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Measure Theory and Square-Mean Pseudo Almost Periodic and Automorphic Process: Application to Stochastic Evolution Equations

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Abstract In this work, we establish a new concept of square-mean pseudo almost periodic and automorphic processes using the measure theory. We use the $\mu-$ergodic process to define the spaces of $\mu-$pseudo almost periodic and automorphic processes in the square-mean sense. We establish many interesting results on those spaces like completeness and composition theorems. Then, we study the existence, the uniqueness, and the stability of the square-mean $\mu-$pseudo almost periodic and automorphic solutions of the stochastic evolution equation. We provide an example to illustrate ours results.

Keywords Measure theory · Ergodicity · $\mu-$Pseudo almost periodic solution · $\mu-$Pseudo almost automorphic solution · Completeness · Composition theorem · Stochastic processes · Stochastic evolution equations

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1 Introduction

The aim of this work is to study the basic properties of the square-mean μ-pseudo almost periodic and automorphic processes using the measure theory and used those results to study the following stochastic evolution equation. First, we study the existence of square-mean $\mu-$pseudo almost periodic and automorphic mild solutions to the following nonhomogeneous linear stochastic evolution equations in a Hilbert space H

$$\mathrm{d}x(t) = Ax(t)\mathrm{d}t + f(t)\mathrm{d}t + \gamma(t)\mathrm{d}W(t) \quad \text{for all} \quad t \in \mathbb{R}, \tag{1.1}$$

where $A : D(A) \subset H$ is the infinitesimal generator of a C_0-semigroup $(T(t))_{t\geq 0}$ on H, $f, \gamma : \mathbb{R} \to L^2(P, H)$ are two stochastic processes and $W(t)$ is a two-sided and standard one-dimensional Brownian motion defined on the filtered probability space $(\Omega, \mathcal{F}, P, \mathcal{F}_t)$ with $\mathcal{F}_t = \sigma\{W(u)-W(v)|u, v \leq t\}$. Second, we prove the uniqueness and the stability of square-mean μ-pseudo almost periodic and automorphic mild solutions for the following nonlinear stochastic evolution equations in a Hilbert space H

$$\mathrm{d}x(t) = Ax(t)\mathrm{d}t + h(t, x(t))\mathrm{d}t + \theta(t, x(t))\mathrm{d}W(t) \quad \text{for all} \quad t \in \mathbb{R}, \tag{1.2}$$

where $h, \theta : \mathbb{R} \times L^2(P, H) \to L^2(P, H)$ are two stochastic processes.

We assume $(H, \|\cdot\|)$ is real separable Hilbert space and $L^2(P, H)$ is the space of all H-valued random variables x such that

$$\mathbb{E}\|x\|^2 = \int_\Omega \|x\|^2 \mathrm{d}P < +\infty.$$

The concept of almost automorphic is a natural generalization of the almost periodicity that was introduced by Bochner [8]. For more details about the almost automorphic functions, we refer the reader to the book [24] where the author gave an important overview about the theory of almost automorphic functions and their applications to differential equations. In the last decade, many authors have produced extensive literature on the theory of almost automorphy and its applications to differential equations, more details can be found in [11,15–17,20–23,29–31] and the references therein. Then a generalization of almost automorphic functions gives pseudo almost automorphic functions. Also weighted pseudo almost automorphic functions which are more general than weighted pseudo almost periodic functions in [12,13] were introduced in first time in [7] by Blot et al. In [7], the authors used the basic properties of such functions to study the existence and uniqueness of weighted pseudo almost automorphic solutions for some abstract differential equations. We say that a continuous function f is $\rho-$weighted pseudo almost automorphic if

$$f = g + \phi,$$

where g is almost automorphic and ϕ is ergodic with respect to some weighted function ρ in the sense that

$$\lim_{r\to\infty}\frac{1}{m(r,\rho)}\int_{-r}^{r}\|\phi(t)\|\rho(t)\mathrm{d}t=0,$$

where $m(r,\rho)=\int_{-r}^{r}\rho(t)\mathrm{d}t$ and ρ is assumed to be positive and locally integrable. When the component g is almost periodic, then f is called weighted almost periodic. Likewise, in [5] the authors used the measure theory to define an ergodic function and give a new concept of μ–pseudo almost automorphic functions. We say that a continuous function f is μ–pseudo almost automorphic if

$$f=g+\phi,$$

where g is almost automorphic and ϕ is μ–ergodic in the sense that

$$\lim_{r\to\infty}\frac{1}{\mu([-r,r])}\int_{-r}^{r}\|\phi(t)\|\mathrm{d}\mu(t)=0,$$

where μ is a positive measure on $\mathbb{R}$, $\mu([-r,r])$ is the measure of the set $[-r,r]$. One can observe that a $\rho-$weighted pseudo almost automorphic function is μ–pseudo almost automorphic, where the measure μ is absolutely continuous with respect to the Lebesgue measure and its Radon–Nikodym derivative is ρ:

$$\frac{\mathrm{d}\mu(t)}{\mathrm{d}t}=\rho(t).$$

The authors investigated many interesting properties of such functions and study the existence and uniqueness of general evolution equations in the space of such functions. In recent years, the study of almost periodic or almost automorphic solutions to some stochastic differential equations have been considerably investigated in lots of publications [1–4,9,14,18,25,27] because of its significance and applications in physics, mechanics, and mathematical biology. Recently, in [19], the concept of square-mean almost automorphic stochastic processes was introduced by Fu and Liu and in [10], authors have studied the space of pseudo almost automorphic process in what they prove the existence of solution of Eq. (1.1) and the uniqueness and the stability of solution of Eq. (1.2) in the square-mean sense. In [10], the authors define the space SBC_0 by all stochastically bounded continuous processes $x(t)$ verifying

$$\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\mathbb{E}\|x(t)\|^2\mathrm{d}t=0.$$

In this paper, instead of the space SBC_0, we use the space of μ–ergodic process to define the square-mean μ–pseudo almost periodic and automorphic processes to study equations Eqs. (1.1) and (1.2). Let us to explain the meaning notion. We say that a continuous stochastic process f is μ–pseudo almost automorphic in the square-mean sense if

$$f=h+\theta,$$

where h is almost automorphic and θ is μ–ergodic in the sense that

$$\lim_{r\to\infty} \frac{1}{\mu([-r,r])} \int_{-r}^{r} \mathbb{E}\|\theta(t)\|^2 \mathrm{d}\mu(t) = 0,$$

where μ is a positive measure on $\mathbb{R}$, $\mu([-r,r])$ is the measure of the set $[-r,r]$. One can observe that a square-mean pseudo almost automorphic process is a square-mean μ–pseudo almost automorphic process in the particular case where the measure μ is the Lebesgue measure on $\mathbb{R}$.
For more details about the μ–ergodic functions, we can refer to [5,6] the papers of J. Blot, P. Cieutat, and K. Ezzinbi where they used the measure theory to define an ergodic function and investigate many interesting properties of such functions.
Now, the classical theory of square-mean pseudo almost automorphy becomes a particular case of our approach. In this work, we investigate many important results on the functional space of μ–square-mean pseudo almost periodic and automorphic processes, and we used them to study the general linear and nonlinear stochastic evolution equation.
The organization of this work is as follows. In Sect. 2, we define the square-mean μ–ergodic process and study some basic properties. In Sect. 3, we give the concepts of μ–pseudo almost periodic and automorphic process, and then we study some of their basic properties like completeness and composition theorems. In Sects. 4 and 5, we prove the existence of the square-mean μ–pseudo almost periodic and automorphic mild solutions of Eq. (1.1) and the uniqueness and the stability of the square-mean μ–pseudo almost periodic and automorphic mild solution of Eq. (1.2). In Sect. 6, we provide an example to illustrate our results.

2 Square-Mean μ–Ergodic Process

Let $\mathcal{B}$ be the Lebesgue σ-field of $\mathbb{R}$ and $\mathcal{M}$ be the set of all positive measures μ on $\mathcal{B}$ satisfying $\mu(\mathbb{R}) = +\infty$ and $\mu([a,b]) < +\infty$ for all $a, b \in \mathbb{R}$ $(a \leq b)$.
$L^2(P, H)$ is Hilbert space equipped with the following norm

$$\|x\|_{L^2} = \left(\int_\Omega \|x\|^2 \mathrm{d}P \right)^{\frac{1}{2}}.$$

Definition 2.1 Let $x : \mathbb{R} \to L^2(P, H)$ be a stochastic process.

(1) x is said to be stochastically bounded if there exists $M > 0$ such that

$$\mathbb{E}\|x(t)\|^2 \leq M \quad \text{for all} \quad t \in \mathbb{R}.$$

(2) x is said to be stochastically continuous if

$$\lim_{t\to s} \mathbb{E}\|x(t) - x(s)\|^2 = 0 \quad \text{for all} \quad s \in \mathbb{R}.$$

Denote by $SBC(\mathbb{R}, L^2(P, H))$, the space of all the stochastically bounded and continuous processes.

The space $SBC(\mathbb{R}, L^2(P, H))$ is a Banach space equipped with the following norm

$$\|x\|_\infty = \sup_{t\in\mathbb{R}} \left(\mathbb{E}\|x(t)\|^2\right)^{\frac{1}{2}}.$$

Definition 2.2 Let $\mu \in \mathcal{M}$. A stochastic process x is said to be square-mean μ–ergodic if $x \in SBC(\mathbb{R}, L^2(P, H))$ and satisfied

$$\lim_{r\to\infty} \frac{1}{\mu([-r, r])} \int_{-r}^{r} \mathbb{E}\|x(t)\|^2 \mathrm{d}\mu(t) = 0.$$

We denote the space of all such process by $\mathcal{E}(\mathbb{R}, L^2(P, H), \mu)$.

Proposition 2.1 *Let $\mu \in \mathcal{M}$. Then $\mathcal{E}(\mathbb{R}, L^2(P, H), \mu)$ is a Banach space with the norm $\|\cdot\|_\infty$.*

The proof is similar to the proof of Proposition 2.13 in [5].

Next result is a characterization of μ–ergodic processes in terms of measure μ.

Lemma 2.1 *Let $\mu \in \mathcal{M}$ and I be a bounded interval (eventually $I = \emptyset$). Assume that $f \in SBC(\mathbb{R}, L^2(P, H))$. Then the following assertions are equivalent:*

(i)

$$f \in \mathcal{E}(\mathbb{R}, L^2(P, H), \mu).$$

(ii)

$$\lim_{r\to\infty} \frac{1}{\mu([-r, r])\backslash I} \int_{[-r,r]\backslash I} \mathbb{E} \parallel f(t) \parallel^2 \mathrm{d}\mu(t) = 0.$$

(iii)

$$\textit{For any } \epsilon > 0, \quad \lim_{r\to+\infty} \frac{\mu\left\{t \in [-r, r]\backslash I : \mathbb{E} \parallel f(t) \parallel^2 > \epsilon\right\}}{\mu\{t \in [-r, r]\backslash I\}} = 0.$$

For the proof of the Lemma 2.1, we have just to use the same arguments in the proof of Theorem 2.14 in [5] with the function $t \mapsto \mathbb{E} \parallel f(t) \parallel^2$.

Example 2.1 Let ρ be a nonnegative $\mathcal{B}-$measure function and μ be the positive measure defined by

$$\mu(A) = \int_A \rho(t)\mathrm{d}t \quad \text{for} \quad A \in \mathcal{B}, \tag{2.1}$$

where dt denotes the Lebesgue measure on $\mathbb{R}$. The function ρ which occurs in (2.1) is called the Radon–Nikodym derivative of μ with respect to the Lebesgue measure on $\mathbb{R}$ [[26], p. 130]. In this case, $\mu \in \mathcal{M}$ if and only if its Radon–Nikodym derivative ρ is locally Lebesgue-integrable on $\mathbb{R}$ and

$$\int_{-\infty}^{+\infty} \rho(t)\mathrm{d}t = +\infty.$$

Definition 2.3 [5] Let μ_1 and $\mu_2 \in \mathcal{M}$. μ_1 is said to be equivalent to μ_2 ($\mu_1 \sim \mu_2$) if there exist constants α and $\beta > 0$ and a bounded interval I (eventually $I = \emptyset$) such that

$$\alpha\mu_1(A) \leq \mu_2(A) \leq \beta\mu_1(A)$$

for $A \in \mathcal{B}$ satisfying $A \cap I = \varnothing$.

Theorem 2.1 *Let μ_1 and $\mu_2 \in \mathcal{M}$. If μ_1 and μ_2 are equivalent. Then $\mathcal{E}(\mathbb{R}, L^2(P, H), \mu_1) = \mathcal{E}(\mathbb{R}, L^2(P, H), \mu_2)$.*

The proof is similar to the proof of Theorem 2.20 in [5].
For $\mu \in \mathcal{M}$ and $\tau \in \mathbb{R}$, we denote by μ_τ the positive measure on $(\mathbb{R}, \mathcal{B})$ defined by

$$\mu_\tau(A) = \mu(a + \tau : a \in A) \quad \text{for} \quad A \in \mathcal{B}.$$

From $\mu \in \mathcal{M}$, we formulate the following hypothesis.

(H) For all $\tau \in \mathbb{R}$, there exists $\beta > 0$ and a bounded interval I such that

$$\mu_\tau(A) \leq \beta\mu(A) \text{ when } A \in \mathcal{B} \text{ satisfies } A \cap I = \varnothing.$$

Lemma 2.2 [5] *Let $\mu \in \mathcal{M}$. Then μ satisfies **(H)** if and only if μ and μ_τ are equivalent for all $\tau \in \mathbb{R}$.*

Lemma 2.3 [5] *Hypothesis **(H)** implies that for all $\sigma > 0$,*

$$\limsup_{r \to +\infty} \frac{\mu([-r - \sigma, r + \sigma])}{\mu([-r, r])} < +\infty.$$

Remark 2.1 [5] For Example 2.1, Hypothesis **(H)** holds if and only if for all $\tau \in \mathbb{R}$, there exists a constant $\beta > 0$ and a bounded interval I such that

$$\rho(t + \tau) \leq \beta\rho(t) \quad \text{a.e} \quad \mathbb{R}\backslash I,$$

where μ is given by

$$\mu(t) = \rho(t)\mathrm{d}t,$$

and ρ satisfies the condition of Example 2.1. Then Hypothesis **(H)** is equivalent to say

$$\text{for all} \quad \tau \in \mathbb{R}, \quad \limsup_{|t|\to+\infty} \frac{\rho(t+\tau)}{\rho(t)} < +\infty.$$

Example 2.2

$$\rho(t) = \exp(-2t) \quad \text{for} \quad t \in \mathbb{R}.$$

In fact

$$\lim_{t\to+\infty} \frac{\rho(t+\tau)}{\rho(t)} = \exp(-2\tau) \quad \text{for} \quad \tau \in \mathbb{R}.$$

Let $f \in SBC(\mathbb{R}, L^2(P, H))$ and $\tau \in \mathbb{R}$. We denote by f_τ the function defined by $f_\tau(t) = f(t+\tau)$ for $t \in \mathbb{R}$.
A subset $\mathfrak{F}$ of $SBC(\mathbb{R}, L^2(P, H))$ is said to translation invariant if for all $f \in \mathfrak{F}$ we have $f_\tau \in \mathfrak{F}$ for all $\tau \in \mathbb{R}$.

Theorem 2.2 *Let $\mu \in \mathcal{M}$ satisfy **(H)**. Then $\mathcal{E}(\mathbb{R}, L^2(P, H), \mu)$ is translation invariant.*

The proof use the same arguments of the proof of Theorem 3.5 in [5] with the function $t \mapsto \mathbb{E} \parallel f(t) \parallel^2$.
Next, we use the results of this section to define the new concepts of μ–pseudo almost periodic and automorphic process in the square-mean sense.

3 Square-Mean μ–Pseudo Almost Periodic and Automorphic Process

In this section, we define the concepts of square-mean μ–pseudo almost periodic and automorphic process and study their basic properties.

3.1 μ–Pseudo Almost Periodic Process

Definition 3.1 [3] Let $x : \mathbb{R} \to L^2(P, H)$ be a continuous stochastic process. x is said be square-mean almost periodic process if for each $\epsilon > 0$ there exists $l > 0$ such that for all $\alpha \in \mathbb{R}$, there exists $\tau \in [\alpha, \alpha + l]$ with

$$\sup_{t\in\mathbb{R}} \mathbb{E}\|x(t+\tau) - x(t)\|^2 < \epsilon.$$

We denote the space off all such stochastic processes by $SAP(\mathbb{R}, L^2(P, H))$.

Theorem 3.1 [3] *$SAP(\mathbb{R}, L^2(P, H))$ equipped with the norm $\|\cdot\|_\infty$ is a Banach space.*

Definition 3.2 Let $\mu \in \mathcal{M}$ and $f : \mathbb{R} \to L^2(P, H)$ be a continuous stochastic process.
f is said be μ–square-mean pseudo almost periodic process if it can be decomposed as follows

$$f = g + \varphi,$$

where $g \in SAP(\mathbb{R}, L^2(P, H))$ and $\varphi \in \mathcal{E}(\mathbb{R}, L^2(P, H), \mu)$.

We denote the space off all such stochastic processes by $SPAP(\mathbb{R}, L^2(P, H), \mu)$. Then we have

$$SPAP(\mathbb{R}, L^2(P, H), \mu) \subset SBC(\mathbb{R}, L^2(P, H)).$$

Theorem 3.2 *Let $\mu \in \mathcal{M}$ satisfy **(H)**. Then $SPAP(\mathbb{R}, L^2(P, H), \mu)$ is translation invariant.*

Proof By using Theorem 2.2, we deduce that $\mathcal{E}(\mathbb{R}, L^2(P, H), \mu)$ is translation invariant. From the definition of the square-mean pseudo almost periodic process, we conclude that $SPAP(\mathbb{R}, L^2(P, H), \mu)$ is also translation invariant. □

Next, we study the completeness of the space of square-mean μ–pseudo almost periodic processes.

Theorem 3.3 *Let $\mu \in \mathcal{M}$ and $f \in SPAP(\mathbb{R}, L^2(P, H), \mu)$ be such that*

$$f = g + \varphi,$$

where $g \in SAP(\mathbb{R}, L^2(P, H))$ and $\varphi \in \mathcal{E}(\mathbb{R}, L^2(P, H), \mu)$. If $SPAP(\mathbb{R}, L^2(P, H), \mu)$ is translation invariant, then

$$\overline{\{f(t)|t \in \mathbb{R}\}} \supset \{g(t)|t \in \mathbb{R}\}.$$

For the proof, we use the same arguments of the proof of Theorem 2.24 in [6] by using the norm $\|\cdot\|_{L^2}$.

Theorem 3.4 *Let $\mu \in \mathcal{M}$. Assume that $SPAP(\mathbb{R}, L^2(P, H), \mu)$ is translation invariant. Then $SPAP(\mathbb{R}, L^2(P, H), \mu)$ is a Banach space with the norm $\|\cdot\|_\infty$.*

The proof use the similar arguments as performed in the proof of Corollary 2.31 in [6].
Next, we study the composition of the space square-mean μ–pseudo almost periodic process.

Definition 3.3 [3] Let $f : \mathbb{R} \times L^2(P, H) \to L^2(P, H), (t, x) \mapsto f(t, x)$ be continuous.

f is said be square-mean almost periodic in $t \in \mathbb{R}$ uniformly in $x \in L^2(P, H)$ if for all compact K of $L^2(P, H)$ and for any $\epsilon > 0$, there exists $l(\epsilon, K) > 0$ such that for all $\alpha \in \mathbb{R}$, there exists $\tau \in [\alpha, \alpha + l(\epsilon, K)]$ with

$$x \in K, \quad \sup_{t\in\mathbb{R}} \mathbb{E}\| f(t + \tau, x) - f(t, x)\|^2 < \epsilon.$$

We denote the following space of stochastic processes by

$g \in SAP(\mathbb{R} \times L^2(P, H), L^2(P, H))$ if $g : \mathbb{R} \times L^2(P, H) \to L^2(P, H)$ such that $g(\cdot, x) \in SAP(\mathbb{R}, L^2(P, H))$ for each $x \in L^2(P, H)$. Similarly

$\varphi \in \mathcal{E}(\mathbb{R} \times L^2(P, H), L^2(P, H), \mu)$ if $\varphi : \mathbb{R} \times L^2(P, H) \to L^2(P, H))$ such that $\varphi(\cdot, x) \in \mathcal{E}(\mathbb{R}, L^2(P, H), \mu)$ for each $x \in L^2(P, H)$.

Theorem 3.5 [3] *Let $f : \mathbb{R} \times L^2(P, H) \to L^2(P, H)$, $(t, x) \mapsto f(t, x)$ be a square almost periodic process in t uniformly in $x \in L^2(P, H)$. Suppose that f is Lipschitz in the following sense:*
there exists a positive number L such that for any $x, y \in L^2(P, H)$,

$$\mathbb{E}\| f(t, x) - f(t, y)\|^2 \le L \cdot \mathbb{E}\|x - y\|^2.$$

Then $t \mapsto f(t, x(t)) \in SAP(\mathbb{R}, L^2(P, H))$ for any $x \in SAP(\mathbb{R}, L^2(P, H))$.

Definition 3.4 Let $\mu \in \mathcal{M}$. A continuous function $f(t, x) : \mathbb{R} \times L^2(P, H) \to L^2(P, H)$ is said to be square-mean μ–pseudo almost periodic in t for any $x \in L^2(P, H)$ if it can be decomposed as $f = g + \varphi$, where $g \in SAP(\mathbb{R} \times L^2(P, H), L^2(P, H))$, $\varphi \in \mathcal{E}(\mathbb{R} \times L^2(P, H), L^2(P, H), \mu)$. Denote the set of all such stochastically continuous processes by $SPAP(\mathbb{R} \times L^2(P, H), L^2(P, H), \mu)$.

Theorem 3.6 *Let $\mu \in \mathcal{M}$ satisfy **(H)**. Suppose that $f \in SPAP(\mathbb{R} \times L^2(P, H), L^2(P, H), \mu)$ and that there exists a positive number L such that, for any $x, y \in L^2(P, H)$,*

$$\mathbb{E}\| f(t, x) - f(t, y)\|^2 \le L \cdot \mathbb{E}\|x - y\|^2$$

for $t \in \mathbb{R}$. Then $t \mapsto f(t, x(t)) \in SPAP(\mathbb{R}, L^2(P, H), \mu)$ for any $x \in SPAP(\mathbb{R}, L^2(P, H), \mu)$.

The theorem above can be proved by using Lemma 2.1 and Theorem 3.3 and Theorem 3.5. For more details about the proof, one may refer to [6].
Now, we study the square-mean μ–Pseudo almost automorphic processes that is a generalization of square-mean μ–Pseudo almost periodic processes.

3.2 μ–Pseudo Almost Automorphic Process

Definition 3.5 Let $x : \mathbb{R} \to L^2(P, H)$ be a continuous stochastic process. x is said be square-mean almost automorphic process if for every sequence of real numbers

$(t_n^{'})_n$, we can extract a subsequence $(t_n)_n$ such that, for some stochastic process $y : \mathbb{R} \to L^2(P, H)$, we have

$$\lim_{n\to+\infty} \mathbb{E}\,\|x(t+t_n) - y(t)\|^2 = 0 \quad \text{for all} \quad t \in \mathbb{R}$$

and

$$\lim_{n\to+\infty} \mathbb{E}\,\|y(t-t_n) - x(t)\|^2 = 0 \quad \text{for all} \quad t \in \mathbb{R}.$$

We denote the space off all such stochastic processes by $SAA(\mathbb{R}, L^2(P, H))$.

Theorem 3.7 [19] *$SAA(\mathbb{R}, L^2(P, H))$ equipped with the norm $\|\cdot\|_\infty$ is a Banach space.*

Definition 3.6 Let $\mu \in \mathcal{M}$ and $f : \mathbb{R} \to L^2(P, H)$ be a continuous stochastic process.
f is said be μ–square-mean pseudo almost automorphic process if it can be decomposed as follows

$$f = g + \varphi,$$

where $g \in SAA(\mathbb{R}, L^2(P, H))$ and $\varphi \in \mathcal{E}(\mathbb{R}, L^2(P, H), \mu)$.

We denote the space off all such stochastic processes by $SPAA(\mathbb{R}, L^2(P, H), \mu)$. Then we have

$$SPAA(\mathbb{R}, L^2(P, H), \mu) \subset SBC(\mathbb{R}, L^2(P, H)).$$

Remark 3.1 A square-mean pseudo almost automorphic process is a square-mean μ–pseudo almost automorphic process in the particular case where the measure μ is the Lebesgue measure on $\mathbb{R}$. For more details on the pseudo almost automorphic process we refer to [10].

Hence, together with Theorem 2.2 and Definition 3.6, we arrive at the following conclusion.

Theorem 3.8 *Let $\mu \in \mathcal{M}$ satisfy **(H)**. Then $SPAA(\mathbb{R}, L^2(P, H), \mu)$ is translation invariant.*

Theorem 3.9 *Let $\mu \in \mathcal{M}$ and $f \in SPAA(\mathbb{R}, L^2(P, H), \mu)$ be such that*

$$f = g + \varphi,$$

where $g \in SAA(\mathbb{R}, L^2(P, H))$ and $\varphi \in \mathcal{E}(\mathbb{R}, L^2(P, H), \mu)$. If $SPAA(\mathbb{R}, L^2(P, H), \mu)$ is translation invariant, then

$$\overline{\{f(t) | t \in \mathbb{R}\}} \supset \{g(t) | t \in \mathbb{R}\}.$$

The proof of Theorem 3.9 is similar to the proof of Theorem 4.1 in [5].

Theorem 3.10 *Let $\mu \in \mathcal{M}$. Assume that $SPAA(\mathbb{R}, L^2(P, H), \mu)$ is translation invariant. Then $SPAA(\mathbb{R}, L^2(P, H), \mu)$ is a Banach space with the norm $\| \cdot \|_\infty$.*

The proof of the theorem above is similar to the proof of Theorem 4.9 in [5].
Next, we study the composition of square-mean μ–pseudo almost automorphic processes that is a generalization of the result of composition of square-mean almost automorphic processes of Miao Miao Fu and Zhen Xin Liu [[19], Theorem 2.6]
We say that $g \in SAA(\mathbb{R} \times L^2(P, H), L^2(P, H)$ if $g : \mathbb{R} \times L^2(P, H) \to L^2(P, H))$ such that $g(\cdot, x) \in SAA(\mathbb{R}, L^2(P, H))$ for each $x \in L^2(P, H)$.

Definition 3.7 Let $\mu \in \mathcal{M}$. A continuous function $f(t, x) : \mathbb{R} \times L^2(P, H) \to L^2(P, H)$ is said to be square-mean μ–pseudo almost automorphic in t for any $x \in L^2(P, H)$ if it can be decomposed as $f = g + \varphi$, where $g \in SAA(\mathbb{R} \times L^2(P, H), L^2(P, H))$, $\varphi \in \mathcal{E}(\mathbb{R} \times L^2(P, H), L^2(P, H), \mu)$. Denote the set of all such stochastically continuous processes by $SPAA(\mathbb{R} \times L^2(P, H), L^2(P, H), \mu)$.

Theorem 3.11 *Let $\mu \in \mathcal{M}$ satisfy **(H)**. Suppose that $f \in SPAA(\mathbb{R} \times L^2(P, H), L^2(P, H), \mu)$ and that there exists a positive number L such that, for any $x, y \in L^2(P, H)$,*

$$\mathbb{E}\| f(t, x) - f(t, y)\|^2 \leq L \cdot \mathbb{E}\|x - y\|^2$$

for $t \in \mathbb{R}$. Then $t \mapsto f(t, x(t)) \in SPAA(\mathbb{R}, L^2(P, H), \mu)$ for any $x \in SPAA(\mathbb{R}, L^2(P, H), \mu)$.

Theorem 3.11 can be proved by using Lemma 2.1, Theorem 3.8, and Theorem 2.6 in [19]. More details about the proof can be found in [5].
Next Sections, we use the results of this section to study the stochastic evolution equations (1.1) and (1.2).

4 Square-Mean μ–Pseudo Almost Periodic Solutions Eq. (1.1)

This section is devoted to the existence of μ–pseudo almost periodic solutions on $\mathbb{R}$ of Eq. (1.1) and the uniqueness and the stability of the square-mean μ–pseudo almost periodic mild solution on $\mathbb{R}$ of Eq. (1.2) in the space $SPAP(\mathbb{R}, L^2(P, H), \mu)$, where $\mu \in \mathcal{M}$.
We assume that $(T(t))_{t\geq 0}$ is exponentially stable which means that there exist constants $K \geq 1$ and $\omega > 0$ such that

$$\|T(t)\| \leq K e^{-\omega t} \quad \text{for all} \quad t \geq 0. \tag{4.1}$$

Definition 4.1 An $\mathcal{F}_t$-progressively measurable process $\{x(t)\}_{t\in\mathbb{R}}$ is called a mild solution on $\mathbb{R}$ of Eq. (1.1) if it satisfies the corresponding stochastic integral equation

$$x(t) = T(t-a)x(a) + \int_a^t T(t-s) f(s) ds + \int_a^t T(t-s)\gamma(s) dW(s) \tag{4.2}$$

for any $t, a \in \mathbb{R}$ such that $t \geq a$.

Theorem 4.1 *Let* $\mu \in \mathcal{M}$ *satisfy* ***(H)****. Then* Eq. (1.1) *has a unique square-mean* μ*–pseudo almost periodic mild solution on* $\mathbb{R}$ *if* $f, \gamma \in SPAP(\mathbb{R}, L^2(P, H), \mu)$.

Proof Since the semigroup $(T(t))_{t\geq 0}$ is exponentially stable, then one can see that Eq. (1.1) has a unique bounded solution on $\mathbb{R}$ that is given by the following formula

$$x(t) = \int_{-\infty}^{t} T(t-s)f(s)\mathrm{d}s + \int_{-\infty}^{t} T(t-s)\gamma(s)\mathrm{d}W(s). \tag{4.3}$$

Since $f, \gamma \in SPAP(\mathbb{R}, L^2(P, H), \mu)$, there exist $g, \alpha \in SAP(\mathbb{R}, L^2(P, H))$ and $\varphi, \beta \in \mathcal{E}(\mathbb{R}, L^2(P, H), \mu)$ such that $f = g + \varphi$, $\gamma = \alpha + \beta$. Hence,

$$\begin{aligned} x(t) &= \int_{-\infty}^{t} T(t-s)[g(s)+\varphi(s)]\mathrm{d}(s) + \int_{-\infty}^{t} T(t-s)[\alpha(s)+\beta(s)]\mathrm{d}W(s) \\ &= \Big[\int_{-\infty}^{t} T(t-s)g(s)\mathrm{d}s + \int_{-\infty}^{t} T(t-s)\alpha(s)\mathrm{d}W(s)\Big] \\ &\quad + \Big[\int_{-\infty}^{t} T(t-s)\varphi(s)\mathrm{d}s + \int_{-\infty}^{t} T(t-s)\beta(s)\mathrm{d}W(s)\Big] \\ &\doteq F(t) + \Phi(t). \end{aligned}$$

The aim is to verify that $F \in SAP(\mathbb{R}, L^2(P, H))$ and $\Phi \in \mathcal{E}(\mathbb{R}, L^2(P, H), \mu)$.
We define

$$\begin{aligned} F(t) &= \int_{-\infty}^{t} T(t-s)g(s)\mathrm{d}s + \int_{-\infty}^{t} T(t-s)\alpha(s)\mathrm{d}W(s) \\ &:= G(t) + P(t). \end{aligned}$$

Using similar arguments as performed in the step 1 of Theorem 3.1 in [10], we obtain the continuity of G and P.
Since $g \in SAP(\mathbb{R}, L^2(P, H))$, then for each $\epsilon > 0$, there exists $l(\epsilon)$ such that for all $a \in \mathbb{R}$, there exists $\tau \in [a, a + l(\epsilon)]$ satisfying

$$\sup_{t\in\mathbb{R}} \mathbb{E}\|g(t+\tau) - g(t)\|^2 < \frac{\omega^2}{K^2}\epsilon.$$

Let $t \in \mathbb{R}$. Then we have

$$\begin{aligned} \mathbb{E}\|G(t+\tau) - G(t)\|^2 &= \mathbb{E}\Big\|\int_{-\infty}^{t+\tau} T(t+\tau-s)g(s)\mathrm{d}s - \int_{-\infty}^{t} T(t-s)g(s)\mathrm{d}s\Big\|^2 \\ &= \mathbb{E}\Big\|\int_{-\infty}^{t} T(t-s)g(s+\tau)\mathrm{d}s - \int_{-\infty}^{t} T(t-s)g(s)\mathrm{d}s\Big\|^2 \\ &= \mathbb{E}\Big\|\int_{-\infty}^{t} T(t-s)[g(s+\tau) - g(s)]\mathrm{d}s\Big\|^2 \end{aligned}$$

$$\leq K^2\mathbb{E}\left(\int_{-\infty}^{t} e^{-\omega(t-s)}\|g(s+\tau)-g(s)\|\mathrm{d}s\right)^2$$
$$\leq K^2\mathbb{E}\left(\int_{-\infty}^{t} (e^{\frac{-\omega(t-s)}{2}})(e^{\frac{-\omega(t-s)}{2}})\|g(s+\tau)-g(s)\|\mathrm{d}s\right)^2.$$

By the Cauchy-Schwartz's inequality, we obtain that

$$\begin{aligned}
&\mathbb{E}\|G(t+\tau)-G(t)\|^2\\
&\leq K^2\mathbb{E}\left[\left(\int_{-\infty}^{t} e^{-\omega(t-s)}\mathrm{d}s\right)\left(\int_{-\infty}^{t} e^{-\omega(t-s)}\|g(s+\tau)-g(s)\|^2\mathrm{d}s\right)\right]\\
&\leq K^2\left(\int_{-\infty}^{t} e^{-\omega(t-s)}\mathrm{d}s\right)\left(\int_{-\infty}^{t} e^{-\omega(t-s)}\mathbb{E}\|g(s+\tau)-g(s)\|^2\mathrm{d}s\right)\\
&\leq K^2\left(\int_{-\infty}^{t} e^{-\omega(t-s)}\mathrm{d}s\right)^2\sup_{s\in\mathbb{R}}\mathbb{E}\|g(s+\tau)-g(s)\|^2\\
&\leq \frac{K^2}{\omega^2}\sup_{s\in\mathbb{R}}\mathbb{E}\|g(s+\tau)-g(s)\|^2\\
&< \epsilon.
\end{aligned}$$

Hence $G \in SAP(\mathbb{R}, L^2(P, H))$. Since $\alpha \in SAP(\mathbb{R}, L^2(P, H))$, then for each $\epsilon > 0$, there exists $l(\epsilon)$ such that for all $a \in \mathbb{R}$, there exists $\tau \in [a, a+l(\epsilon)]$ satisfying

$$\sup_{t\in\mathbb{R}}\mathbb{E}\|\alpha(t+\tau)-\alpha(t)\|^2 < \frac{2\omega}{K^2}\epsilon.$$

Let $\tilde{W}(\sigma) = W(\sigma+\tau) - W(\tau)$ for each $\sigma \in \mathbb{R}$. Then $\tilde{W}$ is also a Wiener process having the same distribution as W. Letting $\sigma = s - \tau$ and using the Ito's isometry property of stochastic integral, we obtain the following estimation

$$\begin{aligned}
&\mathbb{E}\|P(t+\tau)-P(t)\|^2\\
&= \mathbb{E}\|\int_{-\infty}^{t+\tau} T(t+\tau-s)\alpha(s)\mathrm{d}W(s) - \int_{-\infty}^{t} T(t-s)\alpha(s)\mathrm{d}W(s)\|^2\\
&= \mathbb{E}\|\int_{-\infty}^{t} T(t-\sigma)\alpha(\sigma+\tau)d\tilde{W}(\sigma) - \int_{-\infty}^{t} T(t-s)\alpha(s)\mathrm{d}W(s)\|^2\\
&= \mathbb{E}\|\int_{-\infty}^{t} T(t-s)[\alpha(s+\tau)-\alpha(s)]\mathrm{d}W(s)\|^2\\
&= \mathbb{E}\left(\int_{-\infty}^{t} \|T(t-s)[\alpha(s+\tau)-\alpha(s)]\|^2\mathrm{d}s\right)\\
&\leq \int_{-\infty}^{t} \|T(t-s)\|^2\mathbb{E}\|\alpha(s+\tau)-\alpha(s)\|^2\mathrm{d}s
\end{aligned}$$

$$\leq \frac{K^2}{2\omega} \sup_{s\in\mathbb{R}} \mathbb{E}\|\alpha(s+\tau)-\alpha(s)\|^2$$
$$< \epsilon.$$

Therefore $P \in SAP(\mathbb{R}, L^2(P, H))$, hence $F \in SAP(\mathbb{R}, L^2(P, H))$.
Next, we have to show that $\Phi \in \mathcal{E}(\mathbb{R}, L^2(P, H), \mu)$. Using similar arguments as performed in the step 1 of Theorem 3.1 in [10], we obtain the continuity of Φ. We claim that

$$\lim_{n\to+\infty} \frac{1}{\mu([-r,r])} \int_{-r}^{r} \mathbb{E}\|\Phi(t)\|^2 \mathrm{d}\mu(t) = 0.$$

In fact, we have

$$\begin{aligned}
&\frac{1}{\mu([-r,r])} \int_{-r}^{r} \mathbb{E}\|\Phi(t)\|^2 \mathrm{d}\mu(t)\\
&\quad= \frac{1}{\mu([-r,r])} \int_{-r}^{r} \mathbb{E}\Big\| \int_{-\infty}^{t} T(t-s)\varphi(s)\mathrm{d}(s) + \int_{-\infty}^{t} T(t-s)\beta(s)\mathrm{d}W(s)\Big\|^2 \mathrm{d}\mu(t)\\
&\quad\leq \frac{2}{\mu([-r,r])} \int_{-r}^{r} \mathbb{E}\Big\| \int_{-\infty}^{t} T(t-s)\varphi(s)\mathrm{d}(s)\Big\|^2 d\mu(t)\\
&\qquad+ \frac{2}{\mu([-r,r])} \int_{-r}^{r} \mathbb{E}\Big\| \int_{-\infty}^{t} T(t-s)\beta(s)\mathrm{d}W(s)\Big\|^2 \mathrm{d}\mu(t)\\
&\quad\leq \frac{2}{\mu([-r,r])} \int_{-r}^{r} \mathrm{d}\mu(t)\mathbb{E}\left(\int_{-\infty}^{t} \|T(t-s)\|\|\varphi(s)\|\mathrm{d}s\right)^2\\
&\qquad+ \frac{2}{\mu([-r,r])} \int_{-r}^{r} \mathrm{d}\mu(t)\mathbb{E}\Big\| \int_{-\infty}^{t} T(t-s)\beta(s)\mathrm{d}W(s)\Big\|^2.
\end{aligned}$$

By the Cauchy-Schwartz's inequality and the Ito's isometry property of the stochastic integral, we have

$$\begin{aligned}
&\frac{1}{\mu([-r,r])} \int_{-r}^{r} \mathbb{E}\|\Phi(t)\|^2 \mathrm{d}\mu(t)\\
&\quad\leq \frac{2}{\mu([-r,r])} \int_{-r}^{r} \mathrm{d}\mu(t) \int_{-\infty}^{t} Ke^{-\omega(t-s)}\mathrm{d}s \int_{-\infty}^{t} Ke^{-\omega(t-s)}\mathbb{E}\|\varphi(s)\|^2\mathrm{d}s\\
&\qquad+ \frac{2}{\mu([-r,r])} \int_{-r}^{r} \mathrm{d}\mu(t) \int_{-\infty}^{t} K^2e^{-2\omega(t-s)}\mathbb{E}\|\beta(s)\|^2\mathrm{d}s\\
&\quad= \frac{2K}{\omega\mu([-r,r])} \int_{-r}^{r} \mathrm{d}\mu(t) \int_{-\infty}^{t} Ke^{-\omega(t-s)}\mathbb{E}\|\varphi(s)\|^2\mathrm{d}s\\
&\qquad+ \frac{2}{\mu([-r,r])} \int_{-r}^{r} \mathrm{d}\mu(t) \int_{-\infty}^{t} K^2e^{-2\omega(t-s)}\mathbb{E}\|\beta(s)\|^2\mathrm{d}s.
\end{aligned}$$

Using the Fubini's theorem, the first term on the right-hand side of the last equality satisfies the following

$$\begin{aligned}
&\frac{2K}{\omega\mu([-r,r])}\int_{-r}^{r}\mathrm{d}\mu(t)\int_{-\infty}^{t}Ke^{-\omega(t-s)}\mathbb{E}\|\varphi(s)\|^2\mathrm{d}s\\
&\quad=\frac{2K}{\omega\mu([-r,r])}\int_{-r}^{r}\mathrm{d}\mu(t)\int_{0}^{\infty}Ke^{-\omega u}\mathbb{E}\|\varphi(t-u)\|^2\mathrm{d}u \quad (\text{setting} \quad u=t-s)\\
&\quad=\frac{2K^2}{\omega}\int_{0}^{\infty}e^{-\omega u}\frac{1}{\mu([-r,r])}\int_{-r}^{r}\mathbb{E}\|\varphi(t-u)\|^2\mathrm{d}\mu(t)du.
\end{aligned}$$

Since, we have

$$|e^{-\omega u}\frac{1}{\mu([-r,r])}\int_{-r}^{r}\mathbb{E}\|\varphi(t-u)\|^2\mathrm{d}\mu(t)|\leq e^{-\omega u}\|\varphi\|_{\infty}^2$$

and

$$\int_{0}^{\infty}e^{-\omega u}\|\varphi\|_{\infty}^2\mathrm{d}u<\infty.$$

Then, using the Lebesgue dominated convergence theorem and the fact that $\mathcal{E}(\mathbb{R}, L^2(P, H), \mu)$ is translation invariant, we get

$$\lim_{r\to\infty}\frac{2K}{\omega\mu([-r,r])}\int_{-r}^{r}d\mu(t)\int_{-\infty}^{t}Ke^{-\omega(t-s)}\mathbb{E}\|\varphi(s)\|^2\mathrm{d}s=0.$$

We argue as above, we can show that

$$\lim_{r\to\infty}\frac{2}{\mu([-r,r])}\int_{-r}^{r}\mathrm{d}\mu(t)\int_{-\infty}^{t}K^2e^{-2\omega(t-s)}\mathbb{E}\|\beta(s)\|^2\mathrm{d}s=0.$$

Then,

$$\lim_{r\to\infty}\frac{1}{\mu([-r,r])}\int_{-r}^{r}\mathbb{E}\|\Phi(t)\|^2\mathrm{d}\mu(t)=0,$$

which implies that $\Phi\in\mathcal{E}(\mathbb{R}, L^2(P, H), \mu)$.

To end the prove, we have to show the uniqueness. Let u, v be two mild solutions to Eq. (1.1). Setting $w=u-v$, one can easily see that w is bounded and that $w(t)=T(t-s)w(s)$ for $t\geq s$. Now using (4.1), we can obtain that

$$\begin{aligned}
\mathbb{E}\|w(t)\|^2&=\mathbb{E}\|T(t-s)w(s)\|^2\\
&\leq K^2e^{-2\omega(t-s)}\mathbb{E}\|w(s)\|^2\\
&\leq K^2e^{-2\omega(t-s)}\|w\|_{\infty}
\end{aligned}$$

for all $t\geq s$.

Now let $(s_n)_n\in\mathbb{N}$ be a sequence of real numbers such that $s_n\to-\infty$ as $n\to\infty$. Clearly, for any fixed $t\in\mathbb{R}$, there exists a subsequence $(s_{n_k})_{k\in\mathbb{N}}$ of $(s_n)_{n\in\mathbb{N}}$ such that $s_{n_k}<t$ for all $k\in\mathbb{N}$. In view of the above, letting $k\to\infty$ yields $w(t)=u(t)-v(t)=0$. Therefore, $u=v$. This ends the proof. □

Next, we study the uniqueness of the bounded mild solution on $\mathbb{R}$ of Eq. (1.2).

Definition 4.2 An $\mathcal{F}_t$-progressively measurable process $\{x(t)\}_{t\in\mathbb{R}}$ is called a mild solution on $\mathbb{R}$ of Eq. (1.2) provided that it satisfies the corresponding stochastic integral equation

$$x(t) = T(t-a)x(a)+\int_a^t T(t-s)h(s, x(s))\mathrm{d}s+\int_a^t T(t-s)\theta(s, x(s))\mathrm{d}W(s) \quad (4.4)$$

for any $t, a \in \mathbb{R}$ such $t \geq a$.

Theorem 4.2 *Let $\mu \in \mathcal{M}$ satisfy **(H)**. We suppose that $h, \theta \in SPAP(\mathbb{R} \times L^2(P, H), L^2(P, H), \mu)$ and that there exist constants $L, L' > 0$ such that*

$$\mathbb{E} \parallel h(t, x) - h(t, y) \parallel^2 \leq L \cdot \mathbb{E} \parallel x - y \parallel^2$$
$$\mathbb{E} \parallel \theta(t, x) - \theta(t, y) \parallel^2 \leq L' \cdot \mathbb{E} \parallel x - y \parallel^2$$

for all $t \in \mathbb{R}$ and for any $x, y \in L^2(P, H)$.
If

$$\frac{2K^2L}{\omega^2} + \frac{K^2L'}{\omega} < 1,$$

then Eq. (1.2) *has a unique square-mean μ–pseudo almost periodic mild solution on* $\mathbb{R}$.

Proof By the definition 4.2, stochastic process $x : \mathbb{R} \to L^2(P, H)$ is a solution to Eq. (1.2) if and only if it satisfies the stochastic integral equation

$$x(t) = T(t-a)x(a) + \int_a^t T(t-s)h(s, x(s))\mathrm{d}s + \int_a^t T(t-s)\theta(s, x(s))\mathrm{d}W(s).$$

Note that $h(\cdot, x(\cdot)$ and $\theta(\cdot, x(\cdot)$ are L^2-bounded. Therefore, if we let $a \to -\infty$ in above integral equation then by the exponential dissipation condition of T, we obtain the that stochastic process $x : \mathbb{R} \to L^2(P, H)$ is a solution to Eq. (1.2) if and only if x satisfies the stochastic integral equation

$$x(t) = \int_{-\infty}^t T(t-s)h(s, x(s))\mathrm{d}s + \int_{-\infty}^t T(t-s)\theta(s, x(s))\mathrm{d}W(s). \quad (4.5)$$

Define the operator $\mathcal{S}$ by

$$(\mathcal{S}x)(t) = \int_{-\infty}^t T(t-s)h(s, x(s))\mathrm{d}s + \int_{-\infty}^t T(t-s)\theta(s, x(s))\mathrm{d}W(s),$$

for any $x \in SPAP(\mathbb{R}, L^2(P, H), \mu)$. Using Theorem 3.6, we deduce that $h(\cdot, x(\cdot), \theta(\cdot, x(\cdot) \in SPAP(\mathbb{R}, L^2(P, H), \mu)$ and from Theorem 4.1 we conclude that $\mathcal{S}$ is a self-mapping from the space $SPAP(\mathbb{R}, L^2(P, H), \mu)$ to the space $SPAP(\mathbb{R}, L^2(P, H), \mu)$. Now, we have to check that $\mathcal{S}$ is a strict contraction. For $x_1, x_2 \in SPAP(\mathbb{R}, L^2(P, H), \mu)$ and each $t \in \mathbb{R}$, we have

$$\begin{aligned}\mathbb{E}\|(\mathcal{S}x_1)(t) - (\mathcal{S}x_2)(t)\|^2 &= \mathbb{E}\|\int_{-\infty}^{t} T(t-s)[h(s, x_1(s)) - h(s, x_2(s))]\mathrm{d}s \\ &\quad + \int_{-\infty}^{t} T(t-s)[\theta(s, x_1(s)) - \theta(s, x_2(s))]\mathrm{d}W(s)\|^2 \\ &\leq 2K^2\mathbb{E}\left(\int_{-\infty}^{t} e^{-\omega(t-s)}\|h(s, x_1(s)) - h(s, x_2(s))\|\mathrm{d}s\right)^2 \\ &\quad + 2\mathbb{E}\|\int_{-\infty}^{t} T(t-s)[\theta(s, x_1(s)) - \theta(s, x_2(s))]\mathrm{d}W(s)\|^2.\end{aligned}$$

We estimate the first term of the right-hand side by using Cauchy-Schwartz's inequality as follows:

$$\begin{aligned}&\mathbb{E}\left(\int_{-\infty}^{t} e^{-\omega(t-s)}\|h(s, x_1(s)) - h(s, x_2(s))\|\mathrm{d}s\right)^2 \\ &= \mathbb{E}\left(\int_{-\infty}^{t} (e^{\frac{-\omega(t-s)}{2}})(e^{\frac{-\omega(t-s)}{2}})\|h(s, x_1(s)) - h(s, x_2(s))\|\mathrm{d}s\right)^2 \\ &\leq \mathbb{E}\left[\left(\int_{-\infty}^{t} e^{-\omega(t-s)}\mathrm{d}s\right)\left(\int_{-\infty}^{t} e^{-\omega(t-s)}\|h(s, x_1(s)) - h(s, x_2(s))\|^2\mathrm{d}s\right)\right] \\ &\leq \left(\int_{-\infty}^{t} e^{-\omega(t-s)}\mathrm{d}s\right)\left(\int_{-\infty}^{t} e^{-\omega(t-s)}\mathbb{E}\|h(s, x_1(s)) - h(s, x_2(s))\|^2\mathrm{d}s\right) \\ &\leq L\cdot\left(\int_{-\infty}^{t} e^{-\omega(t-s)}\mathrm{d}s\right)\left(\int_{-\infty}^{t} e^{-\omega(t-s)}\mathbb{E}\|x_1(s) - x_2(s)\|^2\mathrm{d}s\right) \\ &\leq L\cdot\left(\int_{-\infty}^{t} e^{-\omega(t-s)}\mathrm{d}s\right)^2 \sup_{s\in\mathbb{R}}\mathbb{E}\|x_1(s) - x_2(s)\|^2 \\ &\leq \frac{L}{\omega^2}\sup_{s\in\mathbb{R}}\mathbb{E}\|x_1(s) - x_2(s)\|^2.\end{aligned}$$

Using Ito's isometry property of stochastic integral, we obtain that

$$\begin{aligned}&\mathbb{E}\|\int_{-\infty}^{t} T(t-s)[\theta(s, x_1(s)) - \theta(s, x_2(s))]\mathrm{d}W(s)\|^2 \\ &= \mathbb{E}[\int_{-\infty}^{t} \|T(t-s)[\theta(s, x_1(s)) - \theta(s, x_2(s))]\|^2\mathrm{d}s \\ &\leq \mathbb{E}[\int_{-\infty}^{t} \|T(t-s)\|^2\|\theta(s, x_1(s)) - \theta(s, x_2(s))\|^2\mathrm{d}s \\ &\leq K^2\int_{-\infty}^{t} e^{-2\omega(t-s)}\mathbb{E}\|\theta(s, x_1(s)) - \theta(s, x_2(s))\|^2\mathrm{d}s\end{aligned}$$

$$\leq K^2L'\left(\int_{-\infty}^{t} e^{-2\omega(t-s)}\mathrm{d}s\right)\sup_{s\in\mathbb{R}}\mathbb{E}\|x_1(s)) - x_2(s)\|^2$$
$$\leq \frac{K^2L'}{2\omega}\sup_{s\in\mathbb{R}}\mathbb{E}\|x_1(s) - x_2(s)\|^2.$$

Thus, it follows that for each $t \in \mathbb{R}$

$$\mathbb{E}\|(\mathcal{S}x_1)(t) - (\mathcal{S}x_2)(t)\|^2 \leq \left(\frac{2K^2L}{\omega^2} + \frac{K^2L'}{\omega}\right)\sup_{s\in\mathbb{R}}\mathbb{E}\|x_1(s) - x_2(s)\|^2$$

that is,

$$\left\|(\mathcal{S}x_1)(t) - (\mathcal{S}x_2)(t)\right\|_{L^2}^2 \leq N\sup_{s\in\mathbb{R}}\left\|x_1(s) - x_2(s)\right\|_{L^2}^2 \leq N\left(\sup_{s\in\mathbb{R}}\left\|x_1(s) - x_2(s)\right\|_{L^2}\right)^2,$$

with $N := \frac{2K^2L}{\omega^2} + \frac{K^2L'}{\omega}$.
Hence

$$\|(\mathcal{S}x_1)(t) - (\mathcal{S}x_2)(t)\|_{L^2} \leq \sqrt{N}\|x_1 - x_2\|_\infty.$$

That implies

$$\|\mathcal{S}x_1 - \mathcal{S}x_2\|_\infty = \sup_{t\in\mathbb{R}}\|(\mathcal{S}x_1)(t) - (\mathcal{S}x_2)(t)\|_{L^2} \leq \sqrt{N}\|x_1 - x_2\|_\infty.$$

Since $N < 1$, it follows $\mathcal{S}$ is a contraction. Therefore by Banach point fixed, there exists a unique fixed point x such that $\mathcal{S}x = x$ which satisfies Eq. (1.2). The proof is complete. □

Now, we investigate the stability of the unique square-mean μ–pseudo almost periodic mild solution on $\mathbb{R}$ of Eq. (1.2). Let us recalling the definition of asymptotic stability.

Definition 4.3 [19] A solution x^* of Eq. (1.2) is said to be stable in square-mean sense, if for arbitrary $\epsilon > 0$, there exists $\delta > 0$ such that

$$\mathbb{E}\left\|x^*(t) - x(t)\right\|^2 < \epsilon, \quad t \geq 0$$

whenever $\mathbb{E}\left\|x^*(0) - x_0\right\|^2 < \delta$, where x is the solution of Eq. (1.1) with initial condition x_0. The solution x^* of Eq. (1.2) is said to be globally asymptotically stable in the square-mean sense if it is stable in square-mean sense and

$$\lim_{t\to\infty}\mathbb{E}\left\|x^*(t) - x(t)\right\|^2 = 0$$

holds for any $x_0 \in L^2(P, H)$.

Remark 4.1 Let $x_0 \in L^2(P, H)$. Then under the assumptions of Theorem 4.2, it easy to show that the solution of Eq. (1.2) with initial value x_0 exists for $t \in [0, +\infty)$.

Theorem 4.3 *Under the assumptions of Theorem 4.2, if*

$$\frac{L}{\omega^2} + \frac{L'}{\omega} < \frac{1}{3K^2},$$

then the unique square-mean μ–pseudo almost periodic mild solution on $\mathbb{R}$ of Eq. (1.2) *is is globally asymptotically stable.*

Proof Let x be a mild solution of Eq. (1.2) with initial value $x(0)$ on $[0, \infty)$. Then, for $t \geq 0$, we have

$$\begin{aligned}\mathbb{E}\|x(t) - x^*(t)\|^2 = \mathbb{E}\|T(t)(x(0) - x^*(0)) \\ + \int_0^t T(t-s)\left[h(s, x(s)) - h\left(s, x^*(s)\right)\right] \mathrm{d}s \\ + \int_0^t T(t-s)\left[\theta(s, x(s)) - \theta\left(s, x^*(s)\right)\right] \mathrm{d}W(s)\|^2\end{aligned}$$

Thus using the Cauchy-Schwartz's inequality, the Ito's isometry property of the stochastic integral and the Fubini's theorem, we have

$$\begin{aligned}\mathbb{E}\|x(t) - x^*(t)\|^2 \leq 3\|T(t)\|^2\mathbb{E}\|x(0) - x^*(0)\|^2 \\ + 3\mathbb{E}\left(\int_0^t \|T(t-s)\left[h(s, x(s)) - h\left(s, x^*(s)\right)\right]\|\mathrm{d}s\right)^2 \\ + 3\mathbb{E}\left(\int_0^t \|T(t-s)\left[\theta(s, x(s)) - \theta\left(s, x^*(s)\right)\right]\|^2\mathrm{d}s\right).\end{aligned}$$

It follows that

$$\begin{aligned}\mathbb{E}\|x(t) - x^*(t)\|^2 \leq 3K^2e^{-2\omega t}\mathbb{E}\|x(0) - x^*(0)\|^2 \\ + 3L\int_0^t Ke^{-\omega(t-s)}\mathrm{d}s \int_0^t Ke^{-\omega(t-s)}\mathbb{E}\|x(s) - x^*(s)\|^2\mathrm{d}s \\ + 3L'\int_0^t K^2e^{-2\omega(t-s)}\mathbb{E}\|x(s) - x^*(s)\|^2\mathrm{d}s.\end{aligned}$$

Consequently, we obtain that

$$\begin{aligned}\mathbb{E}\|x(t) - x^*(t)\|^2 \leq 3K^2e^{-2\omega t}\mathbb{E}\|x(0) - x^*(0)\|^2 \\ + \frac{3K^2L}{\omega}\int_0^t e^{-\omega(t-s)}\mathbb{E}\|x(s) - x^*(s)\|^2\mathrm{d}s \\ + 3K^2L'\int_0^t e^{-2\omega(t-s)}\mathbb{E}\|x(s) - x^*(s)\|^2\mathrm{d}s. \qquad (4.6)\end{aligned}$$

Let $X(t) := \mathbb{E}\left\|x(t)-x^*(t)\right\|^2$ and $k = \frac{3K^2L}{\omega}+3K^2L'$. Since for $t \geq 0, e^{-2\omega t} \leq e^{-\omega t}$. Then by (4.6), we obtain

$$X(t) \leq 3K^2 e^{-\omega t} X(0) + k \int_0^t e^{-\omega(t-s)} X(s)\mathrm{d}s.$$

Hence, we get

$$e^{-\omega t} X(t) \leq 5K^2 X(0) + k \int_0^t e^{\omega s} X(s) ds.$$

Then using Gronvall Lemma, we obtain that

$$X(t) \leq 5K^2 X(0) e^{(k-\omega)t}.$$

That implies that $X(t) \to 0$ exponentially fast if $k - \omega < 0$, that is

$$\frac{L}{\omega^2} + \frac{L'}{\omega} < \frac{1}{3K^2}.$$

Therefore x^* is globally asymptotically stable in the square-mean sense. This completes the proof. □

5 Square-Mean μ–Pseudo Almost Automorphic Solutions

This section is devoted to the existence of μ–pseudo almost automorphic solutions on $\mathbb{R}$ of Eq. (1.1) and the uniqueness and the stability of the square-mean μ–pseudo almost automorphic mild solution on $\mathbb{R}$ of Eq. (1.2) in the space $SPAA(\mathbb{R}, L^2(P, H), \mu)$, where $\mu \in \mathcal{M}$.

Theorem 5.1 *Let $\mu \in \mathcal{M}$ satisfy **(H)**. Then* Eq. (1.1) *has a square-mean μ–pseudo almost automorphic mild solution on $\mathbb{R}$ if $f, \gamma \in SPAA(\mathbb{R}, L^2(P, H), \mu)$.*

Proof We reproduce the proof of Theorem 4.1 in what we take $g, \alpha \in SAA(\mathbb{R}, L^2(P, H))$. Then, we only need to verify that $F \in SAA(\mathbb{R}, L^2(P, H))$. Since it is proven in [10] that $F \in SAA(\mathbb{R}, L^2(P, H))$, then the mild solution of Eq. (1.1) is square-mean μ–pseudo almost automorphic. □

Next, we study the uniqueness of square-mean μ–pseudo almost automorphic mild solution of (1.2).

Theorem 5.2 *Let $\mu \in \mathcal{M}$ satisfy **(H)**. We suppose that $h, \theta \in SPAA(\mathbb{R} \times L^2(P, H), L^2(P, H), \mu)$, and that there exist constants $L, L' > 0$ such that*

$$\mathbb{E} \parallel h(t, x) - h(t, y) \parallel^2 \leq L \cdot \mathbb{E} \parallel x - y \parallel^2$$
$$\mathbb{E} \parallel \theta(t, x) - \theta(t, y) \parallel^2 \leq L' \cdot \mathbb{E} \parallel x - y \parallel^2$$

for all $t \in \mathbb{R}$ and for any $x, y \in L^2(P, H)$.

If

$$\frac{2K^2L}{\omega^2} + \frac{K^2L'}{\omega} < 1,$$

then Eq. (1.2) *has a unique square-mean* μ*–pseudo almost automorphic mild solution on* $\mathbb{R}$.

The arguments of the proof of Theorem 5.2 are the same as performed in the proof of Theorem 4.2.

Theorem 5.3 *Under the assumptions of Theorem* 5.2*, if*

$$\frac{L}{\omega^2} + \frac{L'}{\omega} < \frac{1}{3K^2},$$

then the unique square-mean μ*–pseudo almost automorphic mild solution on* $\mathbb{R}$ *of* Eq. (1.2) *is is globally asymptotically stable.*

The proof uses the same arguments performed in the proof of Theorem 4.3.

6 Example

To apply our theoretical results, we consider the measure μ where its Radon-Nikodym derivative is

$$\rho(t) = \exp(-2t) \quad \text{for} \quad t \in \mathbb{R}.$$

Then $\mu \in \mathcal{M}$ and μ satisfies **(H)** (cf. Example 2.2). Let consider the following one-dimensional stochastic heat equation with the Dirichlet boundary conditions:

$$\begin{cases} du(t,x) = \frac{\partial^2 u(t,x)}{\partial x^2}\mathrm{d}t + \frac{\sqrt{l}}{4}\left[u(t,x)\sin\frac{1}{2+\cos t+\cos\sqrt{2}t} + e^t\cos u(t,x)\right]\mathrm{d}t \\ \qquad + \frac{\sqrt{l}}{4}\left[u(t,x)\sin\frac{1}{2+\cos t+\cos\sqrt{3}t} + e^t\sin u(t,x)\right]\mathrm{d}W(t), \\ (t,x) \in \mathbb{R}\times(0,1), \\ u(t,0) = u(t,1) = 0, \qquad t \in \mathbb{R}, \end{cases} \tag{6.1}$$

where W(t) is a two-sided and standard one-dimensional Brownian motion defined on the filtered probability space $(\Omega, \mathcal{F}, P, \mathcal{F}_t)$ with $\mathcal{F}_t = \sigma\{W(w) - W(v) | w, v \leq t\}$. with $l > 0$.
System (6.1) takes the abstract form (1.2),

$$\mathrm{d}u(t) = Au(t)\mathrm{d}t + h(t,u)\mathrm{d}t + \theta(t,u)\mathrm{d}W(t) \quad \text{for all} \quad t \in \mathbb{R} \tag{6.2}$$

where

$$h(t,u)(x) = h(t,u(t)(x)) = \frac{\sqrt{l}}{4}\left[u(t)(x)\sin\frac{1}{2+\cos t+\cos\sqrt{2}t} + e^t\cos u(t)(x)\right],$$
$$\theta(t,u)(x) = \theta(t,u(t)(x)) = \frac{\sqrt{l}}{4}\left[u(t)(x)\sin\frac{1}{2+\cos t+\cos\sqrt{3}t} + e^t\sin u(t)(x)\right],$$

and

$$\mathcal{D}(A) = \mathbf{H}^2((0,1)) \cap \mathbf{H}_0^1((0,1)),$$
$$Ax(\xi) = x''(\xi) \quad \text{for} \quad \xi \in (0,1) \quad \text{and} \quad x \in D(A).$$

Then, A generates a C_0-semigroup $(T(t))_{t\geq 0}$ on $L^2[0,1]$ given by

$$(T(t)x)(r) = \sum_{n=1}^{\infty} e^{-n^2\pi^2 t}\langle x, e_n\rangle_{L^2} e_n(r),$$

where $e_n(r) = \sqrt{2}\sin(n\pi r)$ for $n = 1, 2, ...$, and $\|T(t)\| \leq e^{-\pi^2 t}$ for all $t \geq 0$. Thus, set $K = 1$ and $\omega = \pi^2$.
$u\sin\frac{1}{2+\cos t+\cos\sqrt{2}t} + e^t\cos u$ belongs to $SPAA(\mathbb{R} \times L^2(P, L^2[0,1]), L^2(P, L^2[0,1]), \mu)$,
where $u\sin\frac{1}{2+\cos t+\cos\sqrt{2}t}$ is the almost automorphic component and $e^t\cos u$ is the μ–ergodic perturbation, since

$$\begin{aligned}\frac{1}{\mu([-r,r])}\int_{-r}^{r}\mathbb{E}\|e^t\cos u\|^2 \mathrm{d}\mu(t) &\leq \frac{-2}{e^{2r}(e^{-4r}-1)}\int_{-r}^{r} e^{2t}e^{-2t}\mathrm{d}t\\ &= \frac{-2}{e^{2r}(e^{-4r}-1)}\int_{-r}^{r}\mathrm{d}t\\ &= \frac{-4r}{e^{2r}(e^{-4r}-1)} \to 0 \quad \text{as} \quad r \to +\infty.\end{aligned}$$

Then $(t,u) \mapsto h(t,u) \in SPAA(\mathbb{R} \times L^2(P, L^2[0,1]), L^2(P, L^2[0,1]), \mu)$. By analogous argument performed above, we have also $(t,u) \mapsto \theta(t,u) \in SPAA(\mathbb{R} \times L^2(P, L^2[0,1]), L^2(P, L^2[0,1]), \mu)$. Clearly, h and θ satisfy the Lipschitz condition in Theorem 5.3 with $L = L' = l$, $K = 1$ and $\omega = \pi^2$. Since

$$l < \frac{1}{3K^2}\left(\frac{1}{\omega^2} + \frac{1}{\omega}\right)^{-1} = \frac{1}{3\pi^2}\left(\frac{1}{\pi^2} + 1\right)^{-1}.$$

Therefore, by Theorems 5.2 and 5.3, the corresponding equation (6.2) has a unique square-mean μ–pseudo almost automorphic mild solution which is globally asymptotically stable.

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