end{aligned}$$

Considering the coefficient of the largest element $g_n^{nk+1}h_{m-1}$ in $(a_n^k e)\alpha^{nk+1}(\beta - b_mh_m)$, we obtain

$$b_{m-1} \in r_R\left(a_n^{(n+1)k+1}\right) = r_R\left(a_n^k\right).$$

Continuing this process yields that $b_j \in r_R(a_n^{nk+1}) = r_R(a_n^k)$ for all $1 \leq j \leq m$. Consider the element α as the sum of two elements a_ng_n and $\gamma = a_1g_1 + a_2g_2 + \cdots + a_{n-1}g_{n-1}$. Then by Lemma 2.17 and Corollary 2.18 we obtain

$$\begin{aligned}
0 &= \alpha^{nk+1}\beta = (a_ng_n + \gamma)^{nk+1}\beta\\
&= (a_ng_n)^{nk+1}\beta + \left[Q^{nk+1}_{nk}(a_ng_n)\gamma\right]\beta + \cdots + \left[Q^{nk+1}_{k-1}(a_ng_n)\gamma\right]\beta + \cdots + \gamma^{nk+1}\beta\\
&= \left[Q^{nk+1}_{k-1}(a_ng_n)\gamma\right]\beta + \left[Q^{nk+1}_{k-2}(a_ng_n)\gamma\right]\beta + \cdots + \gamma^{nk+1}\beta. \qquad (5)
\end{aligned}$$

Multiplying Eq. (5) on the left side by $(a_ng_n)^{k-1}$, and by Corollary 2.18, we obtain $(a_ng_n)^{k-1}\gamma^{nk+1}\beta = 0$ and so $[Q^{nk+1}_{k-1}(a_ng_n)\gamma]\beta\gamma^{k-1} = 0$ since $R[M]$ is reversible. Multiplying Eq. (5) on the right side by γ^{k-1}, we obtain

$$\left[Q_{k-2}^{nk+1}(a_n g_n)\gamma\right]\beta\gamma^{k-1} + \left[Q_{k-3}^{nk+1}(a_n g_n)\gamma\right]\beta\gamma^{k-1} + \cdots + \gamma^{nk+1}\beta\gamma^{k-1} = 0. \tag{6}$$

Multiplying Eq. (6) on the left side by $(a_n g_n)^{k-2}$, we obtain

$$(a_n g_n)^{k-2}\left[Q_1^{nk+1}(a_n g_n)\gamma\right]\beta\gamma^{k-1} + (a_n g_n)^{k-2}\gamma^{nk+1}\beta\gamma^{k-1} = 0.$$

Since $R[M]$ is reversible, $(a_n g_n)^{k-1}\gamma^{nk+1}\beta = 0$ implies

$$(a_n g_n)^{k-2}\left[Q_1^{nk+1}(a_n g_n)\gamma\right]\beta\gamma^{k-1} = 0.$$

Hence we obtain $(a_n g_n)^{k-2}\gamma^{nk+1}\beta\gamma^{k-1} = 0$, and so $[Q_{k-2}^{nk+1}(a_n g_n)\gamma]\beta\gamma^{k-1}\gamma^{k-2} = 0$ since $R[M]$ is reversible. Then multiplying Eq. (6) on the right side by γ^{k-2}, we obtain

$$\left[Q_{k-3}^{nk+1}(a_n g_n)\gamma\right]\beta\gamma^{k-1}\gamma^{k-2} + \left[Q_{k-4}^{nk+1}(a_n g_n)\gamma\right]\beta\gamma^{k-1}\gamma^{k-2} + \cdots + \gamma^{nk+1}\beta\gamma^{k-1}\gamma^{k-2} = 0.$$

Continuing this process, we obtain $\gamma^{nk+1}\beta\gamma^{\frac{k(k-1)}{2}} = 0$ and so $\gamma^{nk+1+\frac{k(k-1)}{2}}\beta = 0$. Using the same way as above, we can show that

$$b_j \in r_R(a_{n-1}^{nk+1+\frac{k(k-1)}{2}}) = r_R(a_{n-1}^k)$$

for all $1 \leq j \leq m$. Using induction on n, we obtain

$$b_j \in r_R\left(a_i^{nk+1}\right) = r_R\left(a_i^k\right)$$

for all $1 \leq j \leq m$ and $1 \leq i \leq n$. Then it is easy to check that $\alpha^{nk}\beta = 0$. Hence $\beta \in r_{R[M]}(\alpha^{nk})$, and so $r_{R[M]}(\alpha^{nk+1}) = r_{R[M]}(\alpha^{nk})$. Therefore $R[M]$ is right strongly Hopfian.

(2) $\Rightarrow$ (1) It is trivial.

Corollary 2.20 ([5, Corollary 5.4]) *Let R be a commutative strongly Hopfian ring, then $R[x, x^{-1}]$ is strongly Hopfian.*

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