Bull. Malays. Math. Sci. Soc. (2016) 39:1499–1513
DOI 10.1007/s40840-015-0251-9

BULLETIN *of the* MALAYSIAN MATHEMATICAL SCIENCES SOCIETY
http://www.springer.com/mathematics/journal/40840

Approximating the Finite Hilbert Transform via Some Companions of Ostrowski's Inequalities

Wenjun Liu[1] · **Xingyue Gao**[1] · **Yaqiong Wen**[1]

Received: 29 December 2013 / Revised: 22 March 2014 / Published online: 29 October 2015

Abstract The finite Hilbert transform is a helpful tool in fields like aerodynamics, the theory of elasticity, and other areas of the engineering sciences. In this paper, by using some companions of Ostrowski's inequalities for absolutely continuous functions due to Dragomir, we give some new inequalities and approximations for the finite Hilbert transform, which may have the better error bounds than the known results.

Keywords Ostrowski's inequality · Absolutely continuous functions · L^p space · Composite quadrature rule

Mathematics Subject Classification 26D15 · 41A55 · 41A80

1 Introduction

Cauchy principal value integrals (finite Hilbert transform) of the form

$$(Tf)(a,b;t) = \frac{1}{\pi} PV \int_a^b \frac{f(\tau)}{\tau - t} \mathrm{d}\tau := \frac{1}{\pi} \lim_{\varepsilon \to 0+} \left[\int_a^{t-\varepsilon} \frac{f(\tau)}{\tau - t} \mathrm{d}\tau + \int_{t+\varepsilon}^b \frac{f(\tau)}{\tau - t} \mathrm{d}\tau \right]. \tag{1.1}$$

Communicated by Rosihan M. Ali, Dato'.

✉ Wenjun Liu
wjliu@nuist.edu.cn

Xingyue Gao
gao_xingyue@163.com

Yaqiong Wen
191857153@163.com

1 College of Mathematics and Statistics, Nanjing University of Information Science and Technology, Nanjing 210044, China

play an important role in many fields like aerodynamics, the theory of elasticity, and other areas of the engineering sciences. When the interval (a, b) coincides with the interval $(-\infty, \infty)$, this integral is called the Hilbert transform of $f(\tau)$, which appears frequently in tomographic image reconstruction [33], signal processing, disordered alloys, magnetic properties of disordered alloys, and phase determination in bright-field electron microscopy [27]. For different approaches in approximating the finite Hilbert transform (1.1) including noninterpolatory, interpolatory, Gaussian, Chebychevian and spline methods, see for example the papers [6–8,10,12–15,18] and the references therein.

In [10], Dragomir proved the following Ostrowski-type inequalities for absolutely continuous functions.

Theorem 1.1 *Let $u : [a, b] \to \mathbb{R}$ be an absolutely continuous function on $[a, b]$. Then we have*

$$\left|u(x)(b-a) - \int_a^b u(t)\mathrm{d}t\right| \le \begin{cases} \left[\frac{1}{4}(b-a)^2 + \left(x - \frac{a+b}{2}\right)^2\right] \|u'\|_{[a,b],\infty}, & u' \in L^\infty[a,b], \\ \frac{1}{(q+1)^{\frac{1}{q}}}\left[(x-a)^{1+\frac{1}{q}} + (b-x)^{1+\frac{1}{q}}\right] \|u'\|_{[a,b],p}, & u' \in L^p[a,b], \\ & p > 1, \frac{1}{p} + \frac{1}{q} = 1, \\ \left[\frac{1}{2}(b-a) + \left|x - \frac{a+b}{2}\right|\right] \|u'\|_{[a,b],1}, & u' \in L^1[a,b], \end{cases} \tag{1.2}$$

where $\|\cdot\|_{[a,b],r}$ $(r \in [1, \infty])$ are the usual Lebesgue norms, i.e., for $c < d$,

$$\|h\|_{[c,d],\infty} := \operatorname{ess}\sup_{t \in [c,d]} |h(t)| \quad \text{and} \quad \|h\|_{[c,d],r} := \left(\int_c^d |h(t)|^r dt\right)^{\frac{1}{r}}, r \ge 1.$$

Then, Dragomir pointed out a new method in approximating the finite Hilbert transform and proved the following estimates for the finite Hilbert transform via the above Ostrowski-type inequalities for absolutely continuous function.

Theorem 1.2 *Let $f : [a, b] \to \mathbb{R}$ be a function such that its derivative $f' : [a, b] \to \mathbb{R}$ is absolutely continuous on $[a, b]$. Then we have the inequality:*

$$\left|(Tf)(a,b;t) - \frac{f(t)}{\pi}\ln\left(\frac{b-t}{t-a}\right) - \frac{b-a}{\pi}[f; \lambda t + (1-\lambda)b, \lambda t + (1-\lambda)a]\right|$$
$$\le \frac{1}{\pi}\begin{cases} \left[\frac{1}{4} + \left(\lambda - \frac{1}{2}\right)^2\right]\left[\frac{1}{4}(b-a)^2 + \left(t - \frac{a+b}{2}\right)^2\right] \|f''\|_{[a,b],\infty}, \quad f'' \in L^\infty[a,b], \\ \frac{q}{(q+1)^{\frac{1}{q}}}\left[\lambda^{1+\frac{1}{q}} + (1-\lambda)^{1+\frac{1}{q}}\right]\left[(t-a)^{1+\frac{1}{q}} + (b-t)^{1+\frac{1}{q}}\right] \|f''\|_{[a,b],p}, \\ \quad f'' \in L^p[a,b],\ p > 1,\ \frac{1}{p} + \frac{1}{q} = 1, \\ \left[\frac{1}{2} + \left|\lambda - \frac{1}{2}\right|\right]\left[\frac{1}{2}(b-a) + \left|t - \frac{a+b}{2}\right|\right] \|f''\|_{[a,b],1}, \quad f'' \in L^1[a,b] \end{cases} \tag{1.3}$$

for any $t \in [a,b]$ *and* $\lambda \in [0,1)$*, where* $[f;\alpha,\beta]$ *is the divided difference, i.e.,*

$$[f;\alpha,\beta] := \frac{f(\alpha)-f(\beta)}{\alpha-\beta}.$$

Moreover, Dragomir obtained the following theorem in approximating the finite Hilbert transform of a differentiable function whose derivative is absolutely continuous.

Theorem 1.3 *Let* $f:[a,b]\to\mathbb{R}$ *be a differentiable function such that its derivative* f' *is absolutely continuous on* $[a,b]$*. If* $\lambda=(\lambda_i)_{i=\overline{0,n-1}}$, $\lambda_i\in[0,1)$ $(i=\overline{0,n-1})$ *and*

$$S_n(f;\lambda,t)=\frac{b-a}{\pi n}\sum_{i=0}^{n-1}\left[f;t+(i+1-\lambda_i)\frac{b-t}{n},t-(i+1-\lambda_i)\frac{a-t}{n}\right], \tag{1.4}$$

then we have

$$(Tf)(a;b,t)=\frac{f(t)}{\pi}\ln\left(\frac{b-t}{t-a}\right)+S_n(f;\lambda,t)+R_n(f;\lambda,t), \tag{1.5}$$

and the remainder $R_n(f;\lambda,t)$ *satisfies the estimate:*

$$|R_n(f;\lambda,t)| \\ \le\frac{1}{\pi}\begin{cases}\frac{1}{n}\left[\frac{1}{4}+\frac{1}{n}\sum\limits_{i=0}^{n-1}\left(\lambda_i-\frac{1}{2}\right)^2\right]\left[\frac{1}{4}(b-a)^2+\left(t-\frac{a+b}{2}\right)^2\right]\|f''\|_{[a,b],\infty}, \\ \quad f''\in L^\infty[a,b], \\ \frac{1}{n}\frac{q}{(q+1)^{1+\frac{1}{q}}}\left[\frac{1}{n}\sum\limits_{i=0}^{n-1}\left(\lambda_i^{1+\frac{1}{q}}+(1-\lambda_i)^{1+\frac{1}{q}}\right)^q\right]^{\frac{1}{q}} \\ \quad\left[(t-a)^{1+\frac{1}{q}}+(b-t)^{1+\frac{1}{q}}\right]\|f''\|_{[a,b],p}, \\ \quad f''\in L^p[a,b],\ p>1,\ \frac{1}{p}+\frac{1}{q}=1, \\ \frac{1}{n}\left[\frac{1}{2}+\max\left|\lambda_i-\frac{1}{2}\right|\right]\left[\frac{1}{2}(b-a)+\left|t-\frac{a+b}{2}\right|\right]\|f''\|_{[a,b],1}, \quad f''\in L^1[a,b];\end{cases} \tag{1.6}$$

$$\le\frac{1}{\pi}\begin{cases}\frac{1}{4n}(b-a)^2\|f''\|_{[a,b],\infty}, & f''\in L^\infty[a,b], \\ \frac{q}{(q+1)^{1+\frac{1}{q}}}(b-a)^{1+\frac{1}{q}}\|f''\|_{[a,b],p}, & f''\in L^p[a,b],\ p>1,\ \frac{1}{p}+\frac{1}{q}=1, \\ \frac{1}{n}(b-a)\|f''\|_{[a,b],1}, & f''\in L^1[a,b].\end{cases} \tag{1.7}$$

Recently, some companions of Ostrowski's inequality were established in [1–3,11,25,28] (see also [4,5,16,17,19–24,26,29–32] for other related Ostrowski-type inequalities).

In this paper, we shall provide some new inequalities and approximations for the finite Hilbert transform by the use of some companions of Ostrowski's inequalities for absolutely continuous functions in various L^p spaces due to Dragomir in [11] (see

Lemma 2.1 below). For our purpose, we will use the idea of Dragomir in [10]. In Sect. 2, we will estimate the finite Hilbert transform on the interval $[a, b]$. Our result can give a smaller error bound than the similar result in [10] (see Remark 1 below). In Sect. 3, we will prove a quadrature formula for equidistant division of $[a, b]$ in approximating the finite Hilbert transform.

2 Some Inequalities in Estimating the Finite Hilbert Transform

For the sake of completeness, we state and prove the following lemma which provides some companions of Ostrowski's inequalities for absolutely continuous functions in various L^p spaces (see [9,11]).

Lemma 2.1 *Let $u : [a, b] \to \mathbb{R}$ be an absolutely continuous function on $[a, b]$. Then, for all $x \in \left[a, \frac{a+b}{2}\right]$, we have*

$$\left|\frac{u(x)+u(a+b-x)}{2}(b-a)-\int_a^b u(t)\mathrm{d}t\right| \le \begin{cases} \left[\frac{1}{8}(b-a)^2+2\left(x-\frac{3a+b}{4}\right)^2\right]\|u'\|_{[a,b],\infty}, & u' \in L^\infty[a,b],\\ \frac{1}{(q+1)^{\frac{1}{q}}}\left[2(x-a)^{1+\frac{1}{q}}+2^{\frac{1}{q}}\left(\frac{a+b}{2}-x\right)^{1+\frac{1}{q}}\right]\|u'\|_{[a,b],p}, & u' \in L^p[a,b],\\ & p>1,\ \frac{1}{p}+\frac{1}{q}=1,\\ \left[\frac{1}{4}(b-a)+\left|x-\frac{3a+b}{4}\right|\right]\|u'\|_{[a,b],1}, & u' \in L^1[a,b], \end{cases} \tag{2.1}$$

where $\|\cdot\|_{[a,b],r}$ $(r \in [1, \infty])$ are the usual Lebesgue norms.

Proof Using the integration by parts formula, we have

$$\frac{u(x)+u(a+b-x)}{2}(b-a)-\int_a^b u(t)\mathrm{d}t = \int_a^x (t-a)u'(t)\mathrm{d}t+\int_x^{a+b-x}\left(t-\frac{a+b}{2}\right)u'(t)\mathrm{d}t+\int_{a+b-x}^b (t-b)u'(t)\mathrm{d}t \tag{2.2}$$

for any $x \in [a, b]$. Taking the modulus, we have

$$\left|\frac{u(x)+u(a+b-x)}{2}(b-a)-\int_a^b u(t)\mathrm{d}t\right| \le \int_a^x (t-a)\left|u'(t)\right|\mathrm{d}t+\int_x^{a+b-x}\left|t-\frac{a+b}{2}\right|\left|u'(t)\right|\mathrm{d}t + \int_{a+b-x}^b (b-t)\left|u'(t)\right|\mathrm{d}t := M(x).$$

Now, it is obvious that

$$
\begin{aligned}
M(x) &\le \|u'\|_{[a,x],\infty} \int_a^x (t-a)\mathrm{d}t + \|u'\|_{[x,a+b-x],\infty} \\
&\quad \int_x^{a+b-x} \left| t - \frac{a+b}{2} \right| \mathrm{d}t + \|u'\|_{[a+b-x,b],\infty} \int_{a+b-x}^{b} (b-t)\mathrm{d}t \\
&= \|u'\|_{[a,x],\infty} \frac{(x-a)^2}{2} + \|u'\|_{[x,a+b-x],\infty} \left(\frac{a+b}{2} - x \right)^2 \mathrm{d}t \\
&\quad + \|u'\|_{[a+b-x,b],\infty} \frac{(x-a)^2}{2} \\
&\le \|u'\|_{[a,b],\infty} \left[\frac{(x-a)^2}{2} + \left(\frac{a+b}{2} - x \right)^2 + \frac{(x-a)^2}{2} \right] \\
&= \|u'\|_{[a,b],\infty} \left[\frac{1}{8}(b-a)^2 + 2\left(x - \frac{3a+b}{4} \right)^2 \right],
\end{aligned}
$$

proving the first part of (2.1).

Using Hölder's inequality, we may write:

$$
\begin{aligned}
M(x) &\le \|u'\|_{[a,x],p} \left(\int_a^x (t-a)^q \mathrm{d}t \right)^{\frac{1}{q}} \\
&\quad + \|u'\|_{[x,a+b-x],p} \left(\int_x^{a+b-x} \left| t - \frac{a+b}{2} \right|^q \mathrm{d}t \right)^{\frac{1}{q}} \\
&\quad + \|u'\|_{[a+b-x,b],p} \left(\int_{a+b-x}^{b} (b-t)^q \mathrm{d}t \right)^{\frac{1}{q}} \\
&= \|u'\|_{[a,x],p} \frac{(x-a)^{1+\frac{1}{q}}}{(q+1)^{\frac{1}{q}}} + \|u'\|_{[x,a+b-x],p} \left[2^{\frac{1}{q}} \frac{\left(\frac{a+b}{2} - x\right)^{1+\frac{1}{q}}}{(q+1)^{\frac{1}{q}}} \right] \\
&\quad + \|u'\|_{[a+b-x,b],p} \frac{(x-a)^{1+\frac{1}{q}}}{(q+1)^{\frac{1}{q}}} \\
&\le \|u'\|_{[a,b],p} \frac{1}{(q+1)^{\frac{1}{q}}} \left[2(x-a)^{1+\frac{1}{q}} + 2^{\frac{1}{q}} \left(\frac{a+b}{2} - x \right)^{1+\frac{1}{q}} \right],
\end{aligned}
$$

proving the second part of (2.1).

Finally, we observe that

$$\begin{aligned} M(x) &\le (x-a)\|u'\|_{[a,x],1} + \left(\frac{a+b}{2}-x\right)\|u'\|_{[x,a+b-x],1} + (x-a)\|u'\|_{[a+b-x,b],1} \\ &\le \max\left\{x-a, \frac{a+b}{2}-x\right\}\|u'\|_{[a,b],1} \\ &= \left[\frac{1}{4}(b-a) + \left|x-\frac{3a+b}{4}\right|\right]\|u'\|_{[a,b],1}, \end{aligned}$$

and the lemma is proved. □

The best inequalities we can get from (2.1) are embodied in the following corollary.

Corollary 2.1 *With the assumption in Lemma 2.1, we have*

$$\begin{aligned} &\left|\frac{u\left(\frac{3a+b}{4}\right)+u\left(\frac{a+3b}{4}\right)}{2}(b-a) - \int_a^b u(t)\mathrm{d}t\right| \\ &\le \begin{cases} \frac{1}{8}(b-a)^2\|u'\|_{[a,b],\infty}, & u' \in L^\infty[a,b], \\ \frac{1}{4(q+1)^{\frac{1}{q}}}(b-a)^{1+\frac{1}{q}}\|u'\|_{[a,b],p}, & u' \in L^p[a,b],\ p>1,\ \frac{1}{p}+\frac{1}{q}=1, \\ \frac{1}{4}(b-a)\|u'\|_{[a,b],1}, & u' \in L^1[a,b]. \end{cases} \end{aligned} \tag{2.3}$$

Using the above inequalities (2.1), we may point out the following result in estimating the finite Hilbert transform.

Theorem 2.1 *Let $f : [a,b] \to \mathbb{R}$ be a function such that its derivative $f' : [a,b] \to \mathbb{R}$ is absolutely continuous on $[a,b]$. Then we have the inequality:*

$$\begin{aligned} &\Bigg|(Tf)(a,b;t) - \frac{f(t)}{\pi}\ln\left(\frac{b-t}{t-a}\right) \\ &\quad - \frac{b-a}{\pi}\frac{[f;\lambda t+(1-\lambda)b, \lambda t+(1-\lambda)a] + [f;\lambda b+(1-\lambda)t, \lambda a+(1-\lambda)t]}{2}\Bigg| \\ &\le \frac{1}{\pi}\begin{cases} \left[\frac{1}{8}+2\left(\lambda-\frac{3}{4}\right)^2\right]\left[\frac{1}{4}(b-a)^2+\left(t-\frac{a+b}{2}\right)^2\right]\|f''\|_{[a,b],\infty}, & f'' \in L^\infty[a,b], \\ \frac{q}{(q+1)^{\frac{1}{q}}}\left[2(1-\lambda)^{1+\frac{1}{q}}+2^{\frac{1}{q}}\left(\frac{1}{2}-\lambda\right)^{1+\frac{1}{q}}\right]\left[(t-a)^{1+\frac{1}{q}}+(b-t)^{1+\frac{1}{q}}\right] & \\ \qquad \|f''\|_{[a,b],p}, \quad f'' \in L^p[a,b],\ p>1,\ \frac{1}{p}+\frac{1}{q}=1, & \\ \left[\frac{1}{4}+\left|\lambda-\frac{3}{4}\right|\right]\left[\frac{1}{2}(b-a)+\left|t-\frac{a+b}{2}\right|\right]\|f''\|_{[a,b],1}, & f'' \in L^1[a,b] \end{cases} \end{aligned} \tag{2.4}$$

for any $t \in [a,b]$ and $\lambda \in [\frac{1}{2}, 1)$, where $[f;\alpha,\beta]$ is the divided difference.

Proof Since f' is bounded on $[a,b]$, it follows that f is Lipschitzian on $[a,b]$ and thus the finite Hilbert transform exists everywhere in (a,b).

As in [10], for the function $f_0 : (a,b) \to \mathbb{R}$, $f_0(t) = 1$, $t \in (a,b)$, we have

$$(Tf)(a,b;t) = \frac{1}{\pi}\ln\left(\frac{b-t}{t-a}\right),\ t \in (a,b),$$

then obviously

$$(Tf)(a,b;t) - \frac{f(t)}{\pi}\ln\left(\frac{b-t}{t-a}\right) = \frac{1}{\pi}PV\int_a^b \frac{f(\tau)-f(t)}{\tau-t}\mathrm{d}\tau. \tag{2.5}$$

Now, if we choose in (2.1), $u = f'$, $x = \lambda c + (1-\lambda)d$, $\lambda \in [\frac{1}{2}, 1]$, $c, d \in [a,b]$, then we get

$$\begin{aligned}
&\left| f(d) - f(c) - \frac{f'(\lambda c + (1-\lambda)d) + f'(\lambda d + (1-\lambda)c)}{2}(d-c) \right| \\
&\le \begin{cases}
(d-c)^2\left[\frac{1}{8} + 2\left(\lambda - \frac{3}{4}\right)^2\right]\|f''\|_{[c,d],\infty}, & f'' \in L^\infty[a,b], \\
\frac{|d-c|^{1+\frac{1}{q}}}{(q+1)^{\frac{1}{q}}}\left[2(1-\lambda)^{1+\frac{1}{q}} + 2^{\frac{1}{q}}\left(\frac{1}{2}-\lambda\right)^{1+\frac{1}{q}}\right]\|f''\|_{[c,d],p}, & f'' \in L^p[a,b], \\
 & p > 1, \frac{1}{p} + \frac{1}{q} = 1, \\
|d-c|\left[\frac{1}{4} + \left|\lambda - \frac{3}{4}\right|\right]\|f''\|_{[c,d],1}, & f'' \in L^1[a,b],
\end{cases}
\end{aligned}$$

which is equivalent to

$$\begin{aligned}
&\left| \frac{f(d) - f(c)}{d-c} - \frac{f'(\lambda c + (1-\lambda)d) + f'(\lambda d + (1-\lambda)c)}{2} \right| \\
&\le \begin{cases}
|d-c|\left[\frac{1}{8} + 2\left(\lambda - \frac{3}{4}\right)^2\right]\|f''\|_{[c,d],\infty}, & f'' \in L^\infty[a,b], \\
\frac{|d-c|^{\frac{1}{q}}}{(q+1)^{\frac{1}{q}}}\left[2(1-\lambda)^{1+\frac{1}{q}} + 2^{\frac{1}{q}}\left(\frac{1}{2}-\lambda\right)^{1+\frac{1}{q}}\right]\|f''\|_{[c,d],p}, & f'' \in L^p[a,b], \\
 & p > 1, \frac{1}{p} + \frac{1}{q} = 1, \\
\left[\frac{1}{4} + \left|\lambda - \frac{3}{4}\right|\right]\|f''\|_{[c,d],1}, & f'' \in L^1[a,b].
\end{cases}
\end{aligned} \tag{2.6}$$

Using (2.6), we may write

$$\begin{aligned}
&\left| \frac{1}{\pi}PV\int_a^b \frac{f(\tau)-f(t)}{\tau-t}\mathrm{d}\tau \right. \\
&\left. -\frac{1}{2\pi}PV\int_a^b [f'(\lambda t + (1-\lambda)\tau) + f'(\lambda\tau + (1-\lambda)t)]\mathrm{d}\tau \right| \\
&\le \begin{cases}
\frac{1}{\pi}\left[\frac{1}{8} + 2\left(\lambda - \frac{3}{4}\right)^2\right] PV\int_a^b |t-\tau|\,\|f''\|_{[t,\tau],\infty}\mathrm{d}\tau, \quad f'' \in L^\infty[a,b], \\
\frac{1}{\pi}\frac{1}{(q+1)^{\frac{1}{q}}}\left[2(1-\lambda)^{1+\frac{1}{q}} + 2^{\frac{1}{q}}\left(\frac{1}{2}-\lambda\right)^{1+\frac{1}{q}}\right] PV\int_a^b |t-\tau|^{\frac{1}{q}}\,\|f''\|_{[t,\tau],p}\mathrm{d}\tau, \\
\qquad f'' \in L^p[a,b],\ p > 1,\ \frac{1}{p} + \frac{1}{q} = 1, \\
\frac{1}{\pi}\left[\frac{1}{4} + \left|\lambda - \frac{3}{4}\right|\right] PV\int_a^b \|f''\|_{[t,\tau],1}\mathrm{d}\tau, \quad f'' \in L^1[a,b];
\end{cases}
\end{aligned}$$

$$\leq \begin{cases} \frac{1}{\pi}\left[\frac{1}{8}+2\left(\lambda-\frac{3}{4}\right)^2\right]\|f''\|_{[a,b],\infty} PV\int_a^b |t-\tau|\,\mathrm{d}\tau, & f''\in L^\infty[a,b], \\ \frac{1}{\pi}\frac{1}{(q+1)^{\frac{1}{q}}}\left[2(1-\lambda)^{1+\frac{1}{q}}+2^{\frac{1}{q}}\left(\frac{1}{2}-\lambda\right)^{1+\frac{1}{q}}\right]\|f''\|_{[a,b],p} PV\int_a^b |t-\tau|^{\frac{1}{q}}\,\mathrm{d}\tau, \\ \quad f''\in L^p[a,b],\ p>1,\ \frac{1}{p}+\frac{1}{q}=1, \\ \frac{1}{\pi}\left[\frac{1}{4}+\left|\lambda-\frac{3}{4}\right|\right] PV\left[\int_a^t \|f''\|_{[\tau,t],1}\mathrm{d}\tau+\int_t^b \|f''\|_{[t,\tau],1}\mathrm{d}\tau\right], & f''\in L^1[a,b]; \end{cases}$$

$$\leq \begin{cases} \frac{1}{\pi}\left[\frac{1}{8}+2\left(\lambda-\frac{3}{4}\right)^2\right]\|f''\|_{[a,b],\infty}\left[\frac{1}{4}(b-a)^2+\left(t-\frac{a+b}{2}\right)^2\right], & f''\in L^\infty[a,b], \\ \frac{1}{\pi}\frac{1}{(q+1)^{\frac{1}{q}}}\left[2(1-\lambda)^{1+\frac{1}{q}}+2^{\frac{1}{q}}\left(\frac{1}{2}-\lambda\right)^{1+\frac{1}{q}}\right]\|f''\|_{[a,b],p} \\ \quad \times\frac{q}{q+1}\left[(t-a)^{1+\frac{1}{q}}+(b-t)^{1+\frac{1}{q}}\right], & f''\in L^p[a,b],\ p>1,\ \frac{1}{p}+\frac{1}{q}=1, \\ \frac{1}{\pi}\left[\frac{1}{4}+\left|\lambda-\frac{3}{4}\right|\right]\left[\frac{1}{2}(b-a)+\left|t-\frac{a+b}{2}\right|\right]\|f''\|_{[a,b],1}, & f''\in L^1[a,b]. \end{cases} \tag{2.7}$$

Since (note that $\lambda \neq 1$) (see [10])

$$\begin{aligned} &\frac{1}{2\pi} PV\int_a^b [f'(\lambda t+(1-\lambda)\tau)+f'(\lambda\tau+(1-\lambda)t)]\mathrm{d}\tau \\ &= \frac{1}{2\pi}\lim_{\varepsilon\to 0+}\left[\int_a^{t-\varepsilon}+\int_{t+\varepsilon}^b\right][f'(\lambda t+(1-\lambda)\tau)+f'(\lambda\tau+(1-\lambda)t)]\mathrm{d}\tau \\ &= \frac{1}{2\pi}\left[\frac{-f(\lambda t+(1-\lambda)a+f(\lambda t+(1-\lambda)b}{1-\lambda}\right. \\ &\quad \left.+\frac{-f(\lambda a+(1-\lambda)t+f(\lambda b+(1-\lambda)t}{\lambda}\right] \\ &= \frac{b-a}{\pi}\frac{[f;\lambda t+(1-\lambda)b,\lambda t+(1-\lambda)a]+[f;\lambda b+(1-\lambda)t,\lambda a+(1-\lambda)t]}{2}. \end{aligned}$$

Using (2.5) and (2.7), we deduce the desired result (2.4). □

The best inequality one may obtain from (2.4) is embodied in the following corollary.

Corollary 2.2 *With the assumption of Theorem 2.1, for* $\lambda=\frac{3}{4}$, *one has the inequality*

$$\left|(Tf)(a,b;t)-\frac{f(t)}{\pi}\ln\left(\frac{b-t}{t-a}\right)-\frac{b-a}{\pi}\frac{\left[f;\frac{3t+b}{4},\frac{3t+a}{4}\right]+\left[f;\frac{3b+t}{4},\frac{3a+t}{4}\right]}{2}\right|$$
$$\leq \frac{1}{\pi}\begin{cases} \frac{1}{8}\left[\frac{1}{4}(b-a)^2+\left(t-\frac{a+b}{2}\right)^2\right]\|f''\|_{[a,b],\infty}, & f''\in L^\infty[a,b], \\ \frac{q}{4^{\frac{p}{q}}(q+1)^{1+\frac{1}{q}}}\left[(t-a)^{1+\frac{1}{q}}+(b-t)^{1+\frac{1}{q}}\right]\|f''\|_{[a,b],p}, & f''\in L^p[a,b], \\ & p>1,\ \frac{1}{p}+\frac{1}{q}=1, \\ \frac{1}{4}\left[\frac{1}{2}(b-a)+\left|t-\frac{a+b}{2}\right|\right]|f''\|_{[a,b],1}, & f''\in L^1[a,b] \end{cases} \tag{2.8}$$

for any $t\in[a,b]$.

Remark 1 As stated in [10], the best estimator one may obtain from (1.3) is, for $\lambda = \frac{1}{2}$,

$$\left|(Tf)(a,b;t) - \frac{f(t)}{\pi}\ln\left(\frac{b-t}{t-a}\right) - \frac{b-a}{\pi}\left[f;\frac{t+b}{2},\frac{t+a}{2}\right]\right| \leq \frac{1}{\pi}\begin{cases} \frac{1}{4}\left[\frac{1}{4}(b-a)^2 + \left(t-\frac{a+b}{2}\right)^2\right]\|f''\|_{[a,b],\infty}, & f'' \in L^{\infty}[a,b], \\ \frac{q}{2^{\frac{p}{q}}(q+1)^{1+\frac{1}{q}}}\left[(t-a)^{1+\frac{1}{q}} + (b-t)^{1+\frac{1}{q}}\right]\|f''\|_{[a,b],p}, & f'' \in L^p[a,b], \\ & p>1,\ \frac{1}{p}+\frac{1}{q}=1, \\ \frac{1}{2}\left[\frac{1}{2}(b-a) + \left|t-\frac{a+b}{2}\right|\right]|f''\|_{[a,b],1}, & f'' \in L^1[a,b]. \end{cases} \tag{2.9}$$

We note that inequality (2.8) we obtained here gives a new estimate of the finite Hilbert transform and a smaller error bound than that of (2.9).

3 A Quadrature Formula for Equidistant Divisions

The following lemma is of interest in itself.

Lemma 3.1 *Let $u : [a,b] \to \mathbb{R}$ be an absolutely continuous function on $[a,b]$. Then for all $n \geq 1$, $\lambda_i \in [\frac{1}{2}, 1)$ $(i = 0, ..., n-1)$ and $t, \tau \in [a,b]$ with $t \neq \tau$, we have the inequality*

$$\left|\frac{1}{\tau-t}\int_t^{\tau} u(s)\mathrm{d}s - \frac{1}{n}\sum_{i=0}^{n-1}\left[\frac{u\left(t+(i+1-\lambda_i)\frac{\tau-t}{n}\right) + u\left(t+(i+\lambda_i)\frac{\tau-t}{n}\right)}{2}\right]\right| \leq \begin{cases} \frac{|t-\tau|}{n}\left[\frac{1}{8} + \frac{2}{n}\sum_{i=0}^{n-1}\left(\lambda_i - \frac{3}{4}\right)^2\right]\|u'\|_{[t,\tau],\infty}, & u' \in L^{\infty}[a,b], \\ \frac{|t-\tau|^{\frac{1}{q}}}{(q+1)^{\frac{1}{q}}n}\left[\frac{1}{n}\sum_{i=0}^{n-1}\left(2(1-\lambda_i)^{1+\frac{1}{q}} + 2^{\frac{1}{q}}\left(\lambda_i - \frac{1}{2}\right)^{1+\frac{1}{q}}\right)^q\right]^{\frac{1}{q}} & \\ \quad \|u'\|_{[t,\tau],p}, & u' \in L^p[a,b], \\ \frac{1}{n}\left[\frac{1}{4} + \max\left|\lambda_i - \frac{3}{4}\right|\right]\|u'\|_{[t,\tau],1}, & u' \in L^1[a,b], \end{cases} \tag{3.1}$$

where $\|\cdot\|_{[t,\tau],r}$ $(r \in [1,\infty])$ are the usual Lebesgue norms, $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$.

Proof Consider the equidistant division of $[t,\tau]$ (if $t < \tau$) or $[\tau, t]$ (if $\tau < t$) given by

$$E_n : x_i = t + i \cdot \frac{\tau - t}{n}, \quad i = \overline{0, n}. \tag{3.2}$$

Then the points $\xi_i = \lambda_i[t + i \cdot \frac{\tau-t}{n}] + (1-\lambda_i)[t + (i+1)\cdot\frac{\tau-t}{n}]$ $(\lambda_i \in [\frac{1}{2}, 1],\ i = \overline{0, n-1})$ are between x_i and x_{i+1}. We observe that we may write for simplicity $\xi_i = t + (i+1-\lambda_i)\frac{\tau-t}{n}$ $(i = \overline{0, n-1})$. We also have

$$
\begin{aligned}
x_i + x_{i+1} - \xi_i &= t + (i+\lambda_i)\frac{\tau - t}{n},\\
\xi_i - \frac{3x_i + x_{i+1}}{4} &= \frac{\tau - t}{n}\left(\frac{3}{4} - \lambda_i\right),\\
\xi_i - x_i &= (1-\lambda_i)\frac{\tau - t}{n}
\end{aligned}
$$

and

$$
\frac{x_i + x_{i+1}}{2} - \xi_i = \left(\lambda_i - \frac{1}{2}\right)\frac{\tau - t}{n}
$$

for any $i = \overline{0, n-1}$.

If we apply the inequality (2.1) on the intervals $[x_i, x_{i+1}]$ and the intermediate points ξ_i $(i = \overline{0, n-1})$, then we may write that

$$
\left|\frac{\tau - t}{n}\left[\frac{u(t + (i+1-\lambda_i)\frac{\tau-t}{n}) + u(t + (i+\lambda_i)\frac{\tau-t}{n})}{2}\right] - \int_{x_i}^{x_{i+1}} u(s)\mathrm{d}s\right|
\leq \begin{cases}
\left[\frac{1}{8}\frac{(t-\tau)^2}{n^2} + 2\frac{(t-\tau)^2}{n^2}\left(\frac{3}{4} - \lambda_i\right)^2\right]\|u'\|_{[x_i,x_{i+1}],\infty}, \qquad u' \in L^\infty[a,b],\\
\frac{1}{(q+1)^{\frac{1}{q}}}\left[\frac{|t-\tau|^{1+\frac{1}{q}}}{n^{1+\frac{1}{q}}}2\,(1-\lambda_i)^{1+\frac{1}{q}} + 2^{\frac{1}{q}}\frac{|t-\tau|^{1+\frac{1}{q}}}{n^{1+\frac{1}{q}}}\left(\lambda_i - \frac{1}{2}\right)^{1+\frac{1}{q}}\right]\|u'\|_{[x_i,x_{i+1}],p},\\
\quad u' \in L^p[a,b],\ p>1,\ \frac{1}{p} + \frac{1}{q} = 1,\\
\left[\frac{1}{4}\frac{|\tau-t|}{n} + \frac{|\tau-t|}{n}\left|\frac{3}{4} - \lambda_i\right|\right]\|u'\|_{[x_i,x_{i+1}],1}, \qquad u' \in L^1[a,b].
\end{cases}
\tag{3.3}
$$

Summing (3.3), we get

$$
\left|\int_t^\tau u(s)ds - \frac{\tau - t}{n}\sum_{i=0}^{n-1}\left[\frac{u\left(t + (i+1-\lambda_i)\frac{\tau-t}{n}\right) + u\left(t + (i+\lambda_i)\frac{\tau-t}{n}\right)}{2}\right]\right|
\leq \begin{cases}
\frac{(t-\tau)^2}{n^2}\sum_{i=0}^{n-1}\left[\frac{1}{8} + 2\left(\lambda_i - \frac{3}{4}\right)^2\right]\|u'\|_{[x_i,x_{i+1}],\infty}, \qquad u' \in L^\infty[a,b],\\
\frac{|t-\tau|^{1+\frac{1}{q}}}{(q+1)^{\frac{1}{q}} n^{1+\frac{1}{q}}}\sum_{i=0}^{n-1}\left[2\,(1-\lambda_i)^{1+\frac{1}{q}} + 2^{\frac{1}{q}}\left(\lambda_i - \frac{1}{2}\right)^{1+\frac{1}{q}}\right]\|u'\|_{[x_i,x_{i+1}],p},\\
\quad u' \in L^p[a,b],\ p>1,\ \frac{1}{p} + \frac{1}{q} = 1,\\
\frac{|\tau-t|}{n}\sum_{i=0}^{n-1}\left[\frac{1}{4} + \left|\lambda_i - \frac{3}{4}\right|\right]\|u'\|_{[x_i,x_{i+1}],1}, \qquad u' \in L^1[a,b].
\end{cases}
\tag{3.4}
$$

However,

$$
\sum_{i=0}^{n-1}\left[\frac{1}{8} + 2\left(\lambda_i - \frac{3}{4}\right)^2\right]\|u'\|_{[x_i,x_{i+1}],\infty} \leq \|u'\|_{[t,\tau],\infty}\left[\frac{1}{8}n + 2\sum_{i=0}^{n-1}\left(\lambda_i - \frac{3}{4}\right)^2\right],
\tag{3.5}
$$

$$\sum_{i=0}^{n-1}\left[2\left(1-\lambda_i\right)^{1+\frac{1}{q}}+2^{\frac{1}{q}}\left(\lambda_i-\frac{1}{2}\right)^{1+\frac{1}{q}}\right]\|u'\|_{[x_i,x_{i+1}],p}$$
$$\leq\left(\sum_{i=0}^{n-1}\left[2\left(1-\lambda_i\right)^{1+\frac{1}{q}}+2^{\frac{1}{q}}\left(\lambda_i-\frac{1}{2}\right)^{1+\frac{1}{q}}\right]^q\right)^{\frac{1}{q}}\left(\sum_{i=0}^{n-1}\|u'\|_{[x_i,x_{i+1}],p}^p\right)^{\frac{1}{p}}$$
$$=\|u'\|_{[t,\tau],p}\left(\sum_{i=0}^{n-1}\left[2\left(1-\lambda_i\right)^{1+\frac{1}{q}}+2^{\frac{1}{q}}\left(\lambda_i-\frac{1}{2}\right)^{1+\frac{1}{q}}\right]^q\right)^{\frac{1}{q}} \tag{3.6}$$

and

$$\sum_{i=0}^{n-1}\left[\frac{1}{4}+\left|\lambda_i-\frac{3}{4}\right|\right]\|u'\|_{[x_i,x_{i+1}],1}\leq\left[\frac{1}{4}+\max\left|\lambda_i-\frac{3}{4}\right|\right]\sum_{i=0}^{n-1}\|u'\|_{[x_i,x_{i+1}],1}$$
$$=\left[\frac{1}{4}+\max\left|\lambda_i-\frac{3}{4}\right|\right]\|u'\|_{[t,\tau],1}. \tag{3.7}$$

Now, using (3.4)–(3.7), we deduce the desired result (3.1). □

We may now state the following theorem in approximating the finite Hilbert transform of a differentiable function whose derivative is absolutely continuous.

Theorem 3.1 *Let $f:[a,b]\to\mathbb{R}$ be a differentiable function such that its derivative f' is absolutely continuous on $[a,b]$. If $\lambda=(\lambda_i)_{i=\overline{0,n-1}}$, $\lambda_i\in[\frac{1}{2},1)$ $(i=\overline{0,n-1})$,*

$$S_n(f;\lambda,t)=\frac{b-a}{\pi n}\sum_{i=0}^{n-1}\left[f;t+(i+1-\lambda_i)\frac{b-t}{n},t+(i+1-\lambda_i)\frac{a-t}{n}\right] \tag{3.8}$$

and

$$T_n(f;\lambda,t)=\frac{b-a}{\pi n}\sum_{i=0}^{n-1}\left[f;t+(i+\lambda_i)\frac{b-t}{n},t+(i+\lambda_i)\frac{a-t}{n}\right], \tag{3.9}$$

then we have

$$(Tf)(a,b;t)=\frac{f(t)}{\pi}\ln\left(\frac{b-t}{t-a}\right)+\frac{S_n(f;\lambda,t)+T_n(f;\lambda,t)}{2}+R_n(f;\lambda,t), \tag{3.10}$$

and the reminder $R_n(f;\lambda,t)$ satisfies the estimate:

$$|R_n(f;\lambda,t)| \le \frac{1}{\pi}\begin{cases} \frac{1}{n}\left[\frac{1}{8}+\frac{2}{n}\sum_{i=0}^{n-1}\left(\lambda_i-\frac{3}{4}\right)^2\right]\left[\frac{1}{4}(b-a)^2+\left(t-\frac{a+b}{2}\right)^2\right]\|f''\|_{[a,b],\infty}, \\ \quad f''\in L^\infty[a,b], \\ \frac{1}{n}\frac{q}{(q+1)^{1+\frac{1}{q}}}\left[\frac{1}{n}\sum_{i=0}^{n-1}\left(2(1-\lambda_i)^{1+\frac{1}{q}}+2^{\frac{1}{q}}\left(\lambda_i-\frac{1}{2}\right)^{1+\frac{1}{q}}\right)^q\right]^{\frac{1}{q}} \\ \quad \times\left[(t-a)^{1+\frac{1}{q}}+(b-t)^{1+\frac{1}{q}}\right]\|f''\|_{[a,b],p}, \\ \quad f''\in L^p[a,b],\ p>1,\ \frac{1}{p}+\frac{1}{q}=1, \\ \frac{1}{n}\left[\frac{1}{4}+\max\left|\lambda_i-\frac{3}{4}\right|\right]\left[\frac{1}{2}(b-a)+\left|t-\frac{a+b}{2}\right|\right] \\ \|f''\|_{[a,b],1}, \qquad f''\in L^1[a,b]. \end{cases} \tag{3.11}$$

Proof Applying Lemma 3.1 for the function f', we may write that

$$\left|\frac{f(\tau)-f(t)}{\tau-t}-\frac{1}{n}\sum_{i=0}^{n-1}\frac{f'\left(t+(i+1-\lambda_i)\frac{\tau-t}{n}\right)+f'\left(t+(i+\lambda_i)\frac{\tau-t}{n}\right)}{2}\right|$$
$$\le\begin{cases} \frac{|t-\tau|}{n}\left[\frac{1}{8}+\frac{2}{n}\sum_{i=0}^{n-1}\left(\lambda_i-\frac{3}{4}\right)^2\right]\|f''\|_{[t,\tau],\infty}, & f''\in L^\infty[a,b], \\ \frac{|t-\tau|^{\frac{1}{q}}}{n(q+1)^{\frac{1}{q}}}\left[\frac{1}{n}\sum_{i=0}^{n-1}\left(2(1-\lambda_i)^{1+\frac{1}{q}}+2^{\frac{1}{q}}\left(\lambda_i-\frac{1}{2}\right)^{1+\frac{1}{q}}\right)^q\right]^{\frac{1}{q}} & \\ \quad\|f''\|_{[t,\tau],p}, & f''\in L^p[a,b], \\ & p>1,\ \frac{1}{p}+\frac{1}{q}=1, \\ \frac{1}{n}\left[\frac{1}{4}+\max\left|\lambda_i-\frac{3}{4}\right|\right]\|f''\|_{[t,\tau],1}, & f''\in L^1[a,b] \end{cases}$$

for any $t,\tau\in[a,b]$, $t\ne\tau$.

Taking the PV, we may write

$$\left|\frac{1}{\pi}PV\int_a^b\frac{f(\tau)-f(t)}{\tau-t}d\tau\right. \tag{3.12}$$
$$\left.-\frac{1}{\pi n}\sum_{i=0}^{n-1}PV\int_a^b\frac{f'\left(t+(i+1-\lambda_i)\frac{\tau-t}{n}\right)+f'\left(t+(i+\lambda_i)\frac{\tau-t}{n}\right)}{2}d\tau\right|$$
$$\le\frac{1}{\pi}\begin{cases} \frac{1}{n}\left[\frac{1}{8}+\frac{2}{n}\sum_{i=0}^{n-1}\left(\lambda_i-\frac{3}{4}\right)^2\right]PV\int_a^b|t-\tau|\,\|f''\|_{[t,\tau],\infty}d\tau, & f''\in L^\infty[a,b], \\ \frac{1}{n(q+1)^{\frac{1}{q}}}\left[\frac{1}{n}\sum_{i=0}^{n-1}\left(2(1-\lambda_i)^{1+\frac{1}{q}}+2^{\frac{1}{q}}\left(\lambda_i-\frac{1}{2}\right)^{1+\frac{1}{q}}\right)^q\right]^{\frac{1}{q}} & \\ \quad\times PV\int_a^b|t-\tau|^{\frac{1}{q}}\,\|f''\|_{[t,\tau],p}d\tau, & f''\in L^p[a,b], \\ & p>1,\ \frac{1}{p}+\frac{1}{q}=1, \\ \frac{1}{n}\left[\frac{1}{4}+\max\left|\lambda_i-\frac{3}{4}\right|\right]PV\int_a^b\|f''\|_{[t,\tau],1}d\tau, & f''\in L^1[a,b]. \end{cases} \tag{3.13}$$

However (see [10]),

$$PV\int_a^b |t-\tau|\,\|f''\|_{[t,\tau],\infty}\mathrm{d}\tau \le \|f''\|_{[a,b],\infty} PV\int_a^b |t-\tau|\,\mathrm{d}\tau$$
$$= \|f''\|_{[a,b],\infty}\left[\frac{(t-a)^2+(b-t)^2}{2}\right],$$
$$PV\int_a^b |t-\tau|^{\frac{1}{q}}\,\|f''\|_{[t,\tau],p}\mathrm{d}\tau \le \|f''\|_{[a,b],p} PV\int_a^b |t-\tau|^{\frac{1}{q}}\,\mathrm{d}\tau$$
$$= \|f''\|_{[a,b],p}\left[\frac{(t+a)^{\frac{1}{q}+1}+(b-t)^{\frac{1}{q}+1}}{\frac{1}{q}+1}\right]$$
$$= \frac{q}{q+1}\left[(t-a)^{\frac{1}{q}+1}+(b-t)^{\frac{1}{q}+1}\right]\|f''\|_{[a,b],p},$$
$$PV\int_a^b \|f''\|_{[t,\tau],1}\mathrm{d}\tau = PV\left[\int_a^t \|f''\|_{[\tau,t],1}\mathrm{d}\tau + \int_a^t \|f''\|_{[t,\tau],1}\mathrm{d}\tau\right]$$
$$\le \max\{t-a, b-t\}\|f''\|_{[a,b],1},$$
$$= \left[\frac{1}{2}(b-a)+\left|t-\frac{a+b}{2}\right|\right]\|f''\|_{[a,b],1},$$

and using the inequality (3.12) we obtain the desired estimate (3.11). □

Remark 2 For $n = 1$, we recapture the inequality (2.4).

For $\lambda_i = \frac{3}{4}$ $(i = \overline{0, n-1})$, the following particular case, which may be easily numerically implemented, holds.

Corollary 3.1 *Let f be as in Theorem 3.1. Define*

$$S_{M,n}(f;t) = \frac{b-a}{\pi n}\sum_{i=0}^{n-1}\left[f; t+\left(i+\frac{1}{4}\right)\frac{b-t}{n}, t+\left(i+\frac{1}{4}\right)\frac{a-t}{n}\right] \quad (3.14)$$

and

$$T_{M,n}(f;t) = \frac{b-a}{\pi n}\sum_{i=0}^{n-1}\left[f; t+\left(i+\frac{3}{4}\right)\frac{b-t}{n}, t+\left(i+\frac{3}{4}\right)\frac{a-t}{n}\right], \quad (3.15)$$

and the remainder $R_{M,n}(f;t)$ satisfies the estimate

$$(Tf)(a,b;t) = \frac{f(t)}{\pi}\ln\left(\frac{b-t}{t-a}\right) + \frac{S_{M,n}(f;t)+T_{M,n}(f;t)}{2} + R_{M,n}(f;t). \quad (3.16)$$

Then we have the representation:

$$\left|R_{M,n}(f;t)\right| \le \frac{1}{\pi}\begin{cases} \frac{1}{8n}\left[\frac{1}{4}(b-a)^2+\left(t-\frac{a+b}{2}\right)^2\right]\|f''\|_{[a,b],\infty}, & f''\in L^\infty[a,b],\\ \frac{1}{n}\frac{(2+2^{\frac{1}{q}})q}{4^{1+\frac{1}{q}}(q+1)^{1+\frac{1}{q}}}\left[(t-a)^{1+\frac{1}{q}}+(b-t)^{1+\frac{1}{q}}\right]\|f''\|_{[a,b],p}, & f''\in L^p[a,b],\\ \frac{1}{4n}\left[\frac{1}{2}(b-a)+\left|t-\frac{a+b}{2}\right|\right]\|f''\|_{[a,b],1}, & f''\in L^1[a,b] \end{cases} \tag{3.17}$$

for any $t\in[a,b]$, *where* $p>1$, $\frac{1}{p}+\frac{1}{q}=1$.

Acknowledgments The authors would like to thank the anonymous referees and the editors for their suggestions which greatly improved the presentation of this paper. This work was supported by the National Natural Science Foundation of China (Grant No. 11301277), the Natural Science Foundation of Jiangsu Province (Grant No. BK20151523), the Six Talent Peaks Project in Jiangsu Province (Grant No. 2015-XCL-020), and the Qing Lan Project of Jiangsu Province.

References

1. Alomari, M.W.: A companion of Ostrowski's inequality with applications. Transylv. J. Math. Mech. **3**(1), 9–14 (2011)
2. Alomari, M.W.: A companion of Ostrowski's inequality for mappings whose first derivatives are bounded and applications in numerical integration. Transylv. J. Math. Mech. **4**(2), 103–109 (2012)
3. Alomari, M.W.: A generalization of companion inequality of Ostrowski's type for mappings whose first derivatives are bounded and applications in numerical integration. Kragujev. J. Math. **36**(1), 77–82 (2012)
4. Anastassiou, G.A.: Basic and s-convexity Ostrowski and Grüss type inequalities involving several functions. Commun. Appl. Anal. **17**(2), 189–212 (2013)
5. Anastassiou, G.A.: General Grüss and Ostrowski type inequalities involving S-convexity. Bull. Allahabad Math. Soc. **28**(1), 101–129 (2013)
6. Criscuolo, G., Mastroianni, G.: Formule gaussiane per il calcolo di integrali a valor principale secondo e Cauchy loro convergenza. Calcolo **22**, 391–411 (1985)
7. Dagnion, C., Demichelis, V., Santi, E.: Numerical integration based on quasi-interpolating splines. Computing **50**, 149–163 (1993)
8. Diethelm, K.: Non-optimality of certain quadrature rules for Cauchy principal value integrals. Z. Angew. Math. Mech. **74**, 689–690 (1994)
9. Dragomir, S.S.: Applications of Ostrowski's inequality to the estimation of error bounds for some special means and to some numerical quadrature rules. Comp. Math. Appl. **33**, 15–20 (1997)
10. Dragomir, S.S.: Approximating the finite Hilbert transform via Ostrowski type inequalities for absolutely continuous functions. Bull. Korean Math. Soc. **39**(4), 543–559 (2002)
11. Dragomir, S.S.: Some companions of Ostrowski's inequality for absolutely continuous functions and applications. Bull. Korean Math. Soc. **42**(2), 213–230 (2005)
12. Elliott, D., Paget, D.F.: On the convergence of a quadrature rule for evaluating certain Cauchy principal value integrals. Numer. Math. **23**, 311–319 (1975)
13. Hasegawa, T., Torii, T.: An automatic quadrature for Cauchy principal value integrals. Math. Comput. **56**, 741–754 (1991)
14. Hasegawa, T., Sugiura, H.: Algorithms for approximating finite Hilbert transform with end-point singularities and its derivatives. J. Comput. Appl. Math. **236**(2), 243–252 (2011)
15. Hunter, D.B.: Some Gauss type formulae for the evaluation of Cauchy principal value integrals. Numer. Math. **19**, 419–424 (1972)
16. Huy, V.N., Ngô, Q.-A.: New bounds for the Ostrowski-like type inequalities. Bull. Korean Math. Soc. **48**(1), 95–104 (2011)

17. Hwang, S.-R., Hsu, K.-C., Tseng, K.-L.: Weighted Ostrowski integral inequalities for mappings whose derivatives belong to $L_P(a, b)$. Appl. Math. Comput. **219**(17), 9516–9523 (2013)
18. Li, X.-J.: An explicit formula for finite Hilbert transforms associated with a polynomial. Indiana Univ. Math. J. **53**(1), 185–203 (2004)
19. Liu, W.: Some weighted integral inequalities with a parameter and applications. Acta Appl. Math. **109**(2), 389–400 (2010)
20. Liu, W., Ngô, Q.A., Chen, W.: Ostrowski type inequalities on time scales for double integrals. Acta Appl. Math. **110**(1), 477–497 (2010)
21. Liu, W., Sun, Y., Zhang, Q.: Some new error inequalities for a generalized quadrature rule of open type. Comput. Math. Appl. **62**(5), 2218–2224 (2011)
22. Liu, W., Tuna, A.: Weighted Ostrowski, trapezoid and Grüss type inequalities on time scales. J. Math. Inequal **6**(3), 381–399 (2012)
23. Liu, W., Tuna, A., Jiang, Y.: On weighted Ostrowski type, Trapezoid type, Grüss type and Ostrowski-Grüss like inequalities on time scales. Appl. Anal. **93**(3), 551–571 (2014)
24. Liu, W., Zhu, Y., Park, J.: Some companions of perturbed Ostrowski-type inequalities based on the quadratic kernel function with three sections and applications. J. Inequal. Appl. **2013**, 14 (2013)
25. Liu, Z.: Some companions of an Ostrowski type inequality and applications, J. Inequal. Pure Appl. Math. **10**(2), Article 52, 12 pp (2009)
26. Masjed-Jamei, M., Dragomir, S.S.: A generalization of the Ostrowski-Grüss Inequality. Anal. Appl. **12**(2), 117–130 (2014)
27. Nandagopal, M., Arunajadai, N.: On the evaluation of finite Hilbert transforms. Comput. Sci. Eng. **9**(6), 90–95 (2007)
28. Park, J.: Some companions of perturbed ostrowski type inequalities for functions whose second derivatives are bounded. Int. J. Appl. Math. Stat. **36**, 95–103 (2013)
29. Qayyum, A., Hussain, S.: A new generalized Ostrowski Grüss type inequality and applications. Appl. Math. Lett. **25**(11), 1875–1880 (2012)
30. Sarikaya, M.Z.: On the Ostrowski type integral inequality. Acta Math. Univ. Comenian. (N.S.) **79**(1), 129–134 (2010)
31. Wang, Z., Vong, S.: On some generalizations of an Ostrowski-Grüss type integral inequality. Appl. Math. Comput. **229**, 239–244 (2014)
32. Vong, S.W.: A note on some Ostrowski-like type inequalities. Comput. Math. Appl. **62**(1), 532–535 (2011)
33. Zeng, G.: Image reconstruction via the finite Hilbert transform of the derivative of the backprojection. Med. Phys. **34**, 2837–2843 (2007)