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Strong Equality Between the 2-Rainbow Domination and Independent 2-Rainbow Domination Numbers in Trees

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Abstract A 2-*rainbow dominating function* (2RDF) on a graph $G = (V, E)$ is a function f from the vertex set V to the set of all subsets of the set $\{1, 2\}$ such that for any vertex $v \in V$ with $f(v) = \emptyset$ the condition $\bigcup_{u\in N(v)} f(u) = \{1, 2\}$ is fulfilled. A 2RDF f is independent (I2RDF) if no two vertices assigned nonempty sets are adjacent. The *weight* of a 2RDF f is the value $\omega(f) = \sum_{v\in V} |f(v)|$. The 2-*rainbow domination number* $\gamma_{r2}(G)$ (respectively, the *independent* 2-*rainbow domination number* $i_{r2}(G)$) is the minimum weight of a 2RDF (respectively, I2RDF) on G. We say that $\gamma_{r2}(G)$ is strongly equal to $i_{r2}(G)$ and denote by $\gamma_{r2}(G) \equiv i_{r2}(G)$, if every 2RDF on G of minimum weight is an I2RDF. In this paper, we provide a constructive characterization of trees T with $\gamma_{r2}(T) \equiv i_{r2}(T)$.

Keywords 2-Rainbow domination number · Independent 2-rainbow domination number · Strong equality · Tree

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1 Introduction

Let G be a simple graph with vertex set $V = V(G)$ and edge set $E = E(G)$. For every vertex $v \in V$, the *open neighborhood* $N(v)$ is the set $\{u \in V \mid uv \in E\}$ and the *closed neighborhood* of v is the set $N[v] = N(v) \cup \{v\}$. The *degree* of a vertex $v \in V$ is $\deg_G(v) = \deg(v) = |N(v)|$. If $A \subseteq V(G)$, then $G[A]$ is the subgraph induced by A. A vertex of degree one is called a *leaf*, and its neighbor is called a *support vertex*. If v is a support vertex, then L_v will denote the set of all leaves adjacent to v. A support vertex v is called *strong support vertex* if $|L_v| > 1$. For $r, s \geq 1$, a double star $S(r, s)$ is a tree with exactly two vertices that are not leaves, with one adjacent to r leaves and the other to s leaves. For a vertex v in a rooted tree T, let $C(v)$ denote the set of children of v, $D(v)$ denote the set of descendants of v and $D[v] = D(v) \cup \{v\}$, and the depth of v, depth(v), is the largest distance from v to a vertex in $D(v)$. The *maximal subtree* at v is the subtree of T induced by $D(v) \cup \{v\}$ and is denoted by T_v. For terminology and notation on graph theory not given here, the reader is referred to [11,13].

For a positive integer k, a *k-rainbow dominating function* (kRDF) of a graph G is a function f from the vertex set $V(G)$ to the set of all subsets of the set $\{1, 2, \ldots, k\}$ such that for any vertex $v \in V(G)$ with $f(v) = \emptyset$ the condition $\bigcup_{u \in N(v)} f(u) = \{1, 2, \ldots, k\}$ is fulfilled. The *weight* of a kRDF f is the value $\omega(f) = \sum_{v \in V} |f(v)|$. The *$k$-rainbow domination number* of a graph G, denoted by $\gamma_{rk}(G)$, is the minimum weight of a kRDF of G. A *$\gamma_{rk}(G)$-function* is a k-rainbow dominating function of G with weight $\gamma_{rk}(G)$. Note that $\gamma_{r1}(G)$ is the classical domination number $\gamma(G)$. The k-rainbow domination number was introduced by Brešar, Henning and Rall [3] and has been studied by several authors (see, e.g., [4–7,12]). To study other domination parameters, we refer the readers to [1,2,14,15].

A k-rainbow dominating function f is called an *independent k-rainbow dominating function* (abbreviated IkRDF) on G if the set $V(G) - \{v \in V \mid f(v) = \emptyset\}$ is independent. The *independent k-rainbow domination number*, denoted by $i_{rk}(G)$, is the minimum weight of an IkRDF on G. An independent k-rainbow dominating function f is called an *$i_{rk}(G)$-function* if $\omega(f) = i_{rk}(G)$. Since each independent k-rainbow dominating function is a k-rainbow dominating function, we have $\gamma_{rk}(G) \leq i_{rk}(G)$.

Clearly if $\gamma_{rk}(G) = i_{rk}(G)$, then every $i_{rk}(G)$-function is also a $\gamma_{rk}(G)$-function. However, not every $\gamma_{rk}(G)$-function is an $i_{rk}(G)$-function, even when $\gamma_{rk}(G) = i_{rk}(G)$. For example, the double star $S(k, k+1)$ has two $\gamma_{rk}(S(k, k+1))$-function, but only one of them is an $i_{rk}(S(k, k+1))$-function. We say that $\gamma_{rk}(G)$ and $i_{rk}(G)$ are *strongly equal* and denote by $\gamma_{rk}(G) \equiv i_{rk}(G)$, if every $\gamma_{rk}(G)$-function is an $i_{rk}(G)$-function.

Haynes and Slater in [10] were the first to introduce strong equality between two parameters. Also in [8,9], Haynes, Henning and Slater gave constructive characterizations of trees with strong equality between some domination parameters.

Our purpose in this paper is to present a constructive characterizations of trees T with $\gamma_{r2}(T) \equiv i_{r2}(T)$.

We make use of the following result in this paper.

Proposition A [5] *Let G be a connected graph. If there is a path $v_3v_2v_1$ in G with* $\deg(v_2) = 2$ *and* $\deg(v_1) = 1$*, then G has a $\gamma_{r2}(G)$-function f such that* $f(v_1) = \{1\}$ *and* $2 \in f(v_3)$.

Corollary 1 *Let T be a tree with $\gamma_{r2}(T) \equiv i_{r2}(T)$. If there is a path $v_3v_2v_1$ in T with* $\deg(v_2) = 2$ *and* $\deg(v_1) = 1$ *such that v_3 is a support vertex, then T has a $\gamma_{r2}(T)$-function f such that* $f(v_3) = \{1, 2\}$, $|f(v_1)| = 1$ *and* $|f(x)| = 0$ *for every* $x \in L_{v_3} \cup \{v_2\}$.

Observation 2 *Let T be a tree and let z be a strong support vertex of T. Then*

(a) *T has a $\gamma_{r2}(T)$-function such that $f(z) = \{1, 2\}$.*
(b) *$\gamma_{r2}(T) \not\equiv i_{r2}(T)$ if and only if T has a $\gamma_{r2}(T)$-function that is not independent and $f(z) = \{1, 2\}$.*

Proof (a) The proof is immediate.
(b) Let $\gamma_{r2}(T) \not\equiv i_{r2}(T)$. Then T has a $\gamma_{r2}(T)$-function that is not independent. If $f(z) = \{1, 2\}$, then we are done. If $|f(z)| = 1$, then $|f(x)| = 1$ for each $x \in L_z$ and the function $g : V(G) \to \mathcal{P}(\{1, 2\})$ defined by $g(z) = \{1, 2\}, g(x) = \emptyset$ for $x \in L_z$ and $g(u) = f(u)$ otherwise, is a 2RDF of T of weight less than $\omega(f)$ which is a contradiction. Let $f(z) = \emptyset$. Then clearly the function $g : V(G) \to \mathcal{P}(\{1, 2\})$ defined by $g(z) = \{1, 2\}, g(x) = \emptyset$ for $x \in L_z$ and $g(u) = f(u)$ otherwise, is a $\gamma_{r2}(T)$-function with the desired property. □

2 Characterizations of Trees with $\gamma_{r2}(T) \equiv i_{r2}(T)$

Let $\mathcal{F}_1$ be the family of trees that can be obtained from $k \geq 1$ disjoint stars $K_{1,2}$ by adding either a new vertex v or a path uv and joining the centers of stars to v. Also let $\mathcal{F}_2$ be the family including P_5 and all trees obtained from $k \geq 2$ disjoint P_3 by adding either a new vertex v or a path uv and joining v to a leaf of each P_3. If T belongs to $\mathcal{F}_1 \cup \mathcal{F}_2 - \{P_5\}$, then we call the vertex v, the *special vertex* of T and if $T = P_5$, then its support vertices are special vertices of T. Note that if $T \in \mathcal{F}_1 \cup \mathcal{F}_2$, then $\gamma_{r2}(T) \equiv i_{r2}(T)$.

Now we provide a constructive characterization of trees T with $\gamma_{r2}(T) \equiv i_{r2}(T)$. For this purpose, we define a family of trees as follows: Let $\mathcal{F}$ be the family of trees such that: $\mathcal{F}$ contains star $K_{1,2}$ and if T is a tree in $\mathcal{F}$, then the tree T' obtained from T by the following seven operations which extend the tree T by attaching a tree to a vertex $y \in V(T)$, called an attacher, is also a tree in $\mathcal{F}$.

- Operation $\mathcal{O}_1$: If z is a strong support vertex of $T \in \mathcal{F}$, then $\mathcal{O}_1$ adds a new vertex x and an edge xz.
- Operation $\mathcal{O}_2$: If z is a vertex of $T \in \mathcal{F}$, then $\mathcal{O}_2$ adds a new tree $T_1 \in \mathcal{F}_1$ with special vertex x and an edge xz provided that if x is a support vertex, then $\gamma_{r2}(T - z) \geq \gamma_{r2}(T)$.
- Operation $\mathcal{O}_3$: If z is a strong support vertex of $T \in \mathcal{F}$, then $\mathcal{O}_3$ adds a path zxy.
- Operation $\mathcal{O}_4$: If z is a vertex of $T \in \mathcal{F}$ which is adjacent to a support vertex of degree 2, then $\mathcal{O}_4$ adds a path zxy.

- Operation $\mathcal{O}_5$: If z is a vertex of $T \in \mathcal{F}$ which is adjacent to a strong support vertex, then $\mathcal{O}_5$ adds a path $zxyw$.
- Operation $\mathcal{O}_6$: If z is a vertex of $T \in \mathcal{F}$, then $\mathcal{O}_6$ adds a new tree $T_2 \in \mathcal{F}_2$ with special vertex x and an edge xz provided that if x is a support vertex, then $\gamma_{r2}(T-z) \geq \gamma_{r2}(T)$.
- Operation $\mathcal{O}_7$: If z is a vertex of $T \in \mathcal{F}$ such that every $\gamma_{r2}(T)$-function assigns $\emptyset$ to z, then $\mathcal{O}_7$ adds the double star $S(1,2)$ and an edge zx where x is a leaf of $S(1,2)$ whose support vertex has degree 3.

Observation 3 *The family $\mathcal{F}$ contains all graphs in $\{K_{1,t} \mid t \geq 2\} \cup \mathcal{F}_1 \cup \mathcal{F}_2$.*

Proof Starting from $K_{1,2} \in \mathcal{F}$ and by applying $t-2$ times Operation $\mathcal{O}_1$, we obtain the star $K_{1,t}$ and hence $\mathcal{F}$ contains all stars. Furthermore, starting from $K_{1,2}$ and by applying Operation $\mathcal{O}_4$, we obtain that $\mathcal{F}$ contains P_5.

Now let $T \in \mathcal{F}_1$. If $|V(T)| = 4$, then $T = K_{1,3}$ and immediately $T \in \mathcal{F}$. If $|V(T)| = 5$, then T can be obtained from $K_{1,2}$ by applying Operation $\mathcal{O}_3$. If $|V(T)| \geq 6$, then T can be obtained from $K_{1,2}$ by applying Operation $\mathcal{O}_2$. Thus $\mathcal{F}$ contains all graphs in $\mathcal{F}_1$.

Finally, let $T \in \mathcal{F}_2 - \{P_5\}$. If $|V(T)| = 7$, then $T = P_7$ and T can be obtained from P_5 by applying Operation $\mathcal{O}_4$ twice and so $T \in \mathcal{F}$. If $|V(T)| \geq 9$, then T can be obtained from $K_{1,2}$ by applying Operation $\mathcal{O}_6$. Thus $\mathcal{F}$ contains all graphs in $\mathcal{F}_2$. □

Lemma 4 *Let T be a tree with $\gamma_{r2}(T) \equiv i_{r2}(T)$ and let T' be the tree obtained from T by Operation $\mathcal{O}_1$. Then $\gamma_{r2}(T') \equiv i_{r2}(T')$.*

Proof Assume z is a strong support vertex of T and let x be a new vertex that is attached to z by applying Operation $\mathcal{O}_1$. By Observation 2(a), T has a γ_{r2}-function f that assigns $\{1,2\}$ to z. Since $\gamma_{r2}(T) \equiv i_{r2}(T)$, f is an I2RDF of T. Now we can extend f to an I2RDF of T' by assigning $\emptyset$ to x, implying that $\gamma_{r2}(T') \leq i_{r2}(T') \leq i_{r2}(T) \leq \gamma_{r2}(T)$. On the other hand, by Observation 2(a), there is a $\gamma_{r2}(T')$-function g which assigns $\{1,2\}$ to z, and clearly the function g, restricted to T, is a 2RDF of T of weight $\gamma_{r2}(T')$, implying that $\gamma_{r2}(T) \leq \gamma_{r2}(T')$. Hence $\gamma_{r2}(T') = i_{r2}(T')$.

It will now be shown that $\gamma_{r2}(T') \equiv i_{r2}(T')$. Suppose h is a $\gamma_{r2}(T')$-function that is not independent. Since $|L_z| \geq 3$, we must have $f(z) = \{1,2\}$. Then the function h, restricted to T, is a $\gamma_{r2}(T)$-function that is not independent which leads to a contradiction. Thus $\gamma_{r2}(T') \equiv i_{r2}(T)$. □

Lemma 5 *Let T be a tree with $\gamma_{r2}(T) \equiv i_{r2}(T)$ and let T' be a tree obtained from T by Operation $\mathcal{O}_2$. Then $\gamma_{r2}(T') \equiv i_{r2}(T')$.*

Proof Let $T_1 \in \mathcal{F}_1$ be the tree which is attached by Operation $\mathcal{O}_2$ to T by the edge xz for obtaining the tree T', where $z \in V(T)$ is the attacher vertex, and let $x_1, x_2, \ldots, x_k \in V(T_1)$ be the strong support vertices of T_1. Assume x is the special vertex of T_1. If x is a support vertex then let y be the leaf that is adjacent to x. Let t be a variant defined by $t = 1$ if x is a support vertex, and $t = 0$ otherwise. Every $i_{r2}(T)$-function can be extended to an I2RDF on T' by assigning $\{1,2\}$ to x_i, $i = 1, 2, \ldots, k$, $\emptyset$ to u for $u \in \cup_{i=1}^{k} N(x_i)$, and $\{1\}$ to y if x is a support vertex. This implies that

$i_{r2}(T') \leq i_{r2}(T) + 2k + t$. Since $\gamma_{r2}(T) \equiv i_{r2}(T)$, we deduce that

$$\gamma_{r2}(T') \leq i_{r2}(T') \leq \gamma_{r2}(T) + 2k + t = i_{r2}(T) + 2k + t. \tag{1}$$

Now we show that $\gamma_{r2}(T') = \gamma_{r2}(T)+2k+t$. Let f be a $\gamma_{r2}(T')$-function. It is easy to see that $\sum_{u\in N[x_i]-\{x\}} |f(u)| \geq 2$, for $i = 1, 2, \ldots, k$, and $|f(x)| + |f(y)| \geq 1$, if $t = 1$. Then $\sum_{u\in V(T_1)} |f(u)| \geq 2k + t$. If $|f(x)| = 0$ then $f|_{V(T)}$ is a 2RDF on T, and so $\sum_{u\in V(T)} |f(u)| \geq \gamma_{r2}(T)$. By adding two recent inequalities, we obtain $\gamma_{r2}(T') = \sum_{u\in V(T')} |f(u)| \geq \gamma_{r2}(T) + 2k + t$. Assume that $|f(x)| \geq 1$. Clearly if $t = 1$ then $|f(x)| + |f(y)| \geq 2$. Thus $\sum_{u\in V(T_1)} |f(u)| \geq 2k + t + 1$. If $|f(z)| \neq 0$ then $f|_{V(T)}$ is a 2RDF on T, and if $|f(z)| = 0$ then the function f_1 defined on $V(T)$ by $f_1(z) = \{1\}$ and $f_1(u) = f(u)$ if $u \in V(T) - \{z\}$ is a 2RDF for T. It follows that $\gamma_{r2}(T') \geq \gamma_{r2}(T) + 2k + t$. Hence, we deduce that

$$\gamma_{r2}(T') = \gamma_{r2}(T) + 2k + t. \tag{2}$$

By (1) and (2), we have

$$i_{r2}(T') = i_{r2}(T) + 2k + t = \gamma_{r2}(T) + 2k + t = \gamma_{r2}(T').$$

It will now be shown that $\gamma_{r2}(T') \equiv i_{r2}(T')$. Assume h is a $\gamma_{r2}(T')$-function that is not independent. We may assume that h assigns $\{1, 2\}$ to each support vertex adjacent to x. If $|h(x)| = 0$ then clearly $h|_{V(T)}$ is a $\gamma_{r2}(T)$-function that is not independent, a contradiction with the assumption $\gamma_{r2}(T) \equiv i_{r2}(T)$. Thus $|h(x)| \geq 1$. Then $|h(z)| = 0$ and $\sum_{v\in V(T_1)} |h(v)| \geq 2k + 1 + t$. If $|h(x)| = 1$, then $\sum_{w\in N_T(z)} |h(w)| \geq 1$ and the function $g : V(T) \to \mathcal{P}(\{1, 2\})$ defined by $g(z) = \{1\}$ and $g(u) = h(u)$ for $u \in V(T) - \{z\}$ is a $\gamma_{r2}(T)$-function that is not independent, contradicting $\gamma_{r2}(T) \equiv i_{r2}(T)$. Thus $|h(x)| = 2$. Then x is a support vertex. Now

$$\gamma_{r2}(T - z) \leq \sum_{u\in V(T-z)} |h(u)| = \gamma_{r2}(T') - 2k - 1 - t < \gamma_{r2}(T).$$

This is a contradiction with the assumption $\gamma_{r2}(T - z) \geq \gamma_{r2}(T)$. Therefore, $\gamma_{r2}(T') \equiv i_{r2}(T')$ and the proof is complete. □

Lemma 6 *If T is a tree with $\gamma_{r2}(T) \equiv i_{r2}(T)$ and T' is a tree obtained from T by Operation $\mathcal{O}_3$, then $\gamma_{r2}(T') \equiv i_{r2}(T')$.*

Proof Let $z \in V(T)$ be a strong support vertex and let zxy be the path added by Operation $\mathcal{O}_3$ to obtain T'. Let f be a $\gamma_{r2}(T)$-function such that $f(z) = \{1, 2\}$ [Observation 2(a)]. Since $\gamma_{r2}(T) \equiv i_{r2}(T)$, f is an I2RDF of T. We can extend f to an I2RDF on T' by assigning $\emptyset$ to x and $\{1\}$ to y, and thus

$$\gamma_{r2}(T') \leq i_{r2}(T') \leq i_{r2}(T) + 1 = \gamma_{r2}(T) + 1. \tag{3}$$

Let now f_1 be a $\gamma_{r2}(T')$-function. We can assume $f_1(z) = \{1, 2\}$ by Observation 2(a). Since f_1 is a $\gamma_{r2}(T')$-function, we must have $|f_1(x)| = 0$ and $|f_1(y)| = 1$. Then

$f_1|_{V(T)}$ is a 2RDF on T, and so

$$\gamma_{r2}(T) \leq \gamma_{r2}(T') - 1. \tag{4}$$

It follows from (3) and (4) that $\gamma_{r2}(T') = i_{r2}(T') = \gamma_{r2}(T) + 1 = i_{r2}(T) + 1$.

Finally, we shall show that $\gamma_{r2}(T') \equiv i_{r2}(T')$. Assume h is a $\gamma_{r2}(T')$-function that it is not independent. First let $|h(x)| \geq 1$. Then $|h(x)| + |h(y)| = 2$. If $|h(z)| \neq 0$ then replace $h(x)$ by $\emptyset$ and $h(y)$ by $\{1\}$ or $\{2\}$ to obtain a 2RDF for T' of weight less than $\gamma_{r2}(T')$, a contradiction. Thus $|h(z)| = 0$. Then clearly $|h(u)| = 1$ for any leaf u adjacent to z and the function $h_1 : V(T') \to \mathcal{P}(\{1, 2\})$ defined by $h_1(y) = \{1\}$, $h_1(z) = \{1, 2\}$, $h_1(u) = \emptyset$ for $u \in L_z \cup \{x\}$ and $h_1(w) = h(w)$ otherwise, is a 2RDF for T' of weight less than $\gamma_{r2}(T')$, a contradiction. Now let $|h(x)| = 0$. Then clearly $|h(y)| = 1$ (else we could make a change to be in the previous case $|h(x)| \geq 1$), and $h|_{V(T)}$ is a $\gamma_{r2}(T)$-function which is not independent, a contradiction. Hence, $\gamma_{r2}(T') \equiv i_{r2}(T')$. This completes the proof. □

Lemma 7 *If T is a tree with $\gamma_{r2}(T) \equiv i_{r2}(T)$ and T' is a tree obtained from T by Operation $\mathcal{O}_4$, then $\gamma_{r2}(T') \equiv i_{r2}(T')$.*

Proof Let $z \in V(T)$ be a vertex which is adjacent to a support vertex of degree 2 such as w, and let Operation $\mathcal{O}_4$ add the path zxy to T.

First let $\deg_T(z) \geq 2$. Let w' be the leaf adjacent to w. Assume f is a $\gamma_{r2}(T)$-function such that $2 \in f(z)$ (Proposition A). Since $\gamma_{r2}(T) \equiv i_{r2}(T)$, f is an $i_{r2}(T)$-function. Now f can be extended to an I2RDF on T' by assigning $\emptyset$ to x and $\{1\}$ to y. Thus

$$\gamma_{r2}(T') \leq i_{r2}(T') \leq i_{r2}(T) + 1 = \gamma_{r2}(T) + 1. \tag{5}$$

On the other hand, if f_1 is a $\gamma_{r2}(T')$-function, then we may assume that $2 \in f_1(z)$ by Proposition A. Clearly $|f_1(x)| + |f_1(y)| \geq 1$ and $f_1|_{V(T)}$ is a 2RDF on T of weight at most $\gamma_{r2}(T') - 1$, implying that $\gamma_{r2}(T') \geq \gamma_{r2}(T) + 1$. It follows from (5) and the recent inequality that $\gamma_{r2}(T') = i_{r2}(T') = i_{r2}(T) + 1 = \gamma_{r2}(T) + 1$.

It will now be shown that $\gamma_{r2}(T') \equiv i_{r2}(T')$. Suppose h is a $\gamma_{r2}(T')$-function which it is not independent. If $|h(z)| > 0$ then we must have $|h(x)| = 0$ and $|h(y)| = 1$, and so $h|_{V(T)}$ is a $\gamma_{r2}(T)$-function which is not independent, a contradiction. Let $|h(z)| = 0$. Then obviously $|h(x)| + |h(y)| = |h(w)| + |h(w')| = 2$. Then the function $g : V(T') \to \mathcal{P}(\{1, 2\})$ defined by $g(x) = g(w) = \emptyset$, $g(y) = g(w') = \{1\}$, $g(z) = \{2\}$ and $g(u) = f(u)$ for $u \in V(T') - \{x, y, w, w', z\}$, is a 2RDF of T' of weight less than $\gamma_{r2}(T')$, a contradiction. Thus $\gamma_{r2}(T') \equiv i_{r2}(T')$.

Now let $\deg_T(z) = 1$, i.e., z is a leaf.

Assume f is a $\gamma_{r2}(T)$-function. By Proposition A, we may assume that $f(z) = \{1\}$. Note that f is an $i_{r2}(T)$-function because $\gamma_{r2}(T) \equiv i_{r2}(T)$. Then f can be extended to an I2RDF on T' by assigning $\emptyset$ to x and $\{2\}$ to y. This implies that

$$\gamma_{r2}(T') \leq i_{r2}(T') \leq i_{r2}(T) + 1 = \gamma_{r2}(T) + 1. \tag{6}$$

On the other hand, if f_1 is a $\gamma_{r2}(T')$-function, then by Proposition A we may assume $f_1(y) = \{1\}$ and $2 \in f_1(z)$. Then $f_1|_{V(T)}$ is a 2RDF of T of weight at most $\gamma_{r2}(T') - 1$, implying that $\gamma_{r2}(T') \geq \gamma_{r2}(T) + 1$. It follows from the last inequality and (6) that $\gamma_{r2}(T') = i_{r2}(T') = \gamma_{r2}(T) + 1 = i_{r2}(T) + 1$.

Next we show that $\gamma_{r2}(T') \equiv i_{r2}(T')$. Assume h is a $\gamma_{r2}(T')$-function that it is not independent. If $|h(z)| > 0$ then we may assume that $|h(x)| = 0$ and $|h(y)| = 1$, and so $h|_{V(T)}$ is a $\gamma_{r2}(T)$-function which is not independent, a contradiction. Let $h(z) = \emptyset$. Then $|h(x)| + |h(y)| \geq 2$. If $|h(w)| = 0$ then $|h(x)| = 2$ and $|h(y)| = 0$, and the function $h_1 : V(T) \to \mathcal{P}(\{1, 2\})$ defined by $h_1(z) = \{1\}$ and $h_1(u) = h(u)$ if $u \in V(T) - \{z\}$ is a $\gamma_{r2}(T)$-function which is not independent, a contradiction. If $|h(w)| \geq 1$ then it follows from $|h(x)| + |h(y)| \geq 2$ that the function $h_1 : V(T) \to \mathcal{P}(\{1, 2\})$ defined above, is a $\gamma_{r2}(T)$-function which is not independent, a contradiction. Hence $\gamma_{r2}(T') \equiv i_{r2}(T')$. □

Lemma 8 *If T is a tree with $\gamma_{r2}(T) \equiv i_{r2}(T)$ and T' is a tree obtained from T by Operation $\mathcal{O}_5$, then $\gamma_{r2}(T') \equiv i_{r2}(T')$.*

Proof Let $z \in V(T)$ be a vertex that has a strong support vertex u in its neighborhood and let Operation $\mathcal{O}_5$ add the path $zxyw$ to T for obtaining T'. Any 2RDF of T can be extended to a 2RDF for T' by assigning $\{1, 2\}$ to y, and $\emptyset$ to x and w. Since $\gamma_{r2}(T) \equiv i_{r2}(T)$, we deduce that

$$\gamma_{r2}(T') \leq i_{r2}(T') \leq i_{r2}(T) + 2 = \gamma_{r2}(T) + 2. \tag{7}$$

Let f be a $\gamma_{r2}(T')$-function. We may assume $f(w) = \{1\}$, $f(y) = \emptyset$ and $2 \in f(x)$, by Proposition A. Also we may assume that $|f(u)| = 2$, since u is a strong support vertex. Then $f|_{V(T)}$ is a 2RDF on T of weight at most $\gamma_{r2}(T') - 2$, and so $\gamma_{r2}(T) \leq \gamma_{r2}(T') - 2$. It follows from (7) that

$$\gamma_{r2}(T') = i_{r2}(T') = i_{r2}(T) + 2 = \gamma_{r2}(T) + 2.$$

To show that $\gamma_{r2}(T') \equiv i_{r2}(T')$, suppose h is a $\gamma_{r2}(T')$-function that it is not independent. Since u is a strong support vertex, we may assume $|h(u)| = 2$. Then clearly $h(z) = \emptyset$ and $|h(x)| + |h(y)| + |h(w)| = 2$, and so $h|_{V(T)}$ is a $\gamma_{r2}(T)$-function which is not independent, a contradiction. Hence $\gamma_{r2}(T') \equiv i_{r2}(T')$ and the proof is completed. □

The proof of next lemma is similar to the proof of Lemma 5, and therefore omitted.

Lemma 9 *If T is a tree with $\gamma_{r2}(T) \equiv i_{r2}(T)$, and T' is a tree obtained from T by Operation $\mathcal{O}_6$, then $\gamma_{r2}(T) \equiv i_{r2}(T)$.*

Lemma 10 *If T is a tree with $\gamma_{r2}(T) \equiv i_{r2}(T)$ and T' is a tree obtained from T by Operation $\mathcal{O}_7$, then $\gamma_{r2}(T') \equiv i_{r2}(T')$.*

Proof Let z be a vertex of T such that every $\gamma_{r2}(T)$-function assign $\emptyset$ to it, and let x be a leaf of double star $S(1, 2)$ whose support vertex has degree 3. Assume that Operation $\mathcal{O}_7$ adds the double star $S(1, 2)$ and the edge xz to obtain T' from T. Let $V(S(1, 2)) = \{x, v, v_0, u, u_0\}$ where $N(v) = \{x, u, v_0\}$ and $u \in N(u_0)$. Any 2RDF of T can be extended to a 2RDF on T' by assigning $\emptyset$ to x, u and v_0, $\{1, 2\}$ to v and $\{1\}$ to u_0. Since $\gamma_{r2}(T) \equiv i_{r2}(T)$, we deduce that

$$\gamma_{r2}(T') \leq i_{r2}(T') \leq i_{r2}(T) + 3 = \gamma_{r2}(T) + 3. \tag{8}$$

Let f be a $\gamma_{r2}(T')$-function such that $f(u_0) = \{1\}$ and $2 \in f(v)$ by Observation A. Clearly $|f(v)| + |f(u_0)| + |f(u)| + |f(v_0)| \geq 3$. We may assume that $|f(x)| = 0$, otherwise we replace $f(x)$ by $\emptyset$ and $f(z)$ by $f(z) \cup f(x)$. Then $f|_{V(T)}$ is a 2RDF of T, implying that $\gamma_{r2}(T) \leq \gamma_{r2}(T') - 3$. By (8), we have $\gamma_{r2}(T') = i_{r2}(T') = \gamma_{r2}(T) + 3 = i_{r2}(T) + 3$.

It will now be shown that $\gamma_{r2}(T') \equiv i_{r2}(T')$. Suppose h is a $\gamma_{r2}(T')$-function which is not independent. Clearly $\sum_{y \in V(S(1,2))} |h(y)| \geq 3$. If $|h(z)| > 0$, then $h|_{V(T)}$ is a $\gamma_{r2}(T)$-function assigning nonempty set to z which leads to a contradiction. Thus $|h(z)| = 0$. If $\sum_{y \in V(S(1,2))} |h(y)| \geq 4$, then we change the values of h on $V(S(1, 2)) \cup \{z\}$ to $h(z) = h(u_0) = \{1\}$, $h(v) = \{1, 2\}$, and $h(x) = h(u) = h(v_0) = \emptyset$, and then the new function plays the role of h which has been considered earlier. Thus we assume that $\sum_{y \in V(S(1,2))} |h(y)| = 3$. Then clearly $|h(x)| = 0$, and $h|_{V(T)}$ is a $\gamma_{r2}(T)$-function which is not independent, a contradiction. Hence $\gamma_{r2}(T') \equiv i_{r2}(T')$. □

Theorem 11 *Each tree T in family $\mathcal{F} \cup \{K_1\}$ satisfies $\gamma_{r2}(T) \equiv i_{r2}(T)$.*

Proof If $T = K_1$, then clearly $\gamma_{r2}(T) \equiv i_{r2}(T)$. Let $T \in \mathcal{F}$. Then T is obtained from a star $K_{1,2}$ by successive operations $\mathcal{T}^1, \ldots, \mathcal{T}^m$, where $\mathcal{T}^i \in \{\mathcal{O}_1, \ldots, \mathcal{O}_7\}$ if $m \geq 1$ and $T = K_{1,2}$ if $m = 0$. The proof is by induction on m. If $m = 0$, then clearly $\gamma_{r2}(K_{1,2}) \equiv i_{r2}(K_{1,2})$. Let $m \geq 1$ and that the statement holds for all trees which are obtained from $K_{1,2}$ by applying $m - 1$ operations in $\{\mathcal{O}_1, \ldots, \mathcal{O}_7\}$. It follows from Lemmas 4, …, 10 that $\gamma_{r2}(T) \equiv i_{r2}(T)$. □

Observation 12 *If $S(p, q)$ is a double star with $q \geq p \geq 1$ and $\gamma_{r2}(S(p, q)) \equiv i_{r2}(S(p, q))$, then $p = 1$ and $q \geq 2$.*

Theorem 13 *Let T be a tree of order n. If $\gamma_{r2}(T) \equiv i_{r2}(T)$, then $T \in \mathcal{F} \cup \{K_1\}$.*

Proof The proof is by induction on n. If $n = 1$ then $T = K_1$. Let the statement holds for all trees of order less than n and let T be a tree of order n with $\gamma_{r2}(T) \equiv i_{r2}(T)$. Since $\gamma_{r2}(P_2) \not\equiv i_{r2}(P_2)$, we may assume that $n \geq 3$. If $\text{diam}(T) = 2$ then T is a star and by Observation 3, $T \in \mathcal{F}$. If $\text{diam}(T) = 3$, then T is a double star $S(p, q)$ with $q \geq p \geq 1$. By Observation 12, we have $p = 1$ and $q \geq 2$. Then T can be obtained from $K_{1,q}$ by Operation $\mathcal{O}_3$ and so $T \in \mathcal{F}$. Therefore, we may assume that $\text{diam}(T) \geq 4$.

Let $v_1 v_2 \ldots v_k$ ($k \geq 5$) be a diametral path in T such that $|L_{v_2}|$ is as large as possible and root T at v_k. Also suppose among paths with this property we choose a path such that $|L_{v_3}|$ is as large as possible.

Assume first that $\deg(v_2) \geq 4$. Let f be a $\gamma_{r2}(T)$-function. Then clearly $f(v_2) = \{1, 2\}$ and so f is a 2RDF of $T - v_1$. Since $\gamma_{r2}(T) \equiv i_{r2}(T)$, f is also an I2RDF of $T - v_1$, implying that $\gamma_{r2}(T) = i_{r2}(T) \geq i_{r2}(T - v_1) \geq \gamma_{r2}(T - v_1)$. On the other hand, by Observation 2(a), $T - v_1$ has a γ_{r2}-function g that assigns $\{1, 2\}$ to

v_2. Then g can be extended to a $\gamma_{r2}(T)$-function by assigning $\emptyset$ to v_1 that yields $\gamma_{r2}(T) \leq \gamma_{r2}(T - v_1)$. Hence $\gamma_{r2}(T) = i_{r2}(T) = i_{r2}(T - v_1) = \gamma_{r2}(T - v_1)$.

We show that $\gamma_{r2}(T - v_1) \equiv i_{r2}(T - v_1)$. Suppose that there is a $\gamma_{r2}(T - v_1)$-function g that is not independent. Since g is a $\gamma_{r2}(T - v_1)$-function, we must have $|g(v_2)| + \sum_{u \in L_{v_2} - \{v_1\}} |g(u)| = 2$. Now the function $h : V(T - v_1) \to \mathcal{P}(\{1, 2\})$ defined by $h(v_2) = \{1, 2\}$, $h(u) = \emptyset$ for $u \in L_{v_2} - \{v_1\}$ and $h(x) = g(x)$ otherwise, is a 2RDF of $T - v_1$ which in not independent. It is clear that h can be extended to a $\gamma_{r2}(T)$-function which is not independent by assigning $\emptyset$ to v_1. This leads to a contradiction with $\gamma_{r2}(T) \equiv i_{r2}(T)$. Thus $\gamma_{r2}(T - v_1) \equiv i_{r2}(T - v_1)$. It follows from the inductive hypothesis that $T - v_1 \in \mathcal{F}$. Now it is clear that T can be obtained from $T - v_1 \in \mathcal{F}$ by applying Operation $\mathcal{O}_1$.

Assume next that $\deg(v_2) = 3$. Let $u \in L_{v_2} - \{v_1\}$. We claim that v_3 is not a strong support vertex. Assume to the contrary that v_3 is a strong support vertex. By Observation 2(a), T has a $\gamma_{r2}(T)$-function f such that $f(v_3) = \{1, 2\}$. Clearly $|f(v_2)| + |f(v_1)| + |f(u)| = 2$. Now the function $g : V(T) \to \mathcal{P}(\{1, 2\})$ defined by $g(v_2) = \{1, 2\}$, $g(v_1) = g(u) = \emptyset$ and $g(x) = f(x)$ for $x \in V(T) - \{u, v_1, v_2\}$ is clearly a $\gamma_{r2}(T)$-function that is not independent, a contradiction with $\gamma_{r2}(T) \equiv i_{r2}(T)$. Thus v_3 is not a strong support vertex. Using Proposition A and an argument similar to that described above, we deduce that v_3 is not adjacent to a support vertex of degree 2. By the choice of the diametral path, we deduce that any child of v_3 is a leaf or a support vertex of degree 3 and at most one of them is leaf. This implies that $T_{v_3} \in \mathcal{F}_1$. Let $T' = T - T_{v_3}$.

We claim that if v_3 is a support vertex, then $\gamma_{r2}(T' - v_4) \geq \gamma_{r2}(T')$. Let v_3 be a support vertex and let to the contrary that $\gamma_{r2}(T' - v_4) < \gamma_{r2}(T')$. Assume h is a $\gamma_{r2}(T' - v_4)$-function and define $g : V(T) \to \mathcal{P}(\{1, 2\})$ by $g(x) = h(x)$ for $x \in V(T') - \{v_4\}$, $g(x) = \{1, 2\}$ for $x \in N[v_3] - (L_{v_3} \cup \{v_4\})$ and $g(x) = \emptyset$ otherwise. Obviously g is a $\gamma_{r2}(T)$-function that is not independent, a contradiction with $\gamma_{r2}(T) \equiv i_{r2}(T)$. Thus $\gamma_{r2}(T' - v_4) \geq \gamma_{r2}(T')$ when v_3 is a support vertex.

It will now be shown that $\gamma_{r2}(T') \equiv i_{r2}(T')$. First we show that $\gamma_{r2}(T') = i_{r2}(T')$. Since every $\gamma_{r2}(T')$-function can be extended to a 2RDF on T by assigning $\{1, 2\}$ to the strong support vertices in $N_{T_{v_3}}(v_3)$, $\{1\}$ to the leaf adjacent to v_3, if any, and $\emptyset$ to the other vertices in T_{v_3}, we deduce that

$$i_{r2}(T) = \gamma_{r2}(T) \leq \gamma_{r2}(T') + 2k + t \leq i_{r2}(T') + 2k + t \tag{9}$$

where k is the number of strong support vertices adjacent to v_3 in T_{v_3} and t is the number of leaf adjacent to v_3. On the other hand, let f be a $\gamma_{r2}(T)$-function. By Observation 2(a), we may assume that f assigns $\{1, 2\}$ to the strong support vertices in T_{v_3}. Since $\gamma_{r2}(T) \equiv i_{r2}(T)$, f is an I2RDF. Then f assigns $\emptyset$ to v_3 and $\{1\}$ or $\{2\}$ to the leaf adjacent to v_3, if any, and $f|_{V(T')}$ is an I2RDF on T' with weight $i_{r2}(T) - 2k - t$. Thus $i_{r2}(T') \leq i_{r2}(T) - 2k - t$. It follows from (9) that $i_{r2}(T) = \gamma_{r2}(T) = \gamma_{r2}(T') + 2k + t = i_{r2}(T') + 2k + t$ and hence $\gamma_{r2}(T') = i_{r2}(T')$.

Now we show that this equality is strong. Suppose h is a $\gamma_{r2}(T')$-function that it is not independent. We can extend h to a 2RDF on T by assigning $\{1, 2\}$ to every strong support vertex of T_{v_3} and $\{1\}$ to the leaf adjacent to v_3, if any, and $\emptyset$ to the other vertices in T_{v_3}, to obtain a $\gamma_{r2}(T)$-function which is not independent, a contradiction with $\gamma_{r2}(T) \equiv i_{r2}(T)$. Therefore $\gamma_{r2}(T') \equiv i_{r2}(T')$. It follows from the induction hypothesis that $T' \in \mathcal{F}$. Then T can be obtained from T' by applying Operation $\mathcal{O}_2$ and hence $T \in \mathcal{F}$.

We thus assume that $\deg(v_2) = 2$. Furthermore, we may assume that every child of v_3 that is a support vertex has degree two. We now consider the following three cases on $|L_{v_3}|$.

Case 1 $|L_{v_3}| \geq 2$.

Let $T' = T - \{v_1, v_2\}$. We show that $\gamma_{r2}(T') \equiv i_{r2}(T')$. Suppose f is a $\gamma_{r2}(T)$-function that assigns $\{1, 2\}$ to v_3 [Observation 2(a)]. Clearly $|f(v_1)| + |f(v_2)| = 1$. Since $\gamma_{r2}(T) \equiv i_{r2}(T)$, f is an $i_{r2}(T)$-function. Hence $|f(v_2)| = 0$ and $f|_{V(T')}$ is an I2RDF on T' implying that

$$\gamma_{r2}(T') \leq i_{r2}(T') \leq i_{r2}(T) - 1 = \gamma_{r2}(T) - 1. \tag{10}$$

Now let g be a $\gamma_{r2}(T')$-function that assigns $\{1, 2\}$ to v_3 [Observation 2(a)]. Then g can be extended to a 2RDF on T by assigning $\emptyset$ to v_2 and $\{1\}$ to v_1. This yields $\gamma_{r2}(T) \leq \gamma_{r2}(T') + 1$, By (10), we have $\gamma_{r2}(T') = i_{r2}(T')$. To show that this equality is strong, assume h is a $\gamma_{r2}(T')$-function that it is not independent. We may assume $h(v_3) = \{1, 2\}$. Now one can extend h to a $\gamma_{r2}(T)$-function which is not independent, by assigning $\emptyset$ to v_2 and $\{1\}$ to v_1, a contradiction with $\gamma_{r2}(T) \equiv i_{r2}(T)$. Thus $\gamma_{r2}(T') \equiv i_{r2}(T')$. By induction hypothesis, $T' \in \mathcal{F}$ and so T can be obtain from T' by Operation $\mathcal{O}_3$.

Case 2 $|L_{v_3}| = 0$.

Then any child of v_3 is a support vertex of degree 2. We consider two subcases.

Subcase 2.1 $\deg(v_3) \geq 3$.

Let z_2 be a child of v_3 different from v_2, and let z_1 be the leaf adjacent to z_2. Suppose $T' = T - \{v_1, v_2\}$. We show that $\gamma_{r2}(T') \equiv i_{r2}(T')$. Let f be a $\gamma_{r2}(T)$-function. We may assume $2 \in f(v_3)$ by Proposition A. Clearly $|f(v_1)| + |f(v_2)| = 1$. Since $\gamma_{r2}(T) \equiv i_{r2}(T)$, f is a $i_{r2}(T)$-function. Clearly $f|_{V(T')}$ is an I2RDF on T', implying that

$$\gamma_{r2}(T') \leq i_{r2}(T') \leq i_{r2}(T) - 1 = \gamma_{r2}(T) - 1. \tag{11}$$

On the other hand, by Proposition A, T' has a $\gamma_{r2}(T')$-function g such that $2 \in g(v_3)$. Then we can extend g on T by assigning $\emptyset$ to v_2 and $\{1\}$ to v_1, to obtain a 2RDF of weight $\gamma_{r2}(T') + 1$. Thus $\gamma_{r2}(T') \geq \gamma_{r2}(T) - 1$. It follows from (11) that $\gamma_{r2}(T') = i_{r2}(T')$.

To show that this equality is strong, assume h is a $\gamma_{r2}(T')$-function that it is not independent. First let $|h(v_3)| > 0$. Assume without loss of generality that $2 \in h(v_3)$. Then the function $h' : V(T) \to \mathcal{P}(\{1, 2\})$ defined by $h'(v_1) = \{1\}, h'(v_2) = \emptyset$ and $h'(x) = h(x)$ for $x \in V(T) - \{v_1, v_2\}$ is a $\gamma_{r2}(T)$-function that is not independent, a contradiction. Let now $|h(v_3)| = 0$. Then $|h(z_2)| + |h(z_1)| = 2$. If $\cup_{x \in N(v_3) - \{z_2\}} h(x) \neq \emptyset$, then we define $g : V(T) \to \mathcal{P}(\{1, 2\})$ by $g(v_3) = \{1\}$, $g(z_2) = g(v_2) = \emptyset$, $g(z_1) = g(v_1) = \{2\}$ and $g(x) = h(x)$ otherwise, to produce a $\gamma_{r2}(T)$-function that is not independent, a contradiction. Let $\cup_{x \in N(v_3) - \{z_2\}} h(x) = \emptyset$. Then to rainbowly dominate v_3, we must have $h(z_2) = \{1, 2\}$ and $|h(z_1)| = 0$. Then the function $h_1 : V(T') \to \mathcal{P}(\{1,2\})$ defined by $h_1(v_3) = \{1\}$, $h_1(z_2) = \emptyset$, $h_1(z_1) = \{2\}$, and $h_1(x) = h(x)$ otherwise, is a $\gamma_{r2}(T')$-function that is not independent and $|h_1(v_3)| > 0$. This leads to a contradiction as above. Thus $\gamma_{r2}(T') \equiv i_{r2}(T')$ and by inductive hypothesis we have $T' \in \mathcal{F}$. Now T can be obtained from T' by Operation $\mathcal{O}_4$.

Subcase 2.2 $\deg(v_3) = 2$.

First let $\deg(v_4) = 2$. Let $T' = T - \{v_1, v_2\}$. We show that $\gamma_{r2}(T') \equiv i_{r2}(T')$. Let f be a $\gamma_{r2}(T)$-function such that $f(v_1) = \{1\}$ and $2 \in f(v_3)$ (Proposition A). This implies that $|f(v_2)| = 0$. Since $\gamma_{r2}(T) \equiv i_{r2}(T)$, f is an $i_{r2}(T)$-function. Obviously the function f, restricted to T', is an I2RDF on T', implying that

$$\gamma_{r2}(T') \le i_{r2}(T') \le i_{r2}(T) - 1 = \gamma_{r2}(T) - 1. \tag{12}$$

Now let g be a $\gamma_{r2}(T')$-function such that $g(v_3) = \{1\}$ by Proposition A. We can extend g to a $\gamma_{r2}(T)$-function by assigning $\emptyset$ to v_2 and $\{2\}$ to v_1. This implies that $\gamma_{r2}(T) \le \gamma_{r2}(T') + 1$, and by (12) we obtain $\gamma_{r2}(T') = i_{r2}(T')$.

Now we show that this equality is strong. Assume h is a $\gamma_{r2}(T')$-function that is not independent. If $|h(v_3)| > 0$, then we can extend h to a $\gamma_{r2}(T)$-function that is not independent by assigning $\emptyset$ to v_2 and $\{1\}$ to v_1 if $2 \in h(v_3)$ and $\{2\}$ to v_1 if $1 \in h(v_3)$, a contradiction with $\gamma_{r2}(T) \equiv i_{r2}(T)$. Let $|h(v_3)| = 0$. Then to rainbowly dominate v_3, we must have $h(v_4) = \{1, 2\}$. Since h is a $\gamma_{r2}(T')$-function and $\deg(v_4) = 2$, we must have $|h(v_5)| = 0$. Then the function $h_1 : V(T) \to \mathcal{P}(\{1,2\})$ defined by $h_1(v_5) = h_1(v_1) = \{1\}$, $h_1(v_3) = \{2\}$, $h_1(v_2) = h_1(v_4) = \emptyset$ and $h_1(x) = h(x)$ otherwise, is a $\gamma_{r2}(T)$-function which is not independent, a contradiction with $\gamma_{r2}(T) \equiv i_{r2}(T)$. Hence $\gamma_{r2}(T') \equiv i_{r2}(T')$ and by inductive hypothesis, $T' \in \mathcal{F}$. Now T can be obtained from T' by Operation $\mathcal{O}_4$.

Next let $\deg(v_4) \ge 3$. By Proposition A, T has a γ_{r2}-function f such that $f(v_1) = \{1\}$, $|f(v_2)| = 0$ and $2 \in f(v_3)$. Also suppose among $\gamma_{r2}(T)$-functions with this property we choose a $\gamma_{r2}(T)$-function such that $|f(v_4)|$ is as large as possible. If $|f(v_3)| = 2$, then the function $g_1 : V(T) \to \mathcal{P}(\{1,2\})$ defined by $g_1(v_1) = \{1\}$, $g_1(v_2) = \emptyset$, $g_1(v_3) = \{2\}$, $g_1(v_4) = \{1\}$ and $g_1(x) = f(x)$ for $x \in V(T) - \{v_1, v_2, v_3, v_4\}$ is a $\gamma_{r2}(T)$-function that is not independent, a contradiction. Therefore $|f(v_3)| = 1$. Since $\gamma_{r2}(T) \equiv i_{r2}(T)$, f is an I2RDF of T and hence $f(v_4) = \emptyset$. This implies that neither v_4 is a strong support vertex nor v_4 has a support vertex of degree 2 in its neighborhood. If there is a path $v_4 y_3 y_2 y_1$ in T_4 where $y_3 \ne v_3$ and $\deg(y_1) = 1$, then by the choice of diametral path $v_1 \ldots v_k$, we have $|L_{v_2}| \ge |L_{y_2}|$ and $|L_{v_3}| \ge |L_{y_3}|$ that implies $\deg(y_2) = 2$ and $|L_{y_3}| = 0$. Hence, if there is a leaf at distance three from v_4 in T_{v_4}, then it plays the same role of v_1. Thus we may assume that each component of $T_{v_4} - v_4$ is isomorphic to P_3, $K_{1,t}$, $(t \ge 2)$ or a single vertex, where v_4 is adjacent to a leaf of each P_3, the center of $K_{1,t}$, or the single vertex, respectively.

Assume first that one of the components of $T_{v_4} - v_4$ is $K_{1,t}$, $(t \ge 2)$. That is, v_4 has a strong support vertex such as z in its neighborhood. Let $T' = T - \{v_1, v_2, v_3\}$ and let f be a $\gamma_{r2}(T)$-function. By Observation 2(a), we may assume $f(z) = \{1, 2\}$. Since $\gamma_{r2}(T) \equiv i_{r2}(T)$, f is a $i_{r2}(T)$-function and hence $|f(v_4)| = 0$. Then clearly

$|f(v_1)| + |f(v_2)| + |f(v_3)| = 2$ and $f|_{V(T')}$ is an I2RDF on T', implying that

$$\gamma_{r2}(T') \le i_{r2}(T') \le i_{r2}(T) - 2 = \gamma_{r2}(T) - 2. \tag{13}$$

On the other hand, let f_1 be a $\gamma_{r2}(T')$-function such that $f_1(z) = \{1, 2\}$ [Observation 2(a)]. We can extend f_1 to a 2RDF on T with weight $\gamma_{r2}(T') + 2$ by assigning $\{2\}$, $\emptyset$ and $\{1\}$ to v_3, v_2 and v_1, respectively. Hence $\gamma_{r2}(T) \le \gamma_{r2}(T') + 2$ and by (13), we have $\gamma_{r2}(T') = i_{r2}(T')$.

If there exists a $\gamma_{r2}(T')$-function h that is not independent, then as above we can extend h to a $\gamma_{r2}(T)$-function that is not independent, a contradiction with $\gamma_{r2}(T) \equiv i_{r2}(T)$. Thus $\gamma_{r2}(T') \equiv i_{r2}(T')$. It follows from inductive hypothesis that $T' \in \mathcal{F}$ and so T can be obtained from T' by Operation $\mathcal{O}_5$.

Now suppose that v_4 has no child which is a strong support vertex. We claim that $|L_{v_4}| \le 1$. Let to the contrary that $|L_{v_4}| \ge 2$. By Proposition A, T has a γ_{r2}-function f that $f(v_1) = \{1\}$ and $2 \in f(v_3)$. Since $|L_{v_4}| \ge 2$, we may assume $f(v_4) = \{1, 2\}$ which contradicts the assumption $\gamma_{r2}(T) \equiv i_{r2}(T)$. Hence $|L_{v_4}| \le 1$. Since $\deg(v_4) \ge 3$, we deduce that $T_{v_4} \in \mathcal{F}_2$. Let $T' = T - T_{v_4}$ and let g be a $\gamma_{r2}(T)$-function with $g(v_1) = \{1\}$ and $2 \in g(v_3)$. By assumption, g is an I2RDF of T and hence $g(v_4) = \emptyset$. Then $g|_{V(T')}$ is an I2RDF of T', implying that

$$\gamma_{r2}(T') \le i_{r2}(T') \le i_{r2}(T) - 2\deg(v_4) + 2 - t = \gamma_{r2}(T) - 2\deg(v_4) + 2 - t, \tag{14}$$

where t is number of leaves adjacent to v_4.

On the other hand, each $\gamma_{r2}(T')$-function f can be extended to a 2RDF of T by assigning $\{2\}$ to v_3, $\{1\}$ to v_1, each vertex of $N(v_4)\backslash(L_{v_4} \cup \{v_5, v_3\})$ and the leaf adjacent to v_4, if any, $\{2\}$ to every vertex in T_{v_4} at distance 3 from v_4 except v_1, and $\emptyset$ to the other vertices of T_{v_4}. It follows that $\gamma_{r2}(T') \ge \gamma_{r2}(T) - 2\deg(v_4) + 2 - t$. By (14) we obtain $\gamma_{r2}(T') = i_{r2}(T')$.

If h is a $\gamma_{r2}(T')$-function that is not independent, then we can easily extend h to a $\gamma_{r2}(T)$-function that is not independent, a contradiction with $\gamma_{r2}(T) \equiv i_{r2}(T)$. Thus $\gamma_{r2}(T') \equiv i_{r2}(T')$. By inductive hypothesis, we have $T' \in \mathcal{F}$. It can be easily seen that $\gamma_{r2}(T' - v_5) \ge \gamma_{r2}(T')$ if v_4 is a support vertex. Now T can be obtained from T' by Operation $\mathcal{O}_6$.

Case 3 $|L_{v_3}| = 1$.

Let w be the leaf adjacent to v_3. We consider the following subcases.

Subcase 3.1 $\deg(v_3) > 3$.

Then v_3 has a child $z_2 \ne v_2$ that is a support vertex of degree 2. Let z_1 be the leaf adjacent to z_2. Set $T' = T - \{v_1, v_2\}$. We show that $\gamma_{r2}(T') \equiv i_{r2}(T')$. Assume that f is a $\gamma_{r2}(T)$-function. We may assume that $f(v_1) = \{1\}$ and $2 \in f(v_3)$ by Proposition A. Clearly $|f(v_2)| = 0$. Since $\gamma_{r2}(T) \equiv i_{r2}(T)$, f is an I2RDF of T. Now $f|_{V(T')}$ is an I2RDF of T' of weight $\gamma_{r2}(T) - 1$ which implies that

$$\gamma_{r2}(T') \le i_{r2}(T') \le i_{r2}(T) - 1 = \gamma_{r2}(T) - 1. \tag{15}$$

On the other hand, if f_1 is a $\gamma_{r2}(T')$-function, then we may assume that $2 \in f_1(v_3)$ by Proposition A, and so f_1 can be extended to a 2RDF of T of weight $\gamma_{r2}(T') + 1$ by assigning $\emptyset$ to v_2 and $\{1\}$ to v_1, implying that $\gamma_{r2}(T) \leq \gamma_{r2}(T') + 1$. By (15) we obtain $\gamma_{r2}(T') = i_{r2}(T')$.

To show that this equality is strong, suppose h is a $\gamma_{r2}(T')$-function which is not independent. We may assume $|h(v_3)| > 0$, for otherwise we must have $|h(w)| = 1$ and $|h(z_2)| + |h(z_1)| = 2$ and the function $g : V(T') \to \mathcal{P}(\{1,2\})$ by $g(v_3) = \{1\}$, $g(z_2) = \emptyset$, $g(z_1) = g(w) = \{2\}$ and $g(x) = h(x)$ otherwise, is a $\gamma_{r2}(T')$-function with the desired property. Then we can easily extend h to a $\gamma_{r2}(T)$-function that is not independent, a contradiction with $\gamma_{r2}(T) \equiv i_{r2}(T)$. Thus $\gamma_{r2}(T') \equiv i_{r2}(T')$ and by inductive hypothesis, $T' \in \mathcal{F}$. Now T can be obtained from T' by Operation $\mathcal{O}_4$.

Subcase 3.2 $\deg(v_3) = 3$.

First let $\deg(v_4) \geq 3$. Let f be a $\gamma_{r2}(T)$-function. By Corollary 1, we may assume $f(v_3) = \{1,2\}$. Since $\gamma_{r2}(T) \equiv i_{r2}(T)$, f is an I2RDF of T. Then $|f(v_4)| = 0$ and $|f(v_1)| = 1$. If $1 \in \cup_{x \in N(v_4)-\{v_3\}} f(x)$ (the case $2 \in \cup_{x \in N(v_4)-\{v_3\}} f(x)$ is similar), then the function $f_1 : V(T) \to \mathcal{P}(\{1,2\})$ defined by $f_1(v_1) = f_1(w) = \{1\}$, $f_1(v_3) = \{2\}$, $f_1(v_2) = \emptyset$ and $f_1(x) = f(x)$ otherwise, is a $\gamma_{r2}(T)$-function which is not independent, a contradiction with $\gamma_{r2}(T) \equiv i_{r2}(T)$. Thus $|\cup_{x \in N[v_4]-\{v_3\}} f(x)| = 0$. This implies that v_4 has no child with depth 0 or 1. Assume that v_4 has a child z with depth 2. Then any leaf of T_z at distance two from z plays the same role of v_1, and thus by the previous arguments, we may assume that $T_z \simeq T_{v_3}$ and as above we can define a $\gamma_{r2}(T)$-function g such that $g(z) = g(v_3) = \{1,2\}$ which leads to a contradiction. Thus $\deg(v_4) = 2$. Suppose $T' = T - T_{v_4}$. We show that $i_{r2}(T') \equiv \gamma_{r2}(T')$. Let f be a $\gamma_{r2}(T)$-function that assigns $\{1,2\}$ to v_3 and $\emptyset$ to v_4, according to Corollary 1. Note that f is also an I2RDF of T because $i_{r2}(T) \equiv \gamma_{r2}(T)$. Then $f|_{V(T')}$ is an I2RDF on T', implying that

$$\gamma_{r2}(T') \leq i_{r2}(T') \leq i_{r2}(T) - 3 = \gamma_{r2}(T) - 3. \tag{16}$$

On the other hand, every $\gamma_{r2}(T')$-function can be extended to a 2RDF of T by assigning $\{1\}$ to v_1, $\emptyset$ to v_2, v_4, w and $\{1,2\}$ to v_3, and thus $\gamma_{r2}(T) \leq \gamma_{r2}(T') + 3$. It follows from (16) that $\gamma_{r2}(T') = i_{r2}(T')$.

If there is a $\gamma_{r2}(T')$-function g that is not independent then as above, we can extend it to a $\gamma_{r2}(T)$-function that is not independent, a contradiction. Thus $\gamma_{r2}(T') \equiv i_{r2}(T')$. By the inductive hypothesis, $T' \in \mathcal{F}$ and T can be obtained from T' by Operation $\mathcal{O}_7$ and the proof is completed. □

Now we are ready to state the main theorem of this paper.

Theorem 14 *Let T be a tree. Then $i_{r2}(T) \equiv \gamma_{r2}(T)$ if and only if $T \in \mathcal{F} \cup \{K_1\}$.*

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