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Positive Solutions of Singular Fractional Boundary Value Problem with p-Laplacian

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Abstract In this paper, we present some new results concerning positive solutions for the singular fractional boundary value problem with p-Laplacian. By imposing some suitable conditions on the nonlinear term f, existence results of positive solutions are obtained. The proof is based upon theory of Leray–Schauder degree. The interesting point is the nonlinear term $f(t, u)$ may be singular at $u = 0$.

Keywords Singular problem · Fractional differential equation · Positive solutions · p-Laplacian

Mathematics Subject Classification 34A08

1 Introduction

The present paper is aimed at a study of the singular fractional boundary value problem with p-Laplacian

$$(\phi_p(D_{0+}^{\alpha}u(t)))' + f(t, u(t)) = 0, \quad 0 < t < 1, \tag{1}$$

$$u'(0) = 0, \quad u(1) - \gamma u(\eta) = 0, \tag{2}$$

where $\phi_p(s) = |s|^{p-2}s$, $p > 1$, $\gamma, \eta \in (0, 1)$, $1 < \alpha \leq 2$, D_{0+}^{α} is the Caputo fractional derivative, and $f(t, u)$ may be singular at $u = 0$. The singular boundary value

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problems have proved to be valuable tools in the modeling of many problems in mathematics and physics such as gas dynamics, chemical reactions, nuclear physics, atomic calculations, and the studies of atomic structures. Because of the physical interests, the singular problems have received a great attention in recent years. Initially, most papers focused on the singular integer order boundary value problem, we refer the reader to [1,4,9–11,13–16]. More recently, fractional differential equations have gained importance due to their applications in various sciences, several authors begin to consider the singular fractional boundary value problem, see [5,6,19] and the references therein. Most of the existing results focused on the fractional boundary value problems with time singularities. However, there are a few papers [2,3,18,20] considering fractional boundary value problems with nonlinearities having singularities in space variables.

In [18], the authors established the existence of positive solutions for the singular fractional boundary value problem

$$\begin{aligned} D_{0+}^{\alpha} u(t) &= f(t, u(t)), \quad 0 < t < 1, \\ u(0) = u(1) &= u'(0) = u'(1) = 0, \end{aligned}$$

where $3 < \alpha \leq 4,\ f(t, u)$ may be singular at $u = 0$.

In [2], the authors investigated positive solutions for the singular fractional boundary value problem

$$\begin{aligned} D^{\alpha} u(t) + f(t, u(t), D^{\mu} u(t)) &= 0, \quad 0 < t < 1, \\ u(0) = u(1) &= 0, \end{aligned}$$

where $1 < \alpha < 2,\ \mu > 0$ are real numbers, $\alpha - \mu \geq 1,\ f(t, x, y)$ is singular at $x = 0$.

In [7,8,12], fractional boundary value problems with p-Laplacian operators have been studied. However, there are very few papers discussing the singular fractional boundary value problems with p-Laplacian.

In [17], using the upper and lower solutions method, the authors got the positive solutions for the following singular fractional boundary value problems with p-Laplacian

$$\begin{aligned} D_{0+}^{\gamma}(\phi_p(D_{0+}^{\alpha} u(t))) &= f(t, u(t)), \quad 0 < t < 1, \\ u(0) = 0,\ u(1) = a u(\xi),\ D_{0+}^{\alpha} u(0) &= 0,\ D_{0+}^{\alpha} u(1) = b D_{0+}^{\alpha} u(\eta), \end{aligned}$$

where $1 < \alpha, \gamma \leq 2,\ 0 \leq a, b \leq 1,\ 0 < \xi, \eta < 1,\ f$ can be singular at $u = 0$.

To our knowledge, there is not a paper in the literature which considers the fractional boundary value problem (1)–(2) [both integer-order derivative and Caputo's fractional-order derivative are included in the Eq. (1)]. The existing literature about the singular boundary value problem discussed directly about the differential equation, then integrated the equation and used the techniques of inequalities. While this classic methods are not applicable to the mixed order (both integer-order derivative and Caputo's fractional-order derivative are included in the equation) Eq. (1). In this paper, by studying the properties of solutions of the fractional boundary value problem (1)–(2) and using the techniques of inequalities, we obtained the existing results.

2 The Preliminary Lemmas

Definition 2.1 The Riemann-Liouville fractional integral of order $\alpha > 0$ for a function $f:\ (0, +\infty) \to R$ is defined by

$$I_{0+}^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)ds,$$

provided that the right-hand side is pointwise defined on $(0, +\infty)$.

Definition 2.2 The Caputo derivative of order $\alpha > 0$ for a function $f:\ (0, +\infty) \to R$ is given by

$$D_{0+}^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{f^{(n)}(s)ds}{(t-s)^{\alpha+1-n}},$$

where $n = [\alpha] + 1$ and $[\alpha]$ means the integer part of α.

Lemma 2.1 *Let $\alpha > 0$ and $u \in C(0,1) \cap L'(0,1)$. Suppose that*

$$D_{0+}^{\alpha} u(t) = 0,$$

then

$$u(t) = c_1 + c_2 t + c_3 t^2 + \cdots + c_n t^{n-1},$$

where $c_i \in R,\ \ i = 1, 2, \cdots, n$ and $n = [\alpha] + 1$ is the unique solution for the above fractional differential equation.

Lemma 2.2 *Let $\alpha > 0$, then*

$$I_{0+}^{\alpha} D_{0+}^{\alpha} u(t) = u(t) + c_1 + c_2 t + c_3 t^2 + \cdots + c_n t^{n-1}$$

for some $c_i \in R,\ \ i = 1, 2, \cdots, n$, and $n = [\alpha] + 1$.

We shall consider the Banach space $E = C[0,1]$ equipped with maximum norm

$$||u|| = \max_{0 \le t \le 1} |u(t)|.$$

We suppose $F:\ [0,1] \times R \to (0, +\infty)$ is continuous.

Lemma 2.3 *For any $x \in E$, the following boundary value problem*

$$(\phi_p(D_{0+}^{\alpha} u(t)))' + F(t, x(t)) = 0, \quad 0 < t < 1, \tag{3}$$

$$u'(0) = a, \ \ u(1) - \gamma u(\eta) = a, \tag{4}$$

has a unique solution

$$u(t) = \int_0^1 G(t,s)\phi_q\left(\int_0^s F(\tau, x(\tau))d\tau\right)ds$$
$$+\frac{\gamma}{1-\gamma}\int_0^1 G(\eta,s)\phi_q\left(\int_0^s F(\tau, x(\tau))d\tau\right)ds + \frac{a\eta\gamma}{1-\gamma} + at, \tag{5}$$

where a is a fixed positive constant and

$$G(t,s) = \begin{cases} \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} - \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}, & 0 \le s \le t \le 1, \\ \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}, & 0 \le t \le s \le 1. \end{cases} \tag{6}$$

Proof Integrating both sides of Eq. (3) on [0, t], we have

$$\phi_p(D_{0+}^{\alpha}u(t)) - \phi_p(D_{0+}^{\alpha}u(0)) = -\int_0^t F(s, x(s))ds,$$

so

$$D_{0+}^{\alpha}u(t) = -\phi_q\left(\int_0^t F(s, x(s))ds\right), \tag{7}$$

then

$$I_{0+}^{\alpha}D_{0+}^{\alpha}u(t) = -I_{0+}^{\alpha}\phi_q\left(\int_0^t F(s, x(s))ds\right),$$

in view of Lemma 2.2, we have

$$u(t) + A_1 + B_1 t = -I_{0+}^{\alpha}\phi_q\left(\int_0^t F(s, x(s))ds\right),$$

that is

$$u(t) = -\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\phi_q\left(\int_0^s F(\tau, x(\tau))d\tau\right)ds + A + Bt.$$

By condition (4), there is $u'(0) = B = a$. Now

$$u(t) = -\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\phi_q\left(\int_0^s F(\tau, x(\tau))d\tau\right)ds + A + at,$$
$$u(1) = -\frac{1}{\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1}\phi_q\left(\int_0^s F(\tau, x(\tau))d\tau\right)ds + A + a,$$
$$u(\eta) = -\frac{1}{\Gamma(\alpha)}\int_0^\eta (\eta-s)^{\alpha-1}\phi_q\left(\int_0^s F(\tau, x(\tau))d\tau\right)ds + A + a\eta,$$

by the condition $u(1) - \gamma u(\eta) = a$, we have

$$A = \tfrac{1}{(1-\gamma)\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1}\phi_q\left(\int_0^s F(\tau, x(\tau))d\tau\right)ds$$
$$- \tfrac{\gamma}{(1-\gamma)\Gamma(\alpha)} \int_0^\eta (\eta-s)^{\alpha-1}\phi_q\left(\int_0^s F(\tau, x(\tau))d\tau\right)ds + \tfrac{a\eta\gamma}{1-\gamma}.$$

So

$$\begin{aligned} u(t) = &-\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\phi_q\left(\int_0^s F(\tau, x(\tau))d\tau\right)ds \\ &+\frac{1}{(1-\gamma)\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1}\phi_q\left(\int_0^s F(\tau, x(\tau))d\tau\right)ds \\ &-\frac{\gamma}{(1-\gamma)\Gamma(\alpha)}\int_0^\eta (\eta-s)^{\alpha-1}\phi_q\left(\int_0^s F(\tau, x(\tau))d\tau\right)ds + \frac{a\eta\gamma}{1-\gamma} + at, \end{aligned}$$

splitting the second integral in two parts of the form

$$\frac{1}{\Gamma(\alpha)} + \frac{\gamma}{(1-\gamma)\Gamma(\alpha)} = \frac{1}{(1-\gamma)\Gamma(\alpha)},$$

we have

$$\begin{aligned} u(t) = &-\tfrac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\phi_q\left(\int_0^s F(\tau, x(\tau))d\tau\right)ds \\ &+\tfrac{1}{\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1}\phi_q\left(\int_0^s F(\tau, x(\tau))d\tau\right)ds \\ &+\tfrac{\gamma}{(1-\gamma)\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1}\phi_q\left(\int_0^s F(\tau, x(\tau))d\tau\right)ds \\ &-\tfrac{\gamma}{(1-\gamma)\Gamma(\alpha)}\int_0^\eta (\eta-s)^{\alpha-1}\phi_q\left(\int_0^s F(\tau, x(\tau))d\tau\right)ds + \tfrac{a\eta\gamma}{1-\gamma} + at \\ = &\int_0^t\left(\tfrac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} - \tfrac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\right)\phi_q\left(\int_0^s F(\tau, x(\tau))d\tau\right)ds \\ &+\int_t^1 \tfrac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}\phi_q\left(\int_0^s F(\tau, x(\tau))d\tau\right)ds \\ &+\tfrac{\gamma}{(1-\gamma)\Gamma(\alpha)}\int_0^\eta ((1-s)^{\alpha-1} - (\eta-s)^{\alpha-1})\phi_q\left(\int_0^s F(\tau, x(\tau))d\tau\right)ds \\ &+\tfrac{\gamma}{(1-\gamma)\Gamma(\alpha)}\int_\eta^1 (1-s)^{\alpha-1}\phi_q\left(\int_0^s F(\tau, x(\tau))d\tau\right)ds + \tfrac{a\eta\gamma}{1-\gamma} + at. \end{aligned}$$

The proof is complete. □

Lemma 2.4 *Fixed $\beta \in (0, 1)$. The function $G(t, s)$ defined by (6) has the following properties.*

1. $0 \le G(t, s) \le G(s, s)$, *for all* $s \in (0, 1)$,

2. $\min_{0\le t\le\beta} G(t,s) \ge \frac{1-\beta^{\alpha-1}}{2} G(s,s)$, *for all* $s \in (0,1)$.

Proof 1. For $1 < \alpha \le 2$ and $0 \le s \le t \le 1$, we get

$$(1-s)^{\alpha-1} > (t-s)^{\alpha-1},$$

this implies $G(t,s) > 0$.

On the other hand, case (1): $0 \le s < t \le 1$,

$$\frac{\partial G(t,s)}{\partial t} = -\frac{(\alpha-1)(t-s)^{\alpha-2}}{\Gamma(\alpha)} < 0,$$

case (2): $0 \le t \le s \le 1$,

$$\frac{\partial G(t,s)}{\partial t} = 0,$$

considering the above two cases, we have the function $G(t,s)$ is nonincreasing of t, thus

$$G(t,s) \le G(s,s), \ \forall s \in (0,1).$$

2. For $0 \le t \le \beta$, in view of the proof of 1, we have

$$\min_{0\le t\le\beta} G(t,s) = G(\beta,s),$$

where

$$G(\beta,s) = \begin{cases} \dfrac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} - \dfrac{(\beta-s)^{\alpha-1}}{\Gamma(\alpha)}, & 0 \le s \le \beta \le 1, \\ \dfrac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}, & 0 \le \beta \le s \le 1. \end{cases}$$

(a) If $0 \le s \le \beta \le 1$,

$$\begin{aligned} \min_{0\le t\le\beta} G(t,s) &= \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} - \frac{(\beta-s)^{\alpha-1}}{\Gamma(\alpha)} \\ &= \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} - \frac{\beta^{\alpha-1}(1-\frac{s}{\beta})^{\alpha-1}}{\Gamma(\alpha)} \\ &\ge \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} - \frac{\beta^{\alpha-1}(1-s)^{\alpha-1}}{\Gamma(\alpha)} \\ &= \frac{(1-\beta^{\alpha-1})(1-s)^{\alpha-1}}{\Gamma(\alpha)} \\ &> \frac{(1-\beta^{\alpha-1})}{2}\frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} = \frac{(1-\beta^{\alpha-1})}{2} G(s,s). \end{aligned}$$

(b) If $0 \le \beta \le s \le 1$,

$$\min_{0\le t\le\beta} G(t,s) = \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} > \frac{(1-\beta^{\alpha-1})}{2}\frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} = \frac{(1-\beta^{\alpha-1})}{2}G(s,s).$$

The proof is complete. □

Consider the following fractional boundary value problem

$$(\phi_p(D_{0+}^{\alpha}u(t)))' + F(t,u(t)) = 0, \quad 0 < t < 1, \tag{8}$$

$$u'(0) = a, \quad u(1) - \gamma u(\eta) = a. \tag{9}$$

Define the operator $T: E \to E$ by

$$Tu(t) = \int_0^1 G(t,s)\phi_q\left(\int_0^s F(\tau,u(\tau))d\tau\right)ds + \frac{\gamma}{1-\gamma}\int_0^1 G(\eta,s)\phi_q\left(\int_0^s F(\tau,u(\tau))d\tau\right)ds + \frac{a\eta\gamma}{1-\gamma} + at. \tag{10}$$

Remark 2.1 By Lemma 2.3, the problem (8)–(9) has a solution $u(t)$ if and only if u is a fixed point of T.

Lemma 2.5 *$T: E \to E$ is completely continuous.*

Proof The continuity of functions $G(t,s)$ and $F(t,u(t))$ implies that $T: E \to E$ is continuous.

Let $\Omega \subset E$ be bounded, that is, there exists $L > 0$ such that $\|u\| \le L$ for all $u \in \Omega$. Set

$$D = \max_{0\le t\le 1,\ 0\le u\le L} |F(t,u)|.$$

Lemmas 2.3 and 2.4 imply that for $u \in \Omega$,

$$\begin{aligned}|Tu(t)| &= \left|\int_0^1 G(t,s)\phi_q\left(\int_0^s F(\tau,u(\tau))d\tau\right)ds\right. \\ &\quad \left.+\frac{\gamma}{1-\gamma}\int_0^1 G(\eta,s)\phi_q\left(\int_0^s F(\tau,u(\tau))d\tau\right)ds + \frac{a\eta\gamma}{1-\gamma} + at\right| \\ &\le \int_0^1 G(s,s)\phi_q\left(\int_0^1 D d\tau\right)ds \\ &\quad +\frac{\gamma}{1-\gamma}\int_0^1 G(\eta,s)\phi_q\left(\int_0^1 D d\tau\right)ds + \frac{a\eta\gamma}{1-\gamma} + a\end{aligned}$$

$$\leq \phi_q(D)\left[\int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}ds + \frac{\gamma}{1-\gamma}\int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}ds\right] + \frac{a\eta\gamma}{1-\gamma} + a$$
$$= \frac{\phi_q(D)}{(1-\gamma)\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1}ds + \frac{a\eta\gamma}{1-\gamma} + a =: d.$$

So $T(\Omega)$ is bounded.

Moreover, let $u \in \Omega,\ t_1,\ t_2 \in [0,1]$ with $t_1 < t_2$, then

$$\begin{aligned}
|Tu(t_2) - Tu(t_1)| &\leq \phi_q(D)\left|\int_0^1 (G(t_2,s) - G(t_1,s))ds\right| + a|t_2 - t_1| \\
&= \phi_q(D)\left|\int_0^{t_2}\left(\frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} - \frac{(t_2-s)^{\alpha-1}}{\Gamma(\alpha)}\right)ds + \int_{t_2}^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}ds\right. \\
&\quad - \int_0^{t_1}\left(\frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} - \frac{(t_1-s)^{\alpha-1}}{\Gamma(\alpha)}\right)ds \\
&\quad \left. - \int_{t_1}^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}ds\right| + a|t_2 - t_1| \\
&= \phi_q(D)\left|-\int_0^{t_2}\frac{(t_2-s)^{\alpha-1}}{\Gamma(\alpha)}ds + \int_0^{t_1}\frac{(t_1-s)^{\alpha-1}}{\Gamma(\alpha)}ds\right| + a|t_2 - t_1| \\
&= \phi_q(D)\frac{1}{\Gamma(\alpha+1)}\left|t_1^\alpha - t_2^\alpha\right| + a|t_2 - t_1| \\
&\leq \phi_q(D)\frac{1}{\Gamma(\alpha)}|t_2 - t_1| + a|t_2 - t_1|.
\end{aligned}$$

We have the right side of the above inequality tends to zero if $t_2 \to t_1$. Using Arzela-Ascoli Theorem, we have T is completely continuous. □

Lemma 2.6 *If $u(t)$ is a solution of the problem (8)–(9), then $u(t) \geq \frac{a\eta\gamma}{1-\gamma}$.*

Proof By lemma 2.3, the solution of the problem (8)–(9) can be written as

$$\begin{aligned}
u(t) &= \int_0^1 G(t,s)\phi_q\left(\int_0^s F(\tau,u(\tau))d\tau\right)ds \\
&\quad + \frac{\gamma}{1-\gamma}\int_0^1 G(\eta,s)\phi_q\left(\int_0^s F(\tau,u(\tau))d\tau\right)ds + \frac{a\eta\gamma}{1-\gamma} + at.
\end{aligned}$$

By lemma 2.4, we get $G(t,s) \geq 0$, this together with $\gamma, \eta \in (0,1),\ F: [0,1]\times R \to (0,+\infty),\ a$ is a fixed positive constant, we have $u(t) \geq \frac{a\eta\gamma}{1-\gamma}$. □

Lemma 2.7 *Suppose that there exists a constant $M > a + \frac{a\eta\gamma}{1-\gamma}$ independent of λ such that for $\lambda \in (0,1),\ \|u\| \neq M$, where $u(t)$ satisfies*

$$\begin{cases} (\phi_p(D_{0+}^\alpha u(t)))' + \lambda F(t,u(t)) = 0, & 0 < t < 1, \\ u'(0) = a, \quad u(1) - \gamma u(\eta) = a. \end{cases} \tag{11$_\lambda$}$$

Then problem $(11)_1$ has at least one solution $u(t)$ with $\|u\| \leq M$.

Proof For any $\lambda \in [0, 1]$, define an operator $A_\lambda : E \to E$ by

$$A_\lambda u(t) = \int_0^1 G(t,s)\phi_q\left(\int_0^s \lambda F(\tau, u(\tau))d\tau\right)ds \\ +\frac{\gamma}{1-\gamma}\int_0^1 G(\eta,s)\phi_q\left(\int_0^s \lambda F(\tau, u(\tau))d\tau\right)ds + \frac{a\eta\gamma}{1-\gamma} + at.$$

In view of Lemma 2.5, A_λ is completely continuous. By remark 2.1, problem $(11)_\lambda$ has a solution $u(t)$ if and only if u is a fixed point of A_λ. Given $\Omega = \{u \in E : \|u\| < M\}$, then Ω is an open set in E. Assume that there exists $u \in \partial\Omega$ such that $A_1 u = u$, then $u(t)$ is a solution of $(11)_1$ with $\|u\| \leq M$. Thus the proof is completed. Otherwise, for any $u \in \partial\Omega$, $A_1 u \neq u$. If $\lambda = 0$, for $u \in \partial\Omega$, $(I - A_0)u(t) = u(t) - A_0 u(t) = u(t) - \frac{a\eta\gamma}{1-\gamma} - at \not\equiv 0$ for $\|u\| = M > a + \frac{a\eta\gamma}{1-\gamma}$. For $\lambda \in (0, 1)$, if the problem $(11)_\lambda$ has a solution $u(t)$, thus we have $\|u\| \neq M$, which is a contradiction to $u \in \partial\Omega$. Therefore, for any $u \in \partial\Omega$ and $\lambda \in [0, 1]$, $A_\lambda u \neq u$. According to the homotopy invariance of Leray- Schauder degree, we have

$$Deg\{I - A_1, \Omega, 0\} = Deg\{I - A_0, \Omega, 0\} = 1.$$

So A_1 has a fixed point u in Ω. That is to say, the problem $(11)_1$ has a solution $u(t)$ with $\|u\| \leq M$. We have completed the proof. □

3 Positive Solutions of the Singular Problem (1), (2)

Define

$$\delta(t) = \int_0^t\left(\frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} - \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\right)\phi_q\left(\int_0^s \Psi_M(\tau)d\tau\right)ds \\ + \int_t^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}\phi_q\left(\int_0^s \Psi_M(\tau)d\tau\right)ds + \frac{\gamma}{1-\gamma}\int_0^1 G(\eta,s)\phi_q\left(\int_0^s \Psi_M(\tau)d\tau\right)ds.$$

Theorem 3.1 *Assume that* (H_1) *for each constant* $H > 0$, *there exists a function* Ψ_H *which is continuous on* $[0, 1]$ *and positive on* $(0, 1)$ *such that* $f(t, u) \geq \Psi_H(t)$ *on* $(0, 1) \times (0, H]$;
(H_2) *there exist nonnegative continuous functions* $f_1(u)$ *and* $f_2(u)$ *such that*

$$0 \leq f(t, u) = f_1(u) + f_2(u) \quad \textit{for } (t, u) \in [0, 1] \times (0, \infty)$$

and $f_1(u) > 0$ *is nonincreasing and* $\frac{f_2(u)}{f_1(u)}$ *is nondecreasing on* $u \in (0, \infty)$;
(H_3) *there exists* $M > 0$ *such that*

$$\frac{1}{(1-\gamma)\Gamma(\alpha+1)}\phi_q\left(f_1(M_0 M)\left\{1 + \frac{f_2(M)}{f_1(M)}\right\}\right) < M,$$

where $M_0 = \min\left((1-\gamma)\frac{1-\beta^{\alpha-1}}{2}, \frac{\eta\gamma}{\eta\gamma+(1-\gamma)}\right)$.
Then the singular fractional boundary value problem (1)-(2) has a positive solution $u(t)$ *with* $\|u\| \leq M$.

Proof Since (H_3) holds, we choose $M > 0$ and $0 < \varepsilon < M$, such that

$$\frac{1}{(1-\gamma)\Gamma(\alpha+1)}\phi_q\left(f_1(M_0M)\left\{1+\frac{f_2(M)}{f_1(M)}\right\}\right)+\varepsilon < M.$$

Let $n_0 \in \{1, 2, 3, \dots, \}$ satisfying that $\frac{\eta\gamma}{n_0(1-\gamma)} + \frac{1}{n_0} \le \varepsilon$, set $N_0 = \{n_0, n_0+1, n_0+2, \dots, \}$.

In what follows, we prove that the following problem

$$\begin{cases} (\phi_p(D_{0+}^{\alpha}u(t)))' + f(t, u(t)) = 0, \quad 0 < t < 1, \\ u'(0) = \frac{1}{m}, \ \ u(1) - \gamma u(\eta) = \frac{1}{m} \end{cases} \tag{11}$$

has a solution for each $m \in N_0$.

In order to obtain a solution of problem (11) for each $m \in N_0$, we discuss the following problem

$$\begin{cases} (\phi_p(D_{0+}^{\alpha}u(t)))' + f^*(t, u(t)) = 0, \quad 0 < t < 1, \\ u'(0) = \frac{1}{m}, \ \ u(1) - \gamma u(\eta) = \frac{1}{m} \end{cases} \tag*{$(11)_m$}$$

where

$$f^*(t,u) = \begin{cases} f(t,u), & u \ge \frac{\eta\gamma}{m(1-\gamma)}, \\ f(t, \frac{\eta\gamma}{m(1-\gamma)}), & u \le \frac{\eta\gamma}{m(1-\gamma)}, \end{cases}$$

clearly $f^* \in C([0,1] \times R, (0, +\infty))$.

According to Lemma 2.7, to obtain a solution of problem $(11)_m$ for each $m \in N_0$, we shall consider the following family of problems

$$\begin{cases} (\phi_p(D_{0+}^{\alpha}u(t)))' + \lambda f^*(t, u(t)) = 0, \quad 0 < t < 1, \\ u'(0) = \frac{1}{m}, \ \ u(1) - \gamma u(\eta) = \frac{1}{m}. \end{cases} \tag*{$(11)_m^{\lambda}$}$$

We claim that any solution u of $(11)_m^{\lambda}$ for any $\lambda \in [0,1]$ must satisfy $\|u\| \ne M$. Otherwise, let $u(t)$ be a solution of $(11)_m^{\lambda}$ for some $\lambda \in [0,1]$ such that $\|u\| = M$. By Lemma 2.6, we get $u(t) \ge \frac{\eta\gamma}{m(1-\gamma)}$ for $t \in [0,1]$. Note that

$$\|u\| \le \frac{1}{m} + \frac{\eta\gamma}{m(1-\gamma)} + \frac{1}{1-\gamma}\int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}\phi_q\left(\int_0^s \lambda f^*(\tau, u(\tau))\mathrm{d}\tau\right)\mathrm{d}s,$$

so for $t \in [0, \beta]$, we have

$$\begin{aligned} u(t) &\ge \frac{\eta\gamma}{m(1-\gamma)} + \int_0^1 \frac{1-\beta^{\alpha-1}}{2}\frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}\phi_q(\int_0^s \lambda f^*(\tau, u(\tau))\mathrm{d}\tau)\mathrm{d}s \\ &\ge \frac{\eta\gamma}{\eta\gamma + (1-\gamma)}\left(\frac{1}{m} + \frac{\eta\gamma}{m(1-\gamma)}\right) \end{aligned}$$

$$
\begin{aligned}
&+(1-\gamma)\frac{1-\beta^{\alpha-1}}{2}\frac{1}{1-\gamma}\int_0^1\frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}\phi_q(\int_0^s\lambda f^*(\tau,u(\tau))\mathrm{d}\tau)\mathrm{d}s\\
&\geq M_0\left(\frac{1}{m}+\frac{\eta\gamma}{m(1-\gamma)}\right)+M_0\left(\frac{1}{1-\gamma}\int_0^1\frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}\phi_q\left(\int_0^s\lambda f^*(\tau,u(\tau)\right)\mathrm{d}\tau)\mathrm{d}s\right)\\
&\geq M_0\|u\|=M_0M.
\end{aligned}
$$

From $(H_2)-(H_3)$, we have for $t\in[0,\beta]$,

$$
\begin{aligned}
u(t)&=\int_0^1 G(t,s)\phi_q\left(\int_0^s\lambda f^*(\tau,u(\tau))\mathrm{d}\tau\right)\mathrm{d}s\\
&\quad+\frac{\gamma}{1-\gamma}\int_0^1 G(\eta,s)\phi_q\left(\int_0^s\lambda f^*(\tau,u(\tau))\mathrm{d}\tau\right)\mathrm{d}s+\frac{\eta\gamma}{m(1-\gamma)}+\frac{t}{m}\\
&=\int_0^1 G(t,s)\phi_q\left(\int_0^s\lambda f(\tau,u(\tau))\mathrm{d}\tau\right)\mathrm{d}s\\
&\quad+\frac{\gamma}{1-\gamma}\int_0^1 G(\eta,s)\phi_q\left(\int_0^s\lambda f(\tau,u(\tau))\mathrm{d}\tau\right)\mathrm{d}s+\frac{\eta\gamma}{m(1-\gamma)}+\frac{t}{m}\\
&\leq\int_0^1\frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}\phi_q\left(\int_0^1 f_1(u(\tau))\left\{1+\frac{f_2(u(\tau))}{f_1(u(\tau))}\right\}\mathrm{d}\tau\right)\mathrm{d}s\\
&\quad+\frac{\gamma}{1-\gamma}\int_0^1\frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}\phi_q\left(\int_0^1 f_1(u(\tau))\left\{1+\frac{f_2(u(\tau))}{f_1(u(\tau))}\right\}\mathrm{d}\tau\right)\mathrm{d}s\\
&\quad+\frac{\eta\gamma}{m(1-\gamma)}+\frac{1}{m}\\
&\leq\frac{1}{1-\gamma}\int_0^1\frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}\phi_q\left(\int_0^1 f_1(M_0M)\left\{1+\frac{f_2(M)}{f_1(M)}\right\}\mathrm{d}\tau\right)\mathrm{d}s\\
&\quad+\frac{\eta\gamma}{n_0(1-\gamma)}+\frac{1}{n_0}\\
&=\frac{1}{(1-\gamma)\Gamma(\alpha+1)}\phi_q\left(f_1(M_0M)\left\{1+\frac{f_2(M)}{f_1(M)}\right\}\right)+\frac{\eta\gamma}{n_0(1-\gamma)}+\frac{1}{n_0}.
\end{aligned}
$$

Therefore

$$
M=\|u\|\leq\frac{1}{(1-\gamma)\Gamma(\alpha+1)}\phi_q\left(f_1(M_0M)\left\{1+\frac{f_2(M)}{f_1(M)}\right\}\right)+\frac{\eta\gamma}{n_0(1-\gamma)}+\frac{1}{n_0}<M.
$$

This is a contradiction, so the claim is proved. Lemma 2.7 guarantees that $(11)_m$ has at least a solution $u^m(t)$ with $\|u^m(t)\|\leq M$ for any fixed m. Lemma 2.6 implies that $u^m(t)\geq\frac{\eta\gamma}{m(1-\gamma)}$, so $f^*(t,u^m(t))=f(t,u^m(t))$. Therefore, $u^m(t)$ is a solution to the fractional boundary value problem (11).

Next, we claim that $u^m(t)$ has a uniform sharper lower bound, i.e., there exists a function $\delta(t)$ which is continuous on [0, 1] and positive on (0, 1), such that

$$
u^m(t)\geq\delta(t),\quad t\in[0,1],
$$

for all $m \in N_0$. Since $0 < \frac{\eta\gamma}{m(1-\gamma)} \leq u^m(t) \leq M,\ t \in [0,1],\ (H_1)$ implies that there exists a continuous function $\Psi_M : (0,1) \to (0,+\infty)$ (independent of m) satisfying

$$f(t, u^m(t)) \geq \Psi_M(t), \quad t \in (0,1),$$

we have for $t \in [0,1]$,

$$\begin{aligned}
u^m(t) &= \int_0^1 G(t,s)\phi_q\left(\int_0^s f(\tau, u^m(\tau))\mathrm{d}\tau\right)\mathrm{d}s \\
&\quad + \frac{\gamma}{1-\gamma}\int_0^1 G(\eta,s)\phi_q\left(\int_0^s f(\tau,u^m(\tau))\mathrm{d}\tau\right)\mathrm{d}s + \frac{\eta\gamma}{m(1-\gamma)} + \frac{t}{m} \\
&\geq \int_0^t \left(\frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} - \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\right)\phi_q\left(\int_0^s \Psi_M(\tau)\mathrm{d}\tau\right)\mathrm{d}s \\
&\quad + \int_t^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}\phi_q\left(\int_0^s \Psi_M(\tau)\mathrm{d}\tau\right)\mathrm{d}s \\
&\quad + \frac{\gamma}{1-\gamma}\int_0^1 G(\eta,s)\phi_q\left(\int_0^s \Psi_M(\tau)\mathrm{d}\tau\right)\mathrm{d}s. \qquad (12)
\end{aligned}$$

From (12), we have for any $m \in N_0$,

$$u^m(t) \geq \delta(t), \quad t \in [0,1].$$

For $t, \tau \in [0,1],\ \tau < t$, we have

$$\begin{aligned}
|u^m(t) - u^m(\tau)| &= \left|\int_0^1 (G(t,s) - G(\tau,s))\phi_q\left(\int_0^s f(r,u^m(r))\mathrm{d}r\right)\mathrm{d}s\right| + \tfrac{1}{m}|t-\tau| \\
&\leq \left|\int_0^1 (G(t,s)-G(\tau,s))\phi_q\left(\int_0^s f_1(\delta(r))\left\{1+\tfrac{f_2(M)}{f_1(M)}\right\}\mathrm{d}r\right)\mathrm{d}s\right| + \tfrac{1}{m}|t-\tau| \\
&= \phi_q\left\{1+\tfrac{f_2(M)}{f_1(M)}\right\}\left|\int_0^1 (G(t,s)-G(\tau,s))\phi_q\left(\int_0^s f_1(\delta(r))\mathrm{d}r\right)\mathrm{d}s\right| + \tfrac{1}{m}|t-\tau| \\
&\leq \phi_q\left\{1+\tfrac{f_2(M)}{f_1(M)}\right\}\phi_q\left(\int_0^1 f_1(\delta(r))\mathrm{d}r\right)\left|\int_0^1 (G(t,s)-G(\tau,s))\mathrm{d}s\right| + \tfrac{1}{m}|t-\tau| \\
&= \phi_q\left\{1+\tfrac{f_2(M)}{f_1(M)}\right\}\phi_q\left(\int_0^1 f_1(\delta(r))\mathrm{d}r\right)\left|\int_0^t \left(\frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} - \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\right)\mathrm{d}s\right. \\
&\quad + \int_t^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}\mathrm{d}s - \int_0^\tau \left(\frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} - \frac{(\tau-s)^{\alpha-1}}{\Gamma(\alpha)}\right)\mathrm{d}s \\
&\quad \left. - \int_\tau^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}\mathrm{d}s\right| + \frac{1}{m}|t-\tau| \\
&= \phi_q\left\{1+\tfrac{f_2(M)}{f_1(M)}\right\}\phi_q\left(\int_0^1 f_1(\delta(r))\mathrm{d}r\right)\left| - \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\right. \\
&\quad \left. + \int_0^\tau \frac{(\tau-s)^{\alpha-1}}{\Gamma(\alpha)}\mathrm{d}s\right| + \frac{1}{m}|t-\tau| \\
&= \phi_q\left\{1+\tfrac{f_2(M)}{f_1(M)}\right\}\phi_q\left(\int_0^1 f_1(\delta(r))\mathrm{d}r\right)\tfrac{1}{\Gamma(\alpha+1)}\left|\tau^\alpha - t^\alpha\right| + \tfrac{1}{m}|t-\tau| \\
&\leq \phi_q\left\{1+\tfrac{f_2(M)}{f_1(M)}\right\}\phi_q\left(\int_0^1 f_1(\delta(r))\mathrm{d}r\right)\tfrac{1}{\Gamma(\alpha)}|t-\tau| + \tfrac{1}{m}|t-\tau|.
\end{aligned}$$

Thus

$$|u^m(t) - u^m(\tau)| \to 0, \quad |t-\tau| \to 0,$$

which implies that $\{u^m(t)\}_{m\in N_0}$ is equicontinuous on [0, 1]. Moreover, from the fact

$$0 < u^m(t) \le M, \quad t \in [0,1],$$

we have $\{u^m(t)\}_{m\in N_0}$ is uniformly bounded on [0, 1].

Now the Arzela-Ascoli Theorem guarantees that there is a subsequence $N_1 \subset N_0$ and a function $u(t)$ such that $\{u^m(t)\}_{m\in N_1}$ converges uniformly on [0, 1] to $u(t)$. From the definition of $u^m(t)$, we have

$$\begin{aligned} u^m(t) &= \int_0^1 G(t,s)\phi_q\left(\int_0^s f(\tau, u^m(\tau))\mathrm{d}\tau\right)\mathrm{d}s \\ &\quad + \frac{\gamma}{1-\gamma}\int_0^1 G(\eta,s)\phi_q\left(\int_0^s f(\tau,u^m(\tau))\mathrm{d}\tau\right)\mathrm{d}s + \frac{\eta\gamma}{m(1-\gamma)} + \frac{t}{m}. \end{aligned} \tag{13}$$

Let $m \to +\infty$ in N_1 in (13). By the continuity of f and Lebesgue's dominated convergence theorem, we have

$$\begin{aligned} u(t) &= \int_0^1 G(t,s)\phi_q\left(\int_0^s f(\tau,u(\tau))\mathrm{d}\tau\right)\mathrm{d}s \\ &\quad + \frac{\gamma}{1-\gamma}\int_0^1 G(\eta,s)\phi_q\left(\int_0^s f(\tau,u(\tau))\mathrm{d}\tau\right)\mathrm{d}s, \end{aligned} \tag{14}$$

hence

$$\begin{aligned} (\phi_p(D_{0+}^{\alpha}u(t)))' + f(t,u(t)) = 0, \quad 0<t<1, \\ u'(0) = 0, \quad u(1) - \gamma u(\eta) = 0. \end{aligned}$$

Therefore, $u(t)$ is a solution of the singular fractional boundary value problem (1)–(2) and satisfies $0 < \|u\| \le M$, which implies that $u(t)$ is a positive solution to the problem (1)–(2). □

4 Example

Example 4.1 We consider the following fractional boundary value problem

$$(D_{0+}^{\frac{3}{2}}u(t))' + u^{-a}(t) + \rho u^b(t) = 0, \quad 0<t<1, \tag{15}$$

$$u'(0) = 0, \quad u(1) - \frac{1}{2}u\left(\frac{1}{2}\right) = 0, \tag{16}$$

where $a, b > 0$, and $\rho > 0$ is a given parameter. Then (15)–(16) has at least one positive solution for each $0 < \rho < \rho_1$, where ρ_1 is some positive constant.

Proof We can easily see that $p = 2, \gamma = \frac{1}{2}, \eta = \frac{1}{2}, \alpha = \frac{3}{2}$, choose $\beta = \frac{1}{4}$, we claim all the assumptions in Theorem 3.1 hold. (H_1) for each constant $H > 0$, there exists $\Psi_H(t) = H^{-a}$ such that $f(t, u) = u^{-a} + \rho u^b \geq \Psi_H(t) = H^{-a}$ on $(0, 1) \times (0, H]$; (H_2)

$$0 < f(t, u) = u^{-a} + \rho u^b = f_1(u) + f_2(u),$$

where $f_1(u) = u^{-a} > 0$ is continuous and nonincreasing on $(0, \infty)$, $f_2(u) = \rho u^b > 0$ is continuous on $(0, \infty)$, $\frac{f_2(u)}{f_1(u)} = \frac{\rho u^b}{u^{-a}}$ is nondecreasing on $u \in (0, \infty)$; (H_3) by calculating, $M_0 = \min\{\frac{1}{3}, \frac{1}{8}\} = \frac{1}{8}$, we choose $\rho < M8^{-a}3\sqrt{\pi} - 8M^{-a} - 8\rho M^b$ for some $M > 0$ and M satisfies $M8^{-a}3\sqrt{\pi} - 8M^{-a} - 8\rho M^b > 0$.

Therefore, (15)–(16) has at least one positive solution for $0 < \rho < \rho_1 := \sup_{M>0} M8^{-a}3\sqrt{\pi} - 8M^{-a} - 8\rho M^b$, where $M > 0$ and satisfies $M8^{-a}3\sqrt{\pi} - 8M^{-a} - 8\rho M^b > 0$. □

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