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Gromov Hyperbolicity of Periodic Graphs

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Abstract Gromov hyperbolicity grasps the essence of both negatively curved spaces and discrete spaces. The hyperbolicity of a geodesic metric space is equivalent to the hyperbolicity of a graph related to it; hence, characterizing hyperbolic graphs is a main problem in the theory of hyperbolicity. Since this is a very ambitious goal, a more achievable problem is to characterize hyperbolic graphs in particular classes of graphs. The main result in this paper is a characterization of the hyperbolicity of periodic graphs.

Keywords Periodic graphs · Gromov hyperbolicity · Infinite graphs · Geodesics

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1 Introduction

Gromov hyperbolicity grasps the essence of both negatively curved spaces and discrete spaces. As observed in [5, Section 1.3], the hyperbolicity of a geodesic metric space is equivalent to the hyperbolicity of a graph related to it. Characterizing hyperbolic graphs is a main problem in the theory of hyperbolicity; since this is a very ambitious goal, a more achievable (yet very difficult) problem is to characterize hyperbolic graphs in particular classes of graphs. The papers [2,4,7–9,11,12,25,27,30–32,37,39] study the hyperbolicity of complement of graphs, chordal graphs, periodic planar graphs, planar graphs, strong product graphs, line graphs, Cartesian product graphs, cubic graphs, short graphs, median graphs, and different generalizations of chordal graphs; however, characterizations of the hyperbolicity in the corresponding classes are obtained only in a few of them. In a previous work, [8], periodic planar graphs were considered. In this work, we shall study how hyperbolicity is affected when considering general periodic graphs, not necessarily planar; a simple characterization of the hyperbolic periodic graphs will be obtained. The key ingredient will be the speed at which points and their images under an isometry separate. The general setting is much more complicated than the planar one and the characterization obtained is totally unexpected. X is a *geodesic metric space* if for every $x, y \in X$ there exists a geodesic joining x and y; denote by $[xy]$ any of such geodesics (since uniqueness of geodesics is not required, this notation is ambiguous, but convenient). It is clear that every geodesic metric space is path-connected. If the metric space X is a graph, $[u, v]$ denotes the edge joining the vertices u and v.

In order to consider a graph G as a geodesic metric space, one must identify any edge $[u, v] \in E(G)$ with the real interval $[0, l]$ $\big(\text{if} l := L([u, v])\big)$; therefore, any point in the interior of any edge is a point of G and, if the edge $[u, v]$ is considered as a graph with just one edge, then it is isometric to $[0, l]$. A connected graph G is naturally equipped with a distance defined on its points, induced by taking shortest paths in G, inducing in G the structure of a metric graph. Note that edges can have arbitrary lengths. As usual, the set of vertices of a graph G will be denoted by $V(G)$.

Let (X, d_X) and (Y, d_Y) be two metric spaces. A map $f : X \longrightarrow Y$ is said to be an (α, β)-*quasi-isometric embedding*, with constants $\alpha \geq 1$, $\beta \geq 0$ if, for every $x, y \in X$:

$$\alpha^{-1} d_X(x, y) - \beta \leq d_Y\big(f(x), f(y)\big) \leq \alpha d_X(x, y) + \beta.$$

The function f is ε-*full* if for each $y \in Y$ there exists $x \in X$ with $d_Y\big(f(x), y\big) \leq \varepsilon$.

A *quasi-isometry* from X to Y is a map $f : X \longrightarrow Y$ that is an ε-full (α, β)-quasi-isometric embedding for some $\alpha \geq 1$ and $\beta, \varepsilon \geq 0$. Two metric spaces X and Y are *quasi-isometric* if there exists a quasi-isometry $f : X \longrightarrow Y$. Quasi-isometry is an equivalence relation on metric spaces.

An (α, β)-*quasigeodesic* of a metric space X is an (α, β)-quasi-isometric embedding $\gamma : I \longrightarrow X$, where I is an interval of $\mathbb{R}$. A *quasigeodesic* is an (α, β)-quasigeodesic for some $\alpha \geq 1$, $\beta \geq 0$. Note that a $(1, 0)$-quasigeodesic is a geodesic. A *geodesic line* is a geodesic with domain $\mathbb{R}$.

This work deals with periodic graphs. A graph G is *periodic* if there exist a geodesic line γ_0 and an isometry T of G with the following properties:

(1) $T\gamma_0 \cap \gamma_0 = \emptyset$,
(2) $G \setminus \gamma_0$ has two connected components,
(3) $G \setminus \{\gamma_0 \cup T\gamma_0\}$ has at least three connected components, two of them, G_1 and G_2, satisfy $\partial G_1 \subset \gamma_0$ and $\partial G_2 \subset T\gamma_0$, and the subgraph $G^* := G \setminus \{G_1 \cup G_2\}$ is connected and $\cup_{n\in\mathbb{Z}} T^n(G^*) = G$.

Such subgraph G^* is a *period graph* of G.

In what follows and throughout the paper, G will denote a periodic graph and G^* a period graph of G. In fact, given a periodic graph G, we will fix a geodesic line γ_0, an isometry T and their corresponding period graph G^*. By η_0, we will denote an arc-length parametrization of γ_0 in G. Let $\eta_k := T^k\eta_0$ be a parametrization of $T^k\gamma_0$ for any $k \in \mathbb{Z}$. Also, for any function $f : G \to \mathbb{R}$ denote by $\limsup_{z\to+\infty, z\in\gamma_0} f(z)$, the limit

$$\limsup_{z\to+\infty, z\in\gamma_0} f(z) := \limsup_{t\to+\infty} f(\eta_0(t)),$$

and analogously for any other limit along the curve.

Our main result is the following:

Theorem 1.1 *Let G be a periodic graph.*

- *If $\inf_{z\in\gamma_0} d_G(z, Tz) > 0$, then G is hyperbolic if only if G^* is hyperbolic and $\lim\limits_{|z|\to\infty, z\in\gamma_0} d_G(z, Tz) = \infty$.*
- *If $\inf_{z\in\gamma_0} d_G(z, Tz) = 0$, then G is hyperbolic if and only if G^* is hyperbolic and G has quasi-exponential decay.*

For the definition of quasi-exponential decay, let G be a periodic graph with $\inf_{z\in\gamma_0} d_G(z, Tz) = 0$, let $\eta_0(t)$ be a parametrization of γ_0 and define $\Phi_{\eta_0}(t)$ as the greatest non-increasing minorant of $F(t)$, where $F(t) := d_G\big(\eta_0(t), T\eta_0(t)\big)$ on $[0, \infty)$. The graph G has *quasi-exponential decay* if there exist a parametrization $\eta_0(t)$ for which $\lim_{t\to-\infty} d_G\big(\eta_0(t), T\eta_0(t)\big) = \infty$ and

$$\sup_{s_2\geq s_1\geq 0} (s_2 - s_1)\frac{\Phi_{\eta_0}(s_2)}{\Phi_{\eta_0}(s_1)} < \infty.$$

In what follows, we will write $\Phi_{\eta_0}(t)$ as $\Phi(t)$.

Note that such condition is satisfied by any exponential function $\Phi(t) = e^{-at}$. Also, on the other hand, if a positive function $\Phi(t)$ satisfies this condition, then $\Phi(t) \leq ke^{-at}$ on $[0, \infty)$ for some $k, a > 0$. Consequently, if G has quasi-exponential decay, then $\lim_{t\to\infty} \Phi(t) = 0$ and $\liminf_{t\to\infty} F(t) = 0$. We obtain an equivalent definition of quasi-exponential decay if we replace $\eta_0(t)$ by $\eta_0(t - t_0)$, i.e., if one considers $t \geq t_0$ instead of $t \geq 0$, for any fixed t_0.

The outline of the paper is as follows. Section 2 states some definitions and background used throughout the paper. In Sect. 3, some technical and basic results on periodic graphs are presented. Section 4 is devoted to the proof of the first part of Theorem 1.1. Finally, the proof of the second part is shown in Sect. 5.

2 Definitions and Background

If X is a geodesic metric space and $J = \{J_1, J_2, \ldots, J_n\}$ is a polygon, with sides $J_j \subseteq X$, the polygon J is *δ-thin* if for every $x \in J_i$ the distance $d(x, \cup_{j \neq i} J_j) \leq \delta$. Denote by $\delta(J)$ the sharp thin constant of J, *i.e.*, $\delta(J) := \inf\{\delta : J \text{ is } \delta\text{-thin}\}$. If $x_1, x_2, x_3 \in X$, a *geodesic triangle* $\mathcal{T} = \{x_1, x_2, x_3\}$ is the union of the three geodesics $[x_1x_2]$, $[x_2x_3]$ and $[x_3x_1]$. The space X is *δ-hyperbolic* if every geodesic triangle in X is δ-thin. Denote by $\delta(X)$ the sharp hyperbolicity constant of X, *i.e.*, $\delta(X) := \sup\{\delta(\mathcal{T}) : \mathcal{T} \text{ is a geodesic triangle in } X\}$. The space X is *hyperbolic* if X is δ-hyperbolic for some δ. Note that if X is δ-hyperbolic, then every geodesic polygon with n sides is $(n-2)\delta$-thin; in particular, every geodesic quadrilateral is 2δ-thin. In the classical references on this subject (see, *e.g.*, [5,17]) appear several different definitions of Gromov hyperbolicity, which are equivalent in the sense that if X is δ-hyperbolic with respect to one definition, then it is δ'-hyperbolic with respect to another definition (for some δ' related to δ), see for example Theorem A in Sect. 5. The definition that we have chosen has a deep geometric meaning (see, *e.g.*, [17]).

Let X be a metric space, Y a non-empty subset of X and ε a positive number. The *ε-neighborhood* of Y in X, denoted by $\mathcal{V}_\varepsilon(Y)$ is the set $\{x \in X : d_X(x, Y) \leq \varepsilon\}$. The *Hausdorff distance* between two non-empty subsets Y and Z of X, denoted by $\mathcal{H}_X(Y, Z)$ or $\mathcal{H}(Y, Z)$, is the number defined by:

$$\inf\{\varepsilon > 0 : Y \subset \mathcal{V}_\varepsilon(Z) \text{ and } Z \subset \mathcal{V}_\varepsilon(Y)\}.$$

A useful property of hyperbolic spaces is the *invariance of hyperbolicity*. Namely, if $f : X \longrightarrow Y$ is an (α, β)-quasi-isometric embedding between the geodesic metric spaces X and Y, and if Y is δ-hyperbolic, then X is δ'-hyperbolic, where δ' is a constant which just depends on δ, α, and β. Besides, if f is ε-full for some $\varepsilon \geq 0$ (a quasi-isometry), then X is hyperbolic if and only if Y is hyperbolic. Furthermore, if X is δ'-hyperbolic, then Y is δ-hyperbolic, where δ is a constant which just depends on δ', α, β, and ε.

Given a geodesic metric space X and a closed connected subset $X_0 \subset X$, the *inner distance* d_{X_0} is defined by minimizing d_X-length of paths contained in X_0.

A subspace X_0 of a geodesic metric space X is an *isometric subspace* if the inner distance d_{X_0} satisfies that $d_{X_0}(x, y) = d_X(x, y)$ for all $x, y \in X_0$. If X_0 is an isometric subspace of X then every geodesic in X_0 is also a geodesic in X, and therefore $\delta(X_0) \leq \delta(X)$.

The following lemma shows that in order to prove the hyperbolicity of a geodesic metric space it suffices to consider geodesic triangles verifying a useful property (see [34, Lemma 2.1]):

Lemma A *In any geodesic metric space X,*

$$\delta(X) = \sup\left\{\delta(\mathcal{T}) : \mathcal{T} \text{ is a geodesic triangle that is a simple closed curve}\right\}.$$

Another fundamental property of hyperbolic spaces is their *geodesic stability*: if X is a δ-hyperbolic geodesic metric space ($\delta \geq 0$), and $\alpha \geq 1$ and $\beta \geq 0$ are given constants, there exists a constant $H = H(\delta, \alpha, \beta)$ such that for any pair of (α, β)-quasigeodesics g, h with the same endpoints, $\mathcal{H}(g, h) \le H$.

In view of this stability, one can extend the thinness to quasigeodesic polygons:

Lemma 2.1 *Let X be a δ-hyperbolic geodesic metric space and P an (α, β)-quasigeodesic polygon with n sides in X. Then P is Δ-thin, where Δ depends only on n, δ, α, β.*

Proof Let P' be a geodesic polygon in X with the same vertices as P. By geodesic stability, the Hausdorff distance between a quasigeodesic side in P and its corresponding geodesic side in P' is less than or equal to the constant $H = H(\delta, \alpha, \beta)$. By splitting P' in $n-2$ geodesic triangles, one can check that P' is $(n-2)\delta$-thin. If p belongs to a side of P, then there exists a point p' on its corresponding geodesic side on P' at distance from p less than or equal to H; since P' is a geodesic polygon with n sides, there exists a point q' on the union of the other $n-1$ geodesic sides in P' at distance from p' less than or equal to $(n-2)\delta$; then, there exists a point q in the union of the corresponding $n-1$ quasigeodesic sides in P at distance from q' less than or equal to H, and $d_G(p, q) \le (n-2)\delta + 2H$. Hence, P is $\big((n-2)\delta + 2H\big)$-thin. □

3 Technical Results on Periodic Graphs

In this section, some definitions and results which will be used throughout the paper are stated.

The following lemmas will be of use in the proof of Theorem 1.1 (see [8, Lemma 3.9] and the proof of [8, Lemma 3.10]):

Lemma B *Let G be a graph and let γ_0 be a geodesic line in G such that $G \setminus \gamma_0$ has two connected components G_1', G_2'. Define $G_1 := G_1' \cup \gamma_0$ and $G_2 := G_2' \cup \gamma_0$. If G is δ-hyperbolic, then G_1, G_2 are δ-hyperbolic. If G_1, G_2 are δ-hyperbolic, then G is 120δ-hyperbolic.*

A geodesic $\gamma = [xy]$ with $x \in T^j G^*$, $y \in T^k G^*$ and $j \le k$ is a *straight geodesic* if $\gamma \cap T^i G^*$ is a connected set for every $j \le i \le k$; note that then $\gamma \subset \cup_{i=j}^k T^i G^*$.

The proof of [8, Lemma 3.11] gives:

Lemma C *Let G be a periodic graph such that G^* is δ^*-hyperbolic and $\lim_{|z| \to \infty, z \in \gamma_0} d_G(z, Tz) = \infty$. Assume also that there exists $z_0 \in \gamma_0$ with $[z_0, Tz_0] \in E(G)$ and $L\big([z_0, Tz_0]\big) = d_G(\gamma_0, T\gamma_0) > 0$. Denote by γ a geodesic joining $x \in T^j G^*$ and $y \in T^k G^*$, $j \le k$. Then:*

(1) *There exists a constant M that depends only on G^* and a straight geodesic γ' joining x and y such that $\mathcal{H}(\gamma, \gamma') \le M$.*
(2) *There exists a constant N that depends only on G^* such that if $\sigma := \cup_{n \in \mathbb{Z}} [T^n z_0, T^{n+1} z_0]$ and $j + 2 \le k$, for each $j < i < k$ there exists a point $z_i \in \gamma'$ with $d_{T^i G^*}(z_i, \sigma \cap T^i G^*) \le N$.*

A geometric consequence of the previous lemma is that two geodesics that start at the same copy of G^* and end at the same copy of G^* are at bounded distance in the intermediate copies of G^*. Namely,

Lemma 3.1 *Under the hypotheses of Lemma C, consider two geodesics $\gamma, \tilde\gamma$ in G from points $x, \tilde x \in T^j G^*$ to points $y, \tilde y \in T^k G^*$, respectively, where $k - j \ge 4$. If $p \in T^i G^* \cap \gamma$ and $q \in T^i G^* \cap \tilde\gamma$ with $j+2 \le i \le k-2$, then $d_G(p,q) \le 2M + 6N + 5d_1$, where $d_1 = L\big([z_0, Tz_0]\big) = d_G(\gamma_0, T\gamma_0)$ and M, N are the constants in Lemma C. Furthermore, if γ and $\tilde\gamma$ are straight geodesics, then $d_G(p,q) \le 6N + 5d_1$.*

Proof By part (1) in Lemma C, it suffices to prove $d_G(p,q) \le 6N + 5d_1$ when γ and $\tilde\gamma$ are straight geodesics. By Lemma C, there exist points $z_i \in T^i G^* \cap \gamma$ and $\tilde z_i \in T^i G^* \cap \tilde\gamma$ so that

$$d_{T^i G^*}(z_i, \sigma \cap T^i G^*),\ d_{T^i G^*}(\tilde z_i, \sigma \cap T^i G^*) \le N$$

for $j + 1 \le i \le k - 1$.

Consider $p \in T^i G^* \cap \gamma$ and $q \in T^i G^* \cap \tilde\gamma$, with $j + 2 \le i \le k - 2$. Then,

$$d_G(p, z_i) \le \max\left\{d_G(z_{i-1}, z_i), d_G(z_i, z_{i+1})\right\} \le 2N + 2d_1.$$

And, identically, $d_G(q, \tilde z_i) \le 2N + 2d_1$. Since $d_G(z_i, \tilde z_i) \le 2N + d_1$, one gets the desired result. □

The following two lemmas will relate distances among points on γ_0 and $T\gamma_0$.

Lemma 3.2 *Let G be a periodic graph. Assume that there exist $a' \in \gamma_0$, $b' \in T\gamma_0$ such that*

$$d_G(a', b') \le \eta_1^{-1}(b') - \eta_0^{-1}(a') = d_G(b', Ta').$$

If $a \in \gamma_0$ so that $\eta_0^{-1}(a) \le \eta_0^{-1}(a')$ then, for every $b \in T\gamma_0$

$$d_G(a, b) \ge \eta_0^{-1}(a) - \eta_1^{-1}(b).$$

Furthermore, if $\eta_1^{-1}(b) \le \eta_0^{-1}(a)$, then $d_G(a, b) \ge d_G(a, Ta)/2$.

Remark By symmetry, if $d_G(a', b') \le \eta_0^{-1}(a') - \eta_1^{-1}(b')$ and if $b \in T\gamma_0$ is so that $\eta_1^{-1}(b) \le \eta_1^{-1}(b')$ then $d_G(a, b) \ge \eta_1^{-1}(b) - \eta_0^{-1}(a)$ for any $a \in \gamma_0$.

Proof Seeking for a contradiction assume that there exist $a \in \gamma_0$ and $b \in T\gamma_0$ with $\eta_0^{-1}(a) - \eta_1^{-1}(b) > d_G(a, b)$ and $\eta_0^{-1}(a) \le \eta_0^{-1}(a')$. Then

$$\begin{aligned} d_G(b, b') &\le d_G(b, a) + d_G(a, a') + d_G(a', b') \\ &< \eta_0^{-1}(a) - \eta_1^{-1}(b) + \eta_0^{-1}(a') - \eta_0^{-1}(a) + \eta_1^{-1}(b') - \eta_0^{-1}(a') \\ &= \eta_1^{-1}(b') - \eta_1^{-1}(b) = d_G(b, b'), \end{aligned}$$

which is a contradiction. Thus, $\eta_0^{-1}(a) - \eta_1^{-1}(b) \le d_G(a, b)$.

If $\eta_1^{-1}(b) \le \eta_0^{-1}(a)$, notice that $d_G(b, Ta) = \eta_0^{-1}(a) - \eta_1^{-1}(b) \le d_G(a, b)$. Hence, $d_G(a, Ta) \le d_G(a, b) + d_G(b, Ta) \le 2d_G(a, b)$. □

The second lemma relating distances among points on the "boundary" of G^* states:

Lemma 3.3 *Let G be a periodic graph and assume that there exist an unbounded sequence $\{\zeta_n\} \subset \gamma_0$ and some constant c_0 with $d_G(\zeta_n, T\zeta_n) \le c_0$ for every $n \in \mathbb{N}$. Then $d_G(z_1, z_2) \le d_G(z_1, Tz_2) + c_0$ for every $z_1, z_2 \in \gamma_0$. Furthermore, $d_G(z_1, Tz_1) \le 2d_G(z_1, Tz_2) + c_0$ and $d_G(z_1, T\gamma_0) \le d_G(z_1, Tz_1) \le 2d_G(z_1, T\gamma_0) + c_0$.*

Proof Fix $z_1, z_2 \in \gamma_0$. Let η_0 be a fixed arc-length parametrization of γ_0 with $\eta_0^{-1}(z_1) \ge \eta_0^{-1}(z_2)$. By hypothesis, there exists $n \in \mathbb{N}$ with either $\eta_0^{-1}(\zeta_n) > \eta_0^{-1}(z_1)$ or $\eta_0^{-1}(\zeta_n) < \eta_0^{-1}(z_2)$. Assume that $\eta_0^{-1}(\zeta_n) > \eta_0^{-1}(z_1)$ (the case $\eta_0^{-1}(\zeta_n) < \eta_0^{-1}(z_2)$ is similar). Hence

$$\begin{aligned} d_G(Tz_2, Tz_1) + d_G(Tz_1, T\zeta_n) = d_G(Tz_2, T\zeta_n) &\le d_G(Tz_2, z_1) + d_G(z_1, \zeta_n) \\ &\quad + d_G(\zeta_n, T\zeta_n)\,, \end{aligned}$$

and, since T is an isometry and $T\gamma_0$ is a geodesic,

$$d_G(z_1, z_2) \le d_G(z_1, Tz_2) + c_0\,.$$

Moreover, $d_G(z_1, Tz_1) \le d_G(z_1, Tz_2) + d_G(Tz_1, Tz_2) \le 2d_G(z_1, Tz_2) + c_0$. □

This last result has two corollaries which will be useful in the proof of the second part of Theorem 1.1. Both give more specific quantitative relations between distances among points. Namely,

Corollary 3.4 *Let G be a periodic graph with $\inf_{z\in\gamma_0} d_G(z, Tz) = 0$. Then $d_G(z_1, z_2) \le d_G(z_1, Tz_2)$ for every $z_1, z_2 \in \gamma_0$. Furthermore, $d_G(z_1, Tz_1) \le 2d_G(z_1, Tz_2)$, $d_G(z_1, T\gamma_0) \le d_G(z_1, Tz_1) \le 2d_G(z_1, T\gamma_0)$ and*

$$\frac{1}{3}\Big(d_G(z_1, z_2)+\max_{i=1,2}\{d_G(z_i, Tz_i)\}\Big) \le d_G(z_1, Tz_2) \le d_G(z_1, z_2)+\min_{i=1,2}\{d_G(z_i, Tz_i)\}. \tag{3.1}$$

Proof In order to prove the inequalities previous to (3.1), it suffices to apply Lemma 3.3 for any $c_0 > 0$ and take the limit as $c_0 \to 0^+$.

The right hand side of (3.1) follows from the triangle inequality and the fact $d_G(Tz_1, Tz_2) = d_G(z_1, z_2)$. The left hand side follows by symmetry and the previous inequalities. □

Some notation is needed for the second corollary. Given $z \in T^m\gamma_0$, $w \in T^n\gamma_0$, define $D_G(z, w)$ as follows: if $m = n$, set $D_G(z, w) := d_G(z, w)$; if $m < n$, then

$$D_G(z, w) := \inf\left\{\sum_{j=m}^{n-1}\Big(d_G(x_j, T^{-1}x_{j+1}) + d_G(T^{-1}x_{j+1}, x_{j+1})\Big) + d_G(x_n, w)\right\},$$

where the infimum is taken among all sets of points $\{x_j\}_{j=m}^n$ with $x_j \in T^j\gamma_0$ and $x_m = z$; finally, if $m > n$ define $D_G(z, w) := D_G(w, z)$. (One can check that the infimum above is in fact a minimum; see, *e.g.*, [6, p. 24]).

Corollary 3.5 *Let G be a periodic graph with $\inf_{z\in\gamma_0} d_G(z, Tz) = 0$. Then $d_G(z_1, z_2) \le d_G(z_1, T^n z_2)$ and $D_G(z_1, T^n z_2)/3 \le d_G(z_1, T^n z_2) \le D_G(z_1, T^n z_2)$ for every $z_1, z_2 \in \gamma_0$ and $n \in \mathbb{Z}$.*

Lemma 3.6 *Let G be a periodic graph. Assume that there exist an unbounded sequence $\{\zeta_n\} \subset \gamma_0$ and some constant c_0 with $d_G(\zeta_n, T\zeta_n) \le c_0$ for every $n \in \mathbb{N}$. Then, for each arc-length parametrization η_0 of γ_0 one of the following situations holds:*

(1) There exists $R \in \mathbb{R}$ such that if $a \in \gamma_0$, $b \in T^m\gamma_0$ ($m \in \mathbb{Z}$) with $\eta_0^{-1}(a), \eta_m^{-1}(b) \ge R$ then $d_G(a, b) \ge \eta_m^{-1}(b) - \eta_0^{-1}(a) - c_0$.
(2) For any $m \ge 0$, $a \in \gamma_0$, $b \in T^m\gamma_0$ then $d_G(a, b) \ge \eta_m^{-1}(b) - \eta_0^{-1}(a)$.
(3) For any $m \le 0$, $a \in \gamma_0$, $b \in T^m\gamma_0$ then $d_G(a, b) \ge \eta_m^{-1}(b) - \eta_0^{-1}(a)$.
(Recall the notation $\eta_m = T^m \circ \eta_0$ for a parametrization of $T^m\gamma_0$.)

Proof Case 1. Suppose that there exists $R \in \mathbb{R}$ so that

$$d_G(z, w) \ge |\eta_0^{-1}(z) - \eta_1^{-1}(w)| \tag{3.2}$$

for all $z \in \eta_0([R, \infty))$ and $w \in \eta_1([R, \infty))$.

Let $a \in \gamma_0$ and $b \in T^m\gamma_0$ with $\eta_m^{-1}(b) \ge \eta_0^{-1}(a) \ge R$ and $m \ge 0$ (if $\eta_m^{-1}(b) < \eta_0^{-1}(a)$, then $d_G(a, b) \ge 0 > \eta_m^{-1}(b) - \eta_0^{-1}(a) - c_0$). Let g be a straight geodesic joining a to b and choose points $u_j \in g \cap T^j\gamma_0$, for $0 \le j \le m$, with $a = u_0$ and $b = u_m$. If $\eta_j^{-1}(u_j) \ge R$ for $0 \le j \le m$ then by (3.2),

$$\begin{aligned} d_G(a, b) &= \sum_{j=0}^{m-1} d_G(u_j, u_{j+1}) \ge \sum_{j=0}^{m-1} \left(\eta_{j+1}^{-1}(u_{j+1}) - \eta_j^{-1}(u_j)\right) \\ &= \eta_m^{-1}(u_m) - \eta_0^{-1}(u_0) = \eta_m^{-1}(b) - \eta_0^{-1}(a)\,. \end{aligned}$$

Otherwise, there exists $0 < j_0 < m$ such that $\eta_j^{-1}(u_j) \ge R$ for all $j_0 < j \le m$ and $\eta_{j_0}^{-1}(u_{j_0}) < R$. Then,

$$d_G(a, b) = \sum_{j=0}^{m-1} d_G(u_j, u_{j+1}) \ge \sum_{j=j_0}^{m-1} d_G(u_j, u_{j+1})\,.$$

By Lemma 3.3,

$$d_G(u_{j_0}, u_{j_0+1}) \ge \eta_{j_0+1}^{-1}(u_{j_0+1}) - \eta_{j_0}^{-1}(u_{j_0}) - c_0\,,$$

and by (3.2),

$$d_G(u_j, u_{j+1}) \ge \eta_{j+1}^{-1}(u_{j+1}) - \eta_j^{-1}(u_j)\,, \quad j_0 < j \le m - 1\,.$$

Therefore,

$$d_G(a,b) \geq \eta_{j_0+1}^{-1}(u_{j_0+1}) - \eta_{j_0}^{-1}(u_{j_0}) - c_0 + \sum_{j=j_0+1}^{m-1} \left(\eta_{j+1}^{-1}(u_{j+1}) - \eta_j^{-1}(u_j)\right)$$
$$= \eta_m^{-1}(u_m) - \eta_{j_0}^{-1}(u_{j_0}) - c_0 \geq \eta_m^{-1}(b) - \eta_0^{-1}(a) - c_0\,,$$

where the last inequality follows from the fact that $\eta_{j_0}^{-1}(u_{j_0}) < R \leq \eta_0^{-1}(a)$. The same argument works when $m < 0$.

Case 2. Suppose that there exist a sequence $R_k \nearrow \infty$ and sequences $z_k \in \eta_0([R_k,\infty))$, $w_k \in \eta_1([R_k,\infty))$ so that $d(z_k,w_k) < \eta_0^{-1}(z_k) - \eta_1^{-1}(w_k)$.

As above, let g be a straight geodesic joining a to b and choose points $u_j \in g \cap T^j\gamma_0$, for $0 \leq j \leq m$, with $a = u_0$ and $b = u_m$. There exists k such that $\eta_j^{-1}(u_j) < R_k$ for every $0 \leq j \leq m$. By (remark after) Lemma 3.2,

$$d_G(u_j,u_{j+1}) \geq \eta_{j+1}^{-1}(u_{j+1}) - \eta_j^{-1}(u_j)$$

and thus,

$$d_G(a,b) = \sum_{j=0}^{m-1} d_G(u_j,u_{j+1}) \geq \sum_{j=0}^{m-1} \left(\eta_{j+1}^{-1}(u_{j+1}) - \eta_j^{-1}(u_j)\right)$$
$$= \eta_m^{-1}(u_m) - \eta_0^{-1}(u_0) = \eta_m^{-1}(b) - \eta_0^{-1}(a)\,.$$

Case 3. Suppose that there exist a sequence $R_k \nearrow \infty$, and sequences $z_k \in \eta_0([R_k,\infty))$, $w_k \in \eta_1([R_k,\infty))$ such that $d(z_k,w_k) < \eta_1^{-1}(w_k) - \eta_0^{-1}(z_k)$. Let g be the straight geodesic from a to b and define points $u_j := g \cap T^{-j}\gamma_0$, for $0 \leq j \leq |m|$, with $a = u_0$ and $b = u_{|m|}$. There exists k such that $\eta_{-j}^{-1}(u_j) < R_k$ for every $0 \leq j \leq |m|$. By Lemma 3.2,

$$d_G(u_j,u_{j+1}) \geq \eta_{-j-1}^{-1}(u_{j+1}) - \eta_{-j}^{-1}(u_j)$$

and thus,

$$d_G(a,b) = \sum_{j=0}^{|m|-1} d_G(u_j,u_{j+1}) \geq \sum_{j=0}^{|m|-1} \left(\eta_{-j-1}^{-1}(u_{j+1}) - \eta_{-j}^{-1}(u_j)\right)$$
$$= \eta_m^{-1}(u_{|m|}) - \eta_0^{-1}(u_0) = \eta_m^{-1}(b) - \eta_0^{-1}(a)\,.$$

□

4 Proof of the First Part of Theorem 1.1

This section is devoted to the proof of the first part of Theorem 1.1. For clarity's sake, we shall begin by stating some lemmas and claims which will be used along the proof.

The first lemma introduces a new graph, G' (quasi-isometric to G) which will guarantee the existence of a *transversal* geodesic.

Lemma 4.1 *Let G be a periodic graph such that $d_G(\gamma_0, T\gamma_0) =: d_1 > 0$. Fix $z_0 \in \gamma_0$ and define G' by adding to G the edges $\big\{[T^n z_0, T^{n+1} z_0]\big\}_{n\in\mathbb{Z}}$ with $L\big([T^n z_0, T^{n+1} z_0]\big) = d_1$ for every $n \in \mathbb{Z}$. Then, the graphs G' and G are quasi-isometric and, moreover, $\cup_{n\in\mathbb{Z}}[T^n z_0, T^{n+1} z_0]$ is a geodesic in G'.*

Proof It is clear that $\cup_{n\in\mathbb{Z}}[T^n z_0, T^{n+1} z_0]$ is a geodesic in G'. It will be shown that the inclusion $i : G \to G'$ is a quasi-isometry. Clearly, the inequality $d_{G'}(x, y) \le d_G(x, y)$ holds for every $x, y \in G$.

Consider $x, y \in G$. If x, y are so that $d_{G'}(x, y) = d_G(x, y)$, then there is nothing to prove. If $d_{G'}(x, y) < d_G(x, y)$, then there exist $m, n \in \mathbb{Z}$ such that $d_{G'}(x, y) = d_G(x, T^m z_0) + d_{G'}(T^m z_0, T^n z_0) + d_G(T^n z_0, y)$. Hence,

$$
\begin{aligned}
d_G(x, y) &\le d_G(x, T^m z_0) + d_G(T^m z_0, T^n z_0) + d_G(T^n z_0, y) \le d_G(x, T^m z_0) \\
&\qquad + |m-n| d_G(z_0, T z_0) + d_G(T^n z_0, y) \\
&\le \frac{d_G(z_0, T z_0)}{d_1}\big(d_G(x, T^m z_0) + |m-n| d_1 + d_G(T^n z_0, y)\big) \\
&= \frac{d_G(z_0, T z_0)}{d_1}\big(d_G(x, T^m z_0) + d_{G'}(T^m z_0, T^n z_0) + d_G(T^n z_0, y)\big) \\
&= \frac{d_G(z_0, T z_0)}{d_1}\, d_{G'}(x, y)\,.
\end{aligned}
$$

Since $L([T^n z_0, T^{n+1} z_0]) = d_1$ for every $n \in \mathbb{Z}$, the map i is $(d_1/2)$-full, and we conclude that G' and G are quasi-isometric. □

The next lemma will show that a certain curve on the graph G is a quasi-geodesic.

Lemma 4.2 *Let G be a periodic graph such that $\inf_{z\in\gamma_0} d_G(z, Tz) =: d_0 > 0$. Let $\zeta \in \gamma_0$ and let σ be a geodesic in G^* joining ζ and $T\zeta$. Then, for each $m \in \mathbb{N}$ the curve $\sigma^m := \bigcup_{j=0}^{m-1} T^j\sigma$ is an (α_0, β_0)-quasi-geodesic in G, with α_0, β_0 depending only on $d_G(\zeta, T\zeta)$, d_0 and $d_G(\gamma_0, T\gamma_0)$.*

In fact, the explicit expressions for α_0 and β_0 will be obtained in the proof of this lemma.

Proof Notice that σ^m is a continuous curve in G joining ζ and $T^m\zeta$. Define $c_0 := d_G(\zeta, T\zeta)$. Fix an arc-length parametrization of σ^m starting at ζ and $s, t \in \mathbb{R}$ in the domain of σ^m with $s < t$. Clearly $d_G(\sigma^m(t), \sigma^m(s)) \le L(\sigma^m|_{[s,t]}) = t - s$. Let $j, r \in \mathbb{N}$ be so that $\sigma^m(s) \in T^j\sigma$ and $\sigma^m(t) \in T^{j+r}\sigma$. The following inequality holds

$$t - s \le (r+1)L(\sigma) = (r+1)d_G(\zeta, T\zeta) = (r+1)c_0\,. \tag{4.1}$$

For the lower bound, notice first that if $d_1 := d_G(\gamma_0, T\gamma_0) > 0$,

$$d_G(\sigma^m(t), \sigma^m(s)) \ge (r-1)d_1 = (r+1)d_1 - 2d_1 \ge \frac{d_1}{c_0}(t-s) - 2d_1\,.$$

Assume next that $d_G(\gamma_0, T\gamma_0) = 0$. Since $d_0 > 0$, there exist monotonous unbounded sequences $\{z'_n\} \subset \gamma_0$ and $\{w'_n\} \subset T\gamma_0$ with $d_G(z'_n, w'_n) < d_0/2$. Fix an arc-length parametrization η_0 of γ_0 such that there exists a subsequence $\{z'_{n_k}\}$ with $\lim_{k\to\infty} \eta_0^{-1}(z'_{n_k}) = \infty$; without loss of generality by replacing $\{z'_n\}$ by the subsequence $\{z'_{n_k}\}$ if necessary, one can assume that $\lim_{k\to\infty} \eta_0^{-1}(z'_n) = \infty$. Recall the notation for η_k.

Assume that $\eta_1^{-1}(w'_n) - \eta_0^{-1}(z'_n) \ge 0$ for infinitely many $n's$ (otherwise, the argument is symmetric). By choosing a subsequence if necessary, one can assume without loss of generality that $\eta_1^{-1}(w'_n) - \eta_0^{-1}(z'_n) \ge 0$ for every n. Then,

$$\begin{aligned}\eta_1^{-1}(w'_n) - \eta_0^{-1}(z'_n) &= d_G(w'_n, Tz'_n) \ge d_G(z'_n, Tz'_n) - d_G(z'_n, w'_n)\\ &> d_0 - \frac{d_0}{2} = \frac{d_0}{2} \ge d_G(z'_n, w'_n)\,. \end{aligned} \tag{4.2}$$

Let $s' \le s \le t \le t'$ such that $\sigma^m(s')$ is the first point of σ^m in $T^j\sigma$ and $\sigma^m(t')$ is the last point of σ^m in $T^{j+r}\sigma$; then $d_G(\sigma^m(s'), \sigma^m(s)) = s - s' \le c_0$ and $d_G(\sigma^m(t'), \sigma^m(t)) = t' - t \le c_0$. Let Γ be a geodesic joining $\sigma^m(s')$ and $\sigma^m(t')$. Define $x_0 := \sigma^m(s') \in T^j\gamma_0$, $x_{r+1} := \sigma^m(t') \in T^{j+r+1}\gamma_0$, and let x_i be any point of Γ in $T^{j+i}\gamma_0$ for $1 \le i \le r$.

Define N_1, N_{21}, N_{22}, as the sets of indices

$$\begin{aligned} N_1 &:= \left\{0 \le i \le r : \quad \eta_{j+i}^{-1}(x_i) \ge \eta_{j+i+1}^{-1}(x_{i+1})\right\},\\ N_{21} &:= \left\{0 \le i \le r : \quad \eta_{j+i}^{-1}(x_i) < \eta_{j+i+1}^{-1}(x_{i+1}) \text{ and } d_G(x_i, x_{i+1}) \ge d_0/2\right\},\\ N_{22} &:= \left\{0 \le i \le r : \quad \eta_{j+i}^{-1}(x_i) < \eta_{j+i+1}^{-1}(x_{i+1}) \text{ and } d_G(x_i, x_{i+1}) < d_0/2\right\}. \end{aligned}$$

Then card N_1 + card N_{21} + card N_{22} $= r + 1$. For $i \in N_1$, $\eta_{j+i}^{-1}(x_i) \ge \eta_{j+i+1}^{-1}(x_{i+1})$. Take $n \in \mathbb{N}$ so that $\eta_0^{-1}(z'_n) > \eta_{j+i}^{-1}(x_i)$. Then, by (4.2) the points x_i and x_{i+1} are under the hypothesis of Lemma 3.2, and hence

$$\begin{aligned} d_G(x_i, x_{i+1}) \ge \eta_{j+i}^{-1}(x_i) - \eta_{j+i+1}^{-1}(x_{i+1}) = d_G(x_{i+1}, Tx_i) \ge d_G(x_i, Tx_i)\\ -d_G(x_{i+1}, x_i) \ge d_0 - d_G(x_i, x_{i+1}) \end{aligned}$$

and conclude $d_G(x_i, x_{i+1}) \ge d_0/2$.

If card N_1 + card $N_{21} \ge (r+1)/2$, then

$$d_G(\sigma^m(s), \sigma^m(t)) + 2c_0 \ge d_G\big(\sigma^m(s'), \sigma^m(t')\big) = \sum_{i=0}^{r} d_G(x_i, x_{i+1}) \ge \frac{d_0}{4}\,(r+1)\,.$$

Hence, by (4.1),

$$d_G(\sigma^m(t), \sigma^m(s)) \ge \frac{d_0}{4}(r+1) - 2c_0 \ge \frac{d_0}{4c_0}(t-s) - 2c_0\,.$$

Assume now that card $N_{22} \geq (r+1)/2$. Note that if $i \in N_{22}$, then

$$\eta_{j+i+1}^{-1}(x_{i+1}) - \eta_{j+i}^{-1}(x_i) = d_G(x_{i+1}, Tx_i) \geq d_G(x_i, Tx_i) - d_G(x_{i+1}, x_i) \\ \geq d_0 - \frac{d_0}{2} = \frac{d_0}{2},$$

and therefore

$$\sum_{i \in N_{22}} \left(\eta_{j+i+1}^{-1}(x_{i+1}) - \eta_{j+i}^{-1}(x_i) \right) \geq \frac{d_0}{2} \text{ card } N_{22} \geq \frac{d_0}{4}(r+1).$$

Note that

$$\sum_{i \in N_{22}} \left(\eta_{j+i+1}^{-1}(x_{i+1}) - \eta_{j+i}^{-1}(x_i) \right) \leq \sum_{i \in N_{22} \cup N_{21}} \left(\eta_{j+i+1}^{-1}(x_{i+1}) - \eta_{j+i}^{-1}(x_i) \right) \\ = \sum_{i \in N_1} \left(\eta_{j+i}^{-1}(x_i) - \eta_{j+i+1}^{-1}(x_{i+1}) \right)$$

since $\eta_{j+r+1}^{-1}(x_{r+1}) = \eta_j^{-1}(x_0)$. Therefore, applying Lemma 3.2,

$$\sum_{i \in N_1} \left(\eta_{j+i}^{-1}(x_i) - \eta_{j+i+1}^{-1}(x_{i+1}) \right) \leq \sum_{i \in N_1} d_G(x_i, x_{i+1}) \\ \leq \sum_{i=0}^{r} d_G(x_i, x_{i+1}) = d_G(\sigma^m(s'), \sigma^m(t')) \\ \leq d_G(\sigma^m(s), \sigma^m(t)) + 2c_0\,.$$

Hence,

$$d_G\left(\sigma^m(t), \sigma^m(s)\right) \geq \frac{d_0}{4}(r+1) - 2c_0 \geq \frac{d_0}{4c_0}(t-s) - 2c_0\,.$$

One concludes that σ^m is an (α_0, β_0)-quasigeodesic (for every m), where $\alpha_0 = c_0/d_1$ if $d_1 > 0$ (note that $c_0 \geq d_0 \geq d_1$), $\alpha_0 = 4c_0/d_0$ if $d_1 = 0$, and $\beta_0 = \max\{2c_0, 2d_1\}$. □

With these previous lemmas established, let us proceed to prove the first part of Theorem 1.1, the main goal of this section.

Proof (First part of Theorem 1.1). Assume first that G is hyperbolic. Since γ_0 and $T\gamma_0$ are geodesic lines, G^* is an isometric subgraph of G and $\delta(G^*) \leq \delta(G)$. Thus, it remains to show that $\lim_{|z|\to\infty, z\in\gamma_0} d_G(z, Tz) = \infty$.

Assume that there exists an unbounded sequence $\{\zeta_n\}_{n\geq 1} \subset \gamma_0$ and a constant c_0 with $d_G(\zeta_n, T\zeta_n) \leq c_0$ for every n. Choosing a subsequence of $\{\zeta_n\}_{n\geq 1}$ if it is necessary, one can assume that there exists an arc-length parametrization η_0 of γ_0 with

$\eta_0^{-1}(\zeta_n) \nearrow \infty$. Let σ_n be a geodesic in G^* joining ζ_n and $T\zeta_n$. Let $\sigma_n^m := \cup_{k=0}^{m-1} T^k\sigma_n$ and γ_0^n be the subcurve of γ_0 joining ζ_{n_0} and ζ_n, where n_0 is chosen as follows: if (1) in Lemma 3.6 holds, take n_0 with $\eta_0^{-1}(\zeta_{n_0}) \ge R$; otherwise, take $n_0 = 1$. Hence, by Lemma 4.2, $Q_{n,m} := \{\gamma_0^n, \sigma_n^m, T^m\gamma_0^n, \sigma_{n_0}^m\}$ is an (α_0, β_0)-quasigeodesic quadrilateral for every n, m, where α_0 and β_0 do not depend on n and m.

Since G is hyperbolic, by Lemma 2.1, $Q_{n,m}$ is $(2\delta(G) + 2H)$-thin, with $H = H(\delta(G), \alpha_0, \beta_0)$ for any n, m. Let M be a constant with $M > 2\delta(G) + 2H$.

Taking $n \in \mathbb{N}$ large enough, $L(\gamma_0^n) > 2M + 4c_0$, and taking $m = m(n)$ large enough, $d_G\left(\gamma_0^n, T^m\gamma_0^n\right) > M$. Choose a point $p \in \gamma_0^n$ so that,

(1) $d_G(p, \zeta_{n_0}) = \eta_0^{-1}(p) - \eta_0^{-1}(\zeta_{n_0}) > M + 2c_0$,
(2) $d_G(p, \zeta_n) = \eta_0^{-1}(\zeta_n) - \eta_0^{-1}(p) > M + 2c_0$.

We also have $d_G(p, T^m\gamma_0^n) \ge d_G(\gamma_0^n, T^m\gamma_0^n) > M$.

Let us proceed to show that $d_G(p, \sigma_{n_0}^m) > M$. Let V^m be the set of points $V^m := \left\{\zeta_{n_0}, T\zeta_{n_0}, T^2\zeta_{n_0}, \dots, T^m\zeta_{n_0}\right\}$. By the triangle inequality, it is enough to show that $d_G(p, V^m) > M + c_0$.

Case I. Assume that (1) in Lemma 3.6 holds. Since $R \le \eta_0^{-1}(\zeta_{n_0}) = \eta_k^{-1}(T^k\zeta_{n_0}) < \eta_0^{-1}(p)$ for $0 \le k \le m$, Lemma 3.6 (1) gives,

$$d_G(p, T^k\zeta_{n_0}) \ge \eta_0^{-1}(p) - \eta_0^{-1}(\zeta_{n_0}) - c_0 > M + c_0\,,$$

thus $d_G(p, V^m) > M + c_0$.

Case II. Suppose that (2) in Lemma 3.6 holds. Then,

$$d_G(p, T^k\zeta_{n_0}) \ge \eta_0^{-1}(p) - \eta_k^{-1}(T^k\zeta_{n_0}) = \eta_0^{-1}(p) - \eta_0^{-1}(\zeta_{n_0}) > M + 2c_0\,,$$

thus $d_G(p, V^m) > M + 2c_0 > M + c_0$.

Case III. If (3) in Lemma 3.6 holds, the argument in case II gives the result, taking now $m \le k \le 0$.

A similar argument shows also that $d_G(p, \sigma_n^m) > M$. Hence, $d_G(p, T^m\gamma_0^n \cup \sigma_{n_0}^m \cup \sigma_n^m) > M$. Since $M > 2\delta(G) + 2H$, the quadrilateral $Q_{n,m}$ is not $(2\delta(G) + 2H)$-thin, which is a contradiction. Therefore, G is not hyperbolic.

Let us prove the converse implication to conclude that G is hyperbolic. Since $\lim_{|z|\to\infty, z\in\gamma_0} d_G(z, Tz) = \infty$, then $d_G(\gamma_0, T\gamma_0) =: d_1 > 0$. By Lemma 4.1, without loss of generality one can assume that there exists a vertex $z_0 \in V(G) \cap \gamma_0$ such that $[z_0, Tz_0] \in E(G)$, with $L([z_0, Tz_0]) = d_G(\gamma_0, T\gamma_0) = d_1$, and so that $\sigma_0 := \cup_{n\in\mathbb{Z}}[T^n z_0, T^{n+1}z_0]$ is a geodesic in G. Define $\delta^* := \delta(G^*)$ and consider a geodesic triangle $\mathcal{T} = \{x_1, x_2, x_3\}$ with $x_i \in T^{j_i}G^*$ and $j_1 \le j_2 \le j_3$. By Lemma C, one can assume that the geodesics of $\mathcal{T}$ are straight.

Suppose first that $\max\{j_2 - j_1, j_3 - j_2\} \le 2$. Then, $\mathcal{T} \subset \cup_{j=j_2-2}^{j_2+2} T^j G^*$ is δ_0-thin, with $\delta_0 = (120)^4\delta^*$ since T^jG^* is δ^*-hyperbolic (apply at most four times Lemma B). Otherwise, $\mathcal{T} \cap \left(T^{j_2-1}\gamma_0 \cup T^{j_2+2}\gamma_0\right) \neq \emptyset$. If $\mathcal{T} \cap \left(T^{j_2-1}\gamma_0\right) \neq \emptyset$, choose $y_1 \in [x_1x_2] \cap T^{j_2-1}\gamma_0$ and $y_2 \in [x_1x_3] \cap T^{j_2-1}\gamma_0$. By Lemma 3.1,

$$d_G(y_1, y_2) \le 6N + 5d_1\,. \tag{4.3}$$

Analogously, if $\mathcal{T} \cap \left(T^{j_2+2}\gamma_0\right) \neq \emptyset$, let $z_1 \in [x_1x_3] \cap T^{j_2+2}\gamma_0$ and $z_2 \in [x_2x_3] \cap T^{j_2+2}\gamma_0$. Again, by Lemma 3.1,

$$d_G(z_1, z_2) \leq 6N + 5d_1 \,. \tag{4.4}$$

Let $p \in \mathcal{T}$. If $p \in T^jG^*$ with $j \in [j_1+2, j_2-2] \cup [j_2+2, j_3-2]$, apply Lemma 3.1 to find $q \in T^jG^*$ on another side of $\mathcal{T}$ with $d_G(p,q) \leq 6N + 5d_1$.

If $p \in T^jG^*$ with $j \in [j_2-1, j_2+1]$, let $\mathcal{P} \subset \cup_{j=j_2-1}^{j_2+1} T^jG^*$ be the geodesic polygon formed by $\mathcal{T} \cap \cup_{j=j_2-1}^{j_2+1} T^jG^*$ and $[y_1y_2] \subset T^{j_2-1}\gamma_0$ and $[z_1z_2] \subset T^{j_2+2}\gamma_0$ whenever they exist. Thus, $\mathcal{P}$ is either a pentagon or a quadrilateral contained in $\cup_{j=j_2-2}^{j_2+2} T^jG^*$ and therefore it is $3\delta_0-$thin. Therefore, there exists a point $q' \in \mathcal{P}$ on another side of $\mathcal{P}$ so that $d_G(p,q') \leq 3\delta_0$. If $q' \notin \mathcal{T}$, then $q' \in [y_1y_2] \cup [z_1z_2]$ and equations (4.3) and (4.4) imply that there is $q \in \mathcal{P} \cap \mathcal{T}$ on another side of $\mathcal{T}$ with $d_G(p,q) \leq 3\delta_0 + 6N + 5d_1$.

If $p \in T^jG^*$ with $j \in \{j_1, j_1+1, j_3-1, j_3\}$, a similar argument with a triangle (in $T^{j_1}G^* \cup T^{j_1+1}G^*$ or $T^{j_3-1}G^* \cup T^{j_3}G^*$) instead of $\mathcal{P}$ gives $d_G(p,q) \leq \delta_0 + 6N + 5d_1$.

Hence, $\delta(\mathcal{T}) \leq 3\delta_0 + 6N + 5d_1$ and Lemma C gives $\delta(G) \leq 2M + 3\delta_0 + 6N + 5d_1$. □

5 Proof of the Second Part of Theorem 1.1

To prove the second part of Theorem 1.1, some auxiliary metric spaces will be defined, and some results relating these new sets with the original one will be given.

Let G be a periodic graph. Sometimes we will require the arc-length parametrization η_0 of γ_0 to also satisfy:

$$0 = \liminf_{t\to\infty} d_G(\eta_0(t), T\eta_0(t)) \leq \limsup_{t\to\infty} d_G(\eta_0(t), T\eta_0(t)) < \infty. \tag{5.1}$$

Fix $t_0 \in \mathbb{R}$ and η_0. Define G_1 as the geodesic metric space given by $G \cup \left(\cup_{n\in\mathbb{Z}, t\geq t_0} U_{n,t}\right)$, where $U_{n,t}$ is a segment joining $T^n\eta_0(t)$ with $T^{n+1}\eta_0(t)$ of length $d_G(\eta_0(t), T\eta_0(t))$. Set G_2 to be the geodesic metric space given by $\left(\cup_{n\in\mathbb{Z}} T^n\eta_0([t_0,\infty))\right) \cup \left(\cup_{n\in\mathbb{Z}, t\geq t_0} U_{n,t}\right)$. The isometry T can be extended to G_1 in an obvious way; also denote this extension by T. Define a *period graph* of G_1 as $G_1^* := G^* \cup \left(\cup_{t\geq t_0} U_{0,t}\right)$. Below, the constant t_0 will be chosen as the constant in Lemma 5.12.

It is clear that G, G_2 are contained in G_1, $G \cup G_2 = G_1$, and G is an isometric subspace of G_1; thus $\delta(G) \leq \delta(G_1)$.

With these definitions in mind, let us state some results on hyperbolicity.

Lemma 5.1 *If a periodic graph G is hyperbolic and satisfies (5.1) and $\liminf_{t\to-\infty} d_G(\eta_0(t), T\eta_0(t)) > 0$, then G_2 is hyperbolic.*

Proof Given any fixed $t_0 \in \mathbb{R}$, the hypotheses imply that there exist constants M, m such that $d_G\big(\eta_0(t), T\eta_0(t)\big) \leq M$ for every $t \in [t_0, \infty)$ and $d_G\big(\eta_0(t), T\eta_0(t)\big) \geq m$ for every $t \in (-\infty, t_0]$; then every segment $U_{n,t}$ has length at most M and $D_G \leq$

$d_{G_2} \le (M/m) D_G$ on $\cup_{n\in\mathbb{Z}} T^n\eta_0([t_0,\infty))$. Consider the map $f : G_2 \to G$ defined by $f(x) = T^n\eta_0(t)$ for every $x \in U_{n,t} \setminus T^{n+1}\eta_0(t)$. By Corollary 3.5, the restriction of f to $\cup_{n\in\mathbb{Z}} T^n\eta_0([t_0,\infty))$ (the identity map) is a $(3M/m, 0)$-quasi-isometric embedding. Since $L(U_{n,t}) \le M$ for every $n \in \mathbb{Z}$, $t \ge t_0$, f is a quasi-isometric embedding and invariance of hyperbolicity gives the result. □

Lemma 5.2 *Consider a periodic graph G satisfying (5.1). Then G^* is hyperbolic if and only if G_1^* is hyperbolic.*

Proof By (5.1), there exists a constant M such that $d_G\big(\eta_0(t), T\eta_0(t)\big) \le M$ for every $t \in [t_0,\infty)$; then every segment $U_{n,t}$ has length at most M. The inclusion map $i : G^* \to G_1^*$ is a $(M/2)$-full $(1,0)$-quasi-isometry, and thus, the invariance of hyperbolicity gives the result. □

Finally, the last auxiliary space will be defined and its hyperbolicity related to that of G will be stated.

Given $t_0 \in \mathbb{R}$ and η_0, define G_3 as the geodesic metric space given by $\big(\cup_{n\in\mathbb{Z}} T^n\eta_0([t_0,\infty))\big) \cup \big(\cup_{n\in\mathbb{Z},t\ge t_0} V_{n,t}\big)$, where $V_{n,t}$ is a segment joining $T^n\eta_0(t)$ with $T^{n+1}\eta_0(t)$ of length $\Phi(t)$, where Φ is the *greatest non-increasing minorant* of $d_G\big(\eta_0(t), T\eta_0(t)\big)$ on $[t_0,\infty)$, i.e., $\Phi(t) = \min\big\{d_G\big(\eta_0(s), T\eta_0(s)\big) : s \in [t_0,t]\big\}$.

Lemma 5.3 *Let G be a periodic graph satisfying (5.1) and* $\sup\big\{t_2 - t_1 : \Phi(t_1) = \Phi(t_2),\ t_2 \ge t_1 \ge t_0\big\} < \infty$. *Then G_2 and G_3 are quasi-isometric.*

Proof Consider the map $f : G_3 \to G_2$ defined as the identity on $\cup_{n\in\mathbb{Z}} T^n\eta_0([t_0,\infty))$ and as a dilation on each $V_{n,t}$ with $f(V_{n,t}) = U_{n,t}$ for every $n \in \mathbb{Z}$, $t \ge t_0$.

Clearly, f is 0-full and $d_{G_2}(f(x), f(y)) \ge d_{G_3}(x,y)$ for every $x, y \in G_3$. By (5.1), there exists a constant M such that $L(U_{n,t}) \le M$ for every $n \in \mathbb{Z}$, $t \ge t_0$. Also $L(V_{n,t}) \le L(U_{n,t}) \le M$ for every $n \in \mathbb{Z}$, $t \ge t_0$. Define $N := \sup\big\{t_2 - t_1 : \Phi(t_1) = \Phi(t_2),\ t_2 \ge t_1 \ge t_0\big\} < \infty$.

Given $x_0 \in T^m\eta_0([t_0,\infty))$ and $y_0 \in T^n\eta_0([t_0,\infty))$ with $m \le n$, let γ be a geodesic in G_3 joining x_0 and y_0 such that $\gamma = [x_0\eta_m(t)] \cup V_{m,t} \cup \cdots \cup V_{n-1,t} \cup [\eta_n(t)y_0]$ for some $t \ge t_0$. Let $t' \ge t$ be defined as $t' := \sup\big\{s : \Phi(s) = \Phi(t),\ s \ge t\big\} \le t + N$; thus $d_{G_3}\big(\eta_0(t'), T\eta_0(t')\big) = \Phi(t') = \Phi(t)$ and $L(V_{k,t}) = L(U_{k,t'})$ for every $k \in \mathbb{Z}$. Consider the curve γ_1 in G_2 joining x_0 and y_0 given by $\gamma_1 := [x_0\eta_m(t')] \cup U_{m,t'} \cup \cdots \cup U_{n-1,t'} \cup [\eta_n(t')y_0]$; then $d_{G_2}(f(x_0), f(y_0)) \le L(\gamma_1) \le L(\gamma) + 2N = d_{G_3}(x_0,y_0) + 2N$.

Finally, since $L(V_{n,t}) \le L(U_{n,t}) \le M$ for every $n \in \mathbb{Z}$, $t \ge t_0$, given $x, y \in G_3$, then $d_{G_2}(f(x), f(y)) \le d_{G_3}(x,y) + 2N + 2M$. □

Lemmas 5.1 and 5.3 and the invariance of hyperbolicity, imply the following result.

Lemma 5.4 *Let G be a periodic graph satisfying (5.1),* $\liminf_{t\to-\infty} d_G(\eta_0(t), T\eta_0(t)) > 0$ *and* $\sup\big\{t_2 - t_1 : \Phi(t_1) = \Phi(t_2),\ t_2 \ge t_1 \ge t_0\big\} < \infty$. *If G is hyperbolic, then G_3 is hyperbolic.*

Recall the definition of quasi-exponential decay given below Theorem 1.1.

Lemma 5.5 *Let G be any periodic graph. If G has quasi-exponential decay, then, for any fixed* t_0, $\sup\{t_2 - t_1 : \Phi(t_1) = \Phi(t_2),\ t_2 \ge t_1 \ge t_0\} < \infty$ *and* (5.1) *holds.*

Proof Fix t_0 and let $K := \sup_{s_2 \ge s_1 \ge t_0} (s_2 - s_1)\Phi(s_2)/\Phi(s_1) < \infty$. If $t_2 \ge t_1 \ge t_0$ and $\Phi(t_1) = \Phi(t_2)$, then $t_2 - t_1 = (t_2 - t_1)\Phi(t_2)/\Phi(t_1) \le K$. Recall that $\liminf_{t\to\infty} F(t) = 0$ and that $\Phi(t) \le F(t)$. Given $\varepsilon > 0$, take $t_\varepsilon = \inf\{t \in \mathbb{R} : \Phi(s) \le \varepsilon$ for all $s \ge t\}$. Clearly, $F(t_\varepsilon) = \Phi(t_\varepsilon) = \varepsilon$. Let $t > t_\varepsilon$. If $F(t) = \Phi(t)$, then $F(t) \le \varepsilon < K + \varepsilon$. Otherwise $F(t) > \Phi(t)$ and there exist t_1, t_2 such that $t_\varepsilon \le t_1 < t < t_2$ and $F(t_1) = \Phi(t_1) = \Phi(t) = \Phi(t_2) = F(t_2) \le \varepsilon$. Then, $F(t) > F(t_1)$ and, since F is Lipschitz, $F(t) - F(t_1) \le 2(t - t_1)$, $F(t) - F(t_2) \le 2(t_2 - t)$, and thus $F(t) \le t_2 - t_1 + F(t_1) \le t_2 - t_1 + \varepsilon$. Using that $t_2 - t_1 \le K$, one deduces $F(t) \le K + \varepsilon$. Consequently, $\limsup_{t\to\infty} F(t) \le K < \infty$ and (5.1) holds. □

Given a periodic graph G, a geodesic in G_3 is a *fundamental geodesic* if it is equal to $\cup_{n=n_1}^{n_2} V_{n,t}$ for some $n_1, n_2 \in \mathbb{Z}, t \ge t_0$. Define $\mathfrak{L}(G_3) := \sup\big\{L(\gamma) : \gamma$ is a fundamental geodesic in $G_3\big\}$.

Lemma 5.6 *Let G be a periodic graph.*

(1) *If* $\mathfrak{L}(G_3) = \infty$, *then* G_3 *is not hyperbolic.*

(2) $\mathfrak{L}(G_3) < \infty$ *if and only if* $\sup_{s_2 \ge s_1 \ge t_0} (s_2 - s_1)\Phi(s_2)/\Phi(s_1) < \infty$. *In fact, if* $\sup_{s_2 \ge s_1 \ge t_0} (s_2 - s_1)\Phi(s_2)/\Phi(s_1) =: K < \infty$, *then* $\mathfrak{L}(G_3) \le 8K$.

Proof (1) Assume first that $\mathfrak{L}(G_3) = \infty$. Note that if $\cup_{n=n_1}^{n_2} V_{n,t}$ is a fundamental geodesic, then $\cup_{n=n_1+k}^{n_2+k} V_{n,t}$ is also a fundamental geodesic for every $k \in \mathbb{Z}$; hence,

$$\mathfrak{L}(G_3) = \sup\{L(\gamma) : \gamma = \cup_{n=0}^{n_2} V_{n,t} \text{ is a fundamental geodesic in } G_3\}.$$

Consider any fixed fundamental geodesic $\sigma = \cup_{n=0}^{n_2} V_{n,t}$ for some $n_2 \in \mathbb{N}, t \ge t_0$, with $L(\sigma) = \ell$. Since $\mathfrak{L}(G_3) = \infty$, one can find $t' \ge t + \ell$ such that $\sigma' = \cup_{n=0}^{n_2} V_{n,t'}$ is also a fundamental geodesic. Define $\sigma_1 := \eta_0([t, t'])$, $\sigma_2 := \eta_{n_2+1}([t, t'])$ and the geodesic quadrilateral $Q := \{\sigma, \sigma_1, \sigma_2, \sigma'\}$.

If $p = \eta_0(t + \ell/4)$, then $d_{G_3}(p, \sigma) = \ell/4$, $d_{G_3}(p, \sigma') \ge 3\ell/4$; choose $s \ge 0$ so that $d_{G_3}(p, \sigma_2) = s + (1 + n_2)\Phi(s + t + \ell/4)$. If $s > \ell/4$, then $d_{G_3}(p, \sigma_2) \ge s > \ell/4$. If $0 \le s \le \ell/4$, then $d_{G_3}(p, \sigma_2) \ge 2(s + \ell/4) - 3\ell/4 + (1 + n_2)\Phi(s + t + \ell/4)$. Since σ is a geodesic, $\ell \le 2(s + \ell/4) + (1 + n_2)\Phi(s + t + \ell/4)$, and therefore, $d_{G_3}(p, \sigma_2) \ge \ell - 3\ell/4 = \ell/4$. Hence, $2\delta(G_3) \ge \delta(Q) \ge \ell/4$ and we conclude that G_3 is not hyperbolic, since $\mathfrak{L}(G_3) = \infty$.

(2) Assume now that $l := \mathfrak{L}(G_3) < \infty$. Let $s_1 \ge t_0$ and $n \in \mathbb{N}$ with $n\Phi(s_1) > l$. Therefore, $\cup_{k=0}^{n-1} V_{k,s_1}$ is not a geodesic joining $\eta_0(s_1)$ and $\eta_n(s_1)$; then there exits $s_{2,n} > s_1$ with $n\Phi(s_1) > 2(s_{2,n} - s_1) + n\Phi(s_{2,n}) = d_{G_3}\big(\eta_0(s_1), \eta_n(s_1)\big)$. It is possible to choose the sequence $\{s_{2,n}\}$ with $s_{2,n+1} \ge s_{2,n}$. Hence, $2(s_{2,n} - s_1) < n\Phi(s_1)$, $\cup_{k=0}^{n-1} V_{k,s_{2,n}}$ is a fundamental geodesic and $n\Phi(s_{2,n}) \le l$. We conclude that $2(s_{2,n} - s_1)\Phi(s_{2,n})/\Phi(s_1) < n\Phi(s_1)\Phi(s_{2,n})/\Phi(s_1) \le l$.

Furthermore, $d_{G_3}\big(\eta_0(s_{2,n}), \eta_{n+1}(s_{2,n})\big) \le (n + 1)\Phi(s_{2,n}) \le 2n\Phi(s_{2,n}) \le 2l$. Since any sub-arc of a geodesic is again a geodesic, it is clear that $2(s_{2,n+1} - s_{2,n}) < 2(s_{2,n+1} - s_{2,n}) + (n+1)\Phi(s_{2,n+1}) \le (n+1)\Phi(s_{2,n}) \le 2l$, and then $s_{2,n+1} < s_{2,n} + l$. If $s_2 \in [s_{2,n}, s_{2,n+1}]$, then

$$(s_2 - s_1)\frac{\Phi(s_2)}{\Phi(s_1)} < (s_{2,n} + l - s_1)\frac{\Phi(s_{2,n})}{\Phi(s_1)} \le \frac{l}{2} + l\,\frac{\Phi(s_{2,n})}{\Phi(s_1)} \le \frac{3l}{2}\,.$$

Let n_0 be the least integer such that $n_0\Phi(s_1) > l$. Thus, $n_0\Phi(s_1) = (n_0-1)\Phi(s_1)+\Phi(s_1) \le l + \Phi(t_0)$ and $2(s_{2,n_0} - s_1) < 2(s_{2,n_0} - s_1) + n_0\Phi(s_{2,n_0}) \le n_0\Phi(s_1) \le l + \Phi(t_0)$. If $s_2 \in [s_1, s_{2,n_0}]$, then

$$(s_2 - s_1)\frac{\Phi(s_2)}{\Phi(s_1)} \le s_{2,n_0} - s_1 \le \frac{1}{2}\big(l + \Phi(t_0)\big),$$

and we conclude, since $\mathfrak{L}(G_3) < \infty$ implies $\lim_{n\to\infty} s_{2,n} = \infty$, that

$$\sup_{s_2\ge s_1\ge t_0} (s_2 - s_1)\frac{\Phi(s_2)}{\Phi(s_1)} \le \max\left\{\frac{3l}{2}\,,\ \frac{1}{2}\big(l + \Phi(t_0)\big)\right\}.$$

For the reverse implication, let $K := \sup_{s_2\ge s_1\ge t_0}(s_2-s_1)\Phi(s_2)/\Phi(s_1) < \infty$. Then, any fundamental geodesic $\cup_{k_1\le n<k_2} V_{n,s}$ satisfies

$$(k_2 - k_1)\Phi(s) \le 2K + (k_2 - k_1)\Phi(s + 2K) + 2K \le 4K + (k_2 - k_1)K\frac{\Phi(s)}{2K}\,,$$
$$L\big(\cup_{k_1\le n<k_2} V_{n,s}\big) = (k_2 - k_1)\Phi(s) \le 8K.$$

Notice that this means that for a fixed s, a fundamental geodesic cannot cross arbitrarily many $T^n\gamma_0(s)$. □

Lemma 5.7 *Let G be any periodic graph with quasi-exponential decay. Then G_3 is hyperbolic.*

Proof It will be enough to show this result for triangles whose sides are certain geodesics which will be introduced below, the *canonical geodesics*, since any other geodesic of G_3 will be close to one of these.

Consider a parametrization η_0 of γ_0 satisfying

$$\sup_{s_2\ge s_1\ge t_0} (s_2 - s_1)\Phi(s_2)/\Phi(s_1) =: K < \infty. \tag{5.2}$$

Let $x_1, x_2 \in \cup_{n\in\mathbb{Z}} T^n\eta_0([t_0,\infty))$. Without loss of generality, $x_1 = T^{n_1}\eta_0(t_1)$ and $x_2 = T^{n_2}\eta_0(t_2)$ with $n_1 \le n_2$. Define $g(t) := t - t_1 + (n_2 - n_1)\Phi(t) + t - t_2$, and let t' be such that

$$g(t') = \inf\big\{g(t) : t \ge \max\{t_1, t_2\}\big\}.$$

Note that this infimum is, in fact, a minimum, and that the curve

$$\gamma_{x_1x_2} := [x_1 T^{n_1}\eta_0(t')] \cup (\cup_{n_1\le n<n_2} V_{n,t'}) \cup [T^{n_2}\eta_0(t')x_2]$$

is a geodesic in G_3 with $d_{G_3}(x_1, x_2) = L(\gamma_{x_1x_2}) = g(t')$, referred to as a *canonical geodesic* joining x_1 and x_2. If $n_1 = n_2$, then $\gamma_{x_1x_2}$ is a segment on $T^{n_1}\gamma_0$.

Any other canonical geodesic σ in G_3 joining x_1 and x_2 will be at a fixed distance from a canonical geodesic: indeed, if there exists another canonical geodesic with $g(t'') = g(t')$ (one can assume that $t'' \geq t'$), then $8K \geq (n_2 - n_1)\Phi(t') = 2(t'' - t') + (n_2 - n_1)\Phi(t'')$ by Lemma 5.6, and hence $t'' - t' \leq 4K$.

More generally, if σ is any geodesic joining x_1 and x_2 which contains just one fundamental geodesic, $\cup_{n_1 \leq n < n_2} V_{n,t}$, for which $t_0 \leq t < \max\{t_1, t_2\} := \tau$, then $\Phi(\tau) = \Phi(t)$ and the curve $\sigma' := [x_1 T^{n_1}\eta_0(\tau)] \cup (\cup_{n_1 \leq n < n_2} V_{n,\tau}) \cup [T^{n_2}\eta_0(\tau)x_2]$ is a canonical geodesic. By (5.2), $\tau - t \leq K$; since $t' - \tau \leq 4K$, $t' - t \leq 5K$, and thus $\mathcal{H}(\sigma, \gamma_{x_1x_2}) \leq 5K + \Phi(t_0)/2$.

Finally, if σ contains at least two fundamental geodesics, applying the same argument one also gets $\mathcal{H}(\sigma, \gamma_{x_1x_2}) \leq 5K + \Phi(t_0)/2$.

Consider a geodesic triangle $\mathcal{T} = \{x_1, x_2, x_3\}$ in G_3 with its vertices lying on $\cup_{n\in\mathbb{Z}} T^n\gamma_0$, concretely, $x_1 = T^{n_1}\eta_0(t_1)$, $x_2 = T^{n_2}\eta_0(t_2)$ and $x_3 = T^{n_3}\eta_0(t_3)$ with $n_1 \leq n_2 \leq n_3$. Let $\mathcal{T}_0$ be the geodesic triangle in G_3 given by $\mathcal{T}_0 = \{\gamma_{x_1x_2}, \gamma_{x_2x_3}, \gamma_{x_1x_3}\}$. If $\mathcal{T}_0$ is δ-thin, then $\mathcal{T}$ is $(\delta + 10K + \Phi(t_0))$-thin.

There exist three fundamental geodesics $g_{12} := \cup_{n_1 \leq n < n_2} V_{n,s_1} \subseteq \gamma_{x_1x_2}$, $g_{23} := \cup_{n_2 \leq n < n_3} V_{n,s_2} \subseteq \gamma_{x_2x_3}$ and $g_{13} := \cup_{n_1 \leq n <, n_3} V_{n,s_3} \subseteq \gamma_{x_1x_3}$. Assume that $s_1 \leq s_2 \leq s_3$ (the other cases are similar). Note that $L(\cup_{n_1 \leq n < n_2} V_{n,s_2}) \leq L(\cup_{n_1 \leq n < n_2} V_{n,s_1}) = L(g_{12}) \leq 8K$; thus $L(\cup_{n_1 \leq n < n_3} V_{n,s_2}) \leq 16K$ and $s_3 - s_2 \leq 8K$. Clearly, from these estimates, if p lies on one side of $\mathcal{T}_0$, then the distance from p to the union of the other two sides is less than $24K$. Any other combination of vertices x_1, x_2, x_3 gives the same estimate.

Hence, $\delta(\mathcal{T}_0) \leq 24K$ and $\delta(\mathcal{T}) \leq 34K + \Phi(t_0)$. Consequently, if H is any geodesic hexagon in G_3 with every vertex in $\cup_{n\in\mathbb{Z}} T^n\eta_0([t_0, \infty))$, then $\delta(H) \leq 4(34K + \Phi(t_0)) = 136K + 4\Phi(t_0)$.

Consider now any fixed geodesic triangle $\mathcal{T} = \{x_1, x_2, x_3\}$ in G_3 that is a simple closed curve. Assume that $x_1, x_2, x_3 \notin \cup_{n\in\mathbb{Z}} T^n\eta_0([t_0, \infty))$ (the other cases are similar). For each x_i, there exist $n_i \in \mathbb{Z}$ and $t_i \geq 0$ such that $x_i \in V_{n_i,t_i}$; let x_i' and x_i'' be the endpoints of V_{n_i,t_i}; since $\mathcal{T}$ is a simple closed curve, $V_{n_i,t_i} \subset \mathcal{T}$. Consider the geodesic hexagon $H = \{x_1', x_1'', x_2', x_2'', x_3', x_3''\}$. Since the vertices of H lie on $\cup_{n\in\mathbb{Z}} T^n\eta_0([t_0, \infty))$, $\delta(H) \leq 136K + 4\Phi(t_0)$.

Given $p \in T$, denote by $\delta(p)$ the distance from p to the union of the two other sides of T. Assume p lies on a side of H that is contained in a side of T. Then, $\delta(p) \leq \delta(H) + L(V_{n_i,t_i})$ for some $i = 1, 2, 3$. Since $L(V_{n_i,t_i}) \leq \Phi(t_i) \leq \Phi(t_0)$, then $\delta(p) \leq \delta(H) + \Phi(t_0) \leq 136K + 5\Phi(t_0)$.

If p lies on V_{n_i,t_i}, $(i = 1, 2, 3)$, then $\delta(p) \leq L(V_{n_i,t_i}) \leq \Phi(t_0)$. Hence, $\delta(p) \leq 136K + 5\Phi(t_0)$ and G_3 is $(136K + 5\Phi(t_0))$-hyperbolic by Lemma A. □

Let G be a periodic graph with quasi-exponential decay. Fix $a \leq b$ in $\{-\infty\} \cup \mathbb{Z} \cup \{\infty\}$. Define $G_3^{a,b} \subseteq G_3$ as the geodesic metric space given by $\big(\cup_{a \leq n \leq b+1} T^n\eta_0([t_0, \infty))\big) \cup \big(\cup_{a \leq n \leq b, t \geq t_0} V_{n,t}\big)$. Lemmas B and 5.7 have the following consequence.

Corollary 5.8 *Let G be any periodic graph with quasi-exponential decay. Then there exists a constant δ such that $G_3^{a,b}$ is δ-hyperbolic for every $a \leq b$ in $\{-\infty\} \cup \mathbb{Z} \cup \{\infty\}$.*

Next, some results on curves which are shown to be quasi-geodesic are given. The aim will be to construct a quasi-geodesic quadrilateral with large δ. Recall the definition of $D_G(z, w)$ given before Corollary 3.5.

Let G be a periodic graph. In the next lemma, for $t \in \mathbb{R}$ and fixed $s_1 < s_2$, define ϕ_t as a geodesic in G joining $\eta_0(s_2+t)$ with $T\eta_0(s_2+t)$, ψ_t as a geodesic joining $\eta_0(s_1-t)$ with $T\eta_0(s_1-t)$, and the curves $\xi_{n,t} := \eta_0\big([s_2, s_2+t]\big) \cup \phi_t \cup T\phi_t \cup \cdots \cup T^{n-1}\phi_t \cup T^n\eta_0\big([s_2, s_2+t]\big)$, $\zeta_{n,t} := \eta_0\big([s_1, s_1-t]\big) \cup \psi_t \cup T\psi_t \cup \cdots \cup T^{n-1}\psi_t \cup T^n\eta_0\big([s_1, s_1-t]\big)$ parameterized by arc-length.

Lemma 5.9 *Let G be a periodic graph with $\inf_{z\in\gamma_0} d_G(z, Tz) = 0$. Let $s_1 < s_2$ and define the constants $c_1 := d_G\big(\eta_0(s_1), T\eta_0(s_1)\big)$, $c_2 := d_G\big(\eta_0(s_2), T\eta_0(s_2)\big)$ and $c^* := \max\{c_1, c_2\}$. Let $n \in \mathbb{N}$ and $c \in \mathbb{R}^+$ be so that $c^*n \le 2(s_2 - s_1)$ and $d_G\big(\eta_0(s), T\eta_0(s)\big) \ge c$ for all $s \in [s_1, s_2]$. If $r, u \ge 0$ satisfy $L(\xi_{n,r}) = \min_{t\ge0} L(\xi_{n,t})$ and $L(\zeta_{n,u}) = \min_{t\ge0} L(\zeta_{n,t})$, then the quadrilateral $Q := \{\eta_0\big([s_1, s_2]\big), \xi_{n,r}, T^n\eta_0\big([s_1, s_2]\big), \zeta_{n,u}\}$ is a $(3c^*/c, 2c^*)$-quasigeodesic quadrilateral and $\delta(Q) \ge c(n-2)/12$. In particular, if n is the integer part of $2(s_2 - s_1)/c^*$, then $\delta(Q) \ge c(s_2 - s_1)/(6c^*) - c/4$.*

Proof To show that Q is a quasi-geodesic quadrilateral, it suffices to show that $\xi_{n,r}$ and $\zeta_{n,t}$ are quasi-geodesics. In fact, by symmetry, it is enough to show it just for, *e.g.*, $\xi_{n,r}$.

Let $\xi_{n,r}(s)$ and $\xi_{n,r}(t)$ be any two points on $\xi_{n,r}$. Without loss of generality, $t \ge s$. Since $\xi_{n,r}$ is parameterized by arc-length, $d_G\big(\xi_{n,r}(s), \xi_{n,r}(t)\big) \le L_G\big(\xi_{n,r}|_{[s,t]}\big) = t - s$.

For the lower bound, suppose $\xi_{n,r}(s) \in T^{j_1}G^*$, $\xi_{n,r}(t) \in T^{j_2-1}G^*$ with $0 \le j_1 < j_2 \le n$. Assume that $\xi_{n,r}(s), \xi_{n,r}(t) \notin \eta_0\big([s_2, s_2+r]\big) \cup T^n\eta_0\big([s_2, s_2+r]\big)$ (the other cases are similar). Let $t_1 \le s \le t \le t_2$ be so that $\xi_{n,r}(t_1) \in T^{j_1}\gamma_0$ and $\xi_{n,r}(t_2) \in T^{j_2}\gamma_0$.

Recall the definition of D_G. By Corollary 3.5, it will be enough to bound D_G below.

Note that $D_G\big(\xi_{n,r}(t_1), \xi_{n,r}(t_2)\big) = \sum_{j=j_1}^{j_2-1} \big(d_G(x_j, T^{-1}x_{j+1}) + d_G(T^{-1}x_{j+1}, x_{j+1})\big) + d_G\big(x_{j_2}, \xi_{n,r}(t_2)\big)$ for appropriate $\{x_j\}$. Choose i so that $j_1 \le i < j_2$ and $d_G\big(T^{-1}x_{i+1}, x_{i+1}\big) = \min_{j_1\le j<j_2} d_G\big(T^{-1}x_{j+1}, x_{j+1}\big)$. Consider $\eta_k := T^k\eta_0$ as a parametrization of $T^k\gamma_0$ for any $k \in \mathbb{Z}$. Then

$$\begin{aligned}
&d_G\big(\xi_{n,r}(t_1), T^{j_1-i-1}x_{i+1}\big)\\
&\quad+(j_2 - j_1)\, d_G\big(T^{-1}x_{i+1}, x_{i+1}\big) + d_G\big(T^{j_2-i-1}x_{i+1}, \xi_{n,r}(t_2)\big)\\
&\quad\le \sum_{j=j_1}^{j_2-1} \big(d_G(x_j, T^{-1}x_{j+1}) + d_G(T^{-1}x_{j+1}, x_{j+1})\big) + d_G(x_{j_2}, \xi_{n,r}(t_2))\\
&\quad\le (j_2 - j_1)\, d_G\big(\eta_0(s_2 + r), T\eta_0(s_2 + r)\big)\,.
\end{aligned} \tag{5.3}$$

If the second inequality in (5.3) is an equality, then $D_G(\xi_{n,r}(t_1), \xi_{n,r}(t_2)) = t_2 - t_1$ and $d_G(\xi_{n,r}(t_1), \xi_{n,r}(t_2)) \ge (t_2 - t_1)/3$. Otherwise, the second inequality in (5.3) is strict.

Define $a := \eta_{i+1}^{-1}(x_{i+1})$. Then (5.3) gives that $L(\xi_{n,a-s_2}) < L(\xi_{n,r})$. Therefore $a < s_2$ by the definition of $\xi_{n,r}$. Also, $a > s_1$, since otherwise $L(\xi_{n,r}) > L(\xi_{n,a-s_2}) > 2(s_2 - s_1) \ge c_2 n = L(\xi_{n,0}) \ge L(\xi_{n,r})$.

Hence $s_1 < a < s_2$ and then $d_G(T^{-1}x_{i+1}, x_{i+1}) \ge c = d_G\big(\eta_0(s_2), T\eta_0(s_2)\big)c/c_2$ and (5.3) gives

$$\begin{aligned} D_G\big(\xi_{n,r}(t_1), \xi_{n,r}(t_2)\big) &\ge d_G\big(\xi_{n,r}(t_1), T^{j_1-i-1}x_{i+1}\big) + (j_2 - j_1)\, d_G\big(T^{-1}x_{i+1}, x_{i+1}\big) \\ &\quad + d_G\big(T^{j_2-i-1}x_{i+1}, \xi_{n,r}(t_2)\big) \\ &\ge \frac{c}{c_2}(j_2 - j_1)\, d_G\big(\eta_0(s_2), T\eta_0(s_2)\big) \\ &\ge \frac{c}{c_2}\,(j_2 - j_1)\, d_G\big(\eta_0(s_2 + r), T\eta_0(s_2 + r)\big) \\ &= \frac{c}{c_2}\,(t_2 - t_1)\,. \end{aligned}$$

By Corollary 3.5, $(t_2 - t_1)c/(3c_2) \le d_G\big(\xi_{n,r}(t_1), \xi_{n,r}(t_2)\big)$, and, by the triangle inequality,

$$\begin{aligned} d_G\big(\xi_{n,r}(s), \xi_{n,r}(t)\big) &\ge d_G\big(\xi_{n,r}(t_1), \xi_{n,r}(t_2)\big) - 2c_2 \ge \frac{c}{3c_2}(t_2 - t_1) - 2c_2 \\ &\ge \frac{c}{3c_2}(t - s) - 2c_2\,. \end{aligned}$$

Any other case gives the same inequality. Thus, $\xi_{n,r}$ is a $(3c_2/c, 2c_2)$-quasigeodesic.

Finally, let us estimate $\delta(Q)$.

Let p be the midpoint in $\eta_0([s_1, s_2])$. By Corollary 3.5,

$$d_G\big(p, \xi_{n,r} \cap (\cup_k T^k\gamma_0)\big) \ge d_G(p, \eta_0(s_2)) = \frac{s_2 - s_1}{2} \ge \frac{c^*n}{4}\,.$$

Therefore,

$$\begin{aligned} d_G(p, \xi_{n,r}) &\ge d_G\big(p, \xi_{n,r} \cap (\cup_k T^k\gamma_0)\big) - (1/2)d_G\big(\eta_0(s_2 + r), T\eta_0(s_2 + r)\big) \\ &\ge d_G\big(p, \xi_{n,r} \cap (\cup_k T^k\gamma_0)\big) - (1/2)d_G\big(\eta_0(s_2), T\eta_0(s_2)\big) \ge \frac{c^*n}{4} - \frac{c^*}{2} \\ &= \frac{c^*(n-2)}{4}\,. \end{aligned}$$

Similarly, $d_G(p, \zeta_{n,u}) \ge c^*(n-2)/4$.

As above, $D_G\big(p, T^n\eta_0([s_1, s_2])\big) \ge \min\{cn,\ (s_2 - s_1)/2\} \ge \min\{cn,\ c^*n/4\} \ge cn/4$ and then, by Corollary 3.5, $d_G\big(p, T^n\eta_0([s_1, s_2])\big) \ge cn/12$ and, since $c \le c^*$, $\delta(Q) \ge c(n-2)/12$. □

For Lemma 5.10 below, it will be useful to keep in mind the definition of fine triangles. Given a geodesic triangle $T = \{x, y, z\}$ in a geodesic metric space X, let T_E be a Euclidean triangle with sides of the same length than T. Since there is no possible confusion, denote the corresponding points in T and T_E by the same letters. The maximum inscribed circle in T_E meets the side $[xy]$ (respectively $[yz]$, $[zx]$) in a point z' (respectively x', y') such that $d(x, z') = d(x, y')$, $d(y, x') = d(y, z')$

and $d(z,x') = d(z,y')$. We call the points x', y', z', the *internal points* of $\{x,y,z\}$. There is a unique isometry f of the triangle $\{x,y,z\}$ onto a tripod (a star graph with one vertex w of degree 3, and three vertices x_0, y_0, z_0 of degree one, such that $d(x_0,w) = d(x,z') = d(x,y')$, $d(y_0,w) = d(y,x') = d(y,z')$, and $d(z_0,w) = d(z,x') = d(z,y')$). The triangle $\{x,y,z\}$ is *δ-fine* if $f(p) = f(q)$ implies that $d(p,q) \le \delta$. The space X is *δ-fine* if every geodesic triangle in X is δ-fine.

There are two definitions of Gromov hyperbolicity (the second one is the definition of fine space) whose equivalence will be useful to quantify (see, *e.g*, [17, Proposition 2.21, p. 41]):

Theorem A *Let us consider a geodesic metric space X.*
(1) *If X is δ-hyperbolic, then it is 4δ-fine.*
(2) *If X is δ-fine, then it is δ-hyperbolic.*

Finally, for Lemma 5.10 below, some notation needs to be introduced. Let G be a periodic graph. Fix a parametrization η_0 of γ_0 and $t_0 \in \mathbb{R}$. Consider points $x \in T^nG^*$, $y \in T^{n+k}G^*$, with $n \in \mathbb{N}$, $k \ge 4$, so that if γ is a straight geodesic in G from x to y, then there exists $x_j \in \gamma \cap T^{n+j}\gamma_0$ with $s_j := \eta_{n+j}^{-1}(x_j) \ge t_0$ for $2 \le j \le k-1$.

In G_1, consider the curves $g_j := U_{n+j,s_j} \cup [x_{j+1}Tx_j]$ joining x_j and x_{j+1} for $2 \le j \le k-2$, and the curve $g := [xx_1] \cup [x_1x_2] \cup (\cup_{(2\le j\le k-2)} g_j) \cup [x_{k-1}x_k] \cup [x_ky]$ joining x and y in G_1.

Lemma 5.10 *With the above notation, if G satisfies (5.1) and G^* is hyperbolic, then g with its arc-length parametrization is an (α,β)-quasi-geodesic in G_1 and $\mathcal{H}_{G_1}(g,\gamma) \le H$, where α,β and H are constants depending just on $\delta(G_1^*)$ and $M := \sup_{t\ge t_0} d_G\big(\eta_0(t), T\eta_0(t)\big)$. In fact, $(\alpha,\beta) = \big(3, 8\delta(G_1^*) + 6M\big)$.*

Proof Let $\gamma : [0,l_0] \to G$ be an arc-length parametrization of γ and let $g : [0,l] \to G_1$ be an arc-length parametrization of g; then $d_{G_1}\big(g(t_1), g(t_2)\big) \le |t_1 - t_2|$ for every $t_1, t_2 \in [0,l]$.

To obtain a lower bound, note that $M < \infty$ by (5.1); then every segment $U_{n,t}$ with $t \ge t_0$ has length at most M. Fix $t_1, t_2 \in [0,l]$ with $t_1 < t_2$. Assume first that $g(t_1), g(t_2) \in T^{n+j}G_1^*$ for some j with $2 \le j \le k-2$. Consider the geodesic triangle $\mathcal{T}_j = \{[x_jx_{j+1}], U_{n+j,s_j}, [x_{j+1}Tx_j]\}$ in $T^{n+j}G_1^*$. Since G^* is hyperbolic, G_1^* is hyperbolic by Lemma 5.2 and the triangle $\mathcal{T}_j$ is $4\delta(G_1^*)$-fine by Theorem A.

Let $[a_0,b_0] := \gamma^{-1}\big([x_jx_{j+1}]\big)$, $[a,b] := g^{-1}(g_j)$ and $c := g^{-1}(Tx_j)$. By the triangle inequality, $b_0 - a_0 \le b - a$, thus one can choose $c_1, c_2 \in [a,b]$ such that $c - c_1 = c_2 - c > 0$ satisfying $(c_1 - a) + (b - c_2) = b_0 - a_0$. Finally, pick $c_0 \in [a_0,b_0]$ with $c_1 - a = c_0 - a_0$ and $b - c_2 = b_0 - c_0$.

Define $u : [a,b] \to [a_0,b_0]$ as the piecewise linear continuous function

$$u(t) := \begin{cases} t - a + a_0, & \text{if } t \in [a,c_1], \\ c_0, & \text{if } t \in (c_1,c_2), \\ t - b + b_0, & \text{if } t \in [c_2,b]. \end{cases}$$

Since $\mathcal{T}_j$ is $4\delta(G_1^*)$-fine, $d_{G_1}\big(g(t), \gamma(u(t))\big) \le 4\delta(G_1^*) + c - c_1 \le 4\delta(G_1^*) + M$.

Therefore, by the triangle inequality,

$$\begin{aligned} d_{G_1}\big(g(t_1), g(t_2)\big) &\geq d_{G_1}\big(\gamma(u(t_1)), \gamma(u(t_2))\big) - 8\delta(G_1^*) - 2M \\ &= u(t_2) - u(t_1) - 8\delta(G_1^*) - 2M \\ &\geq t_2 - t_1 - (c_2 - c_1) - 8\delta(G_1^*) - 2M \geq t_2 - t_1 - 8\delta(G_1^*) - 4M. \end{aligned}$$

Since $[xx_1] \cup [x_1x_2]$ and $[x_{k-1}x_k] \cup [x_k y]$ are geodesics in G_1, the above inequality also holds if $g(t_1), g(t_2) \in T^{n+j}G_1^*$ for some $j \in \{0, 1, k-1, k\}$.

Assume now that $g(t_1) \in T^{n+j_1}G_1^*$ and $g(t_2) \in T^{n+j_2}G_1^*$ with $j_1 < j_2$. Let $r_1, r_2 \in [t_1, t_2]$ such that $g(r_1) = x_{j_1+1}$ and $g(r_2) = x_{j_2}$. The previous argument with the function u provides t_1^*, t_2^* satisfying $\gamma(t_1^*) \in T^{n+j_1}G_1^*$, $\gamma(t_2^*) \in T^{n+j_2}G_1^*$, $d_{G_1}\big(g(t_1), \gamma(t_1^*)\big) \leq 4\delta(G_1^*) + M$, $d_{G_1}\big(g(t_2), \gamma(t_2^*)\big) \leq 4\delta(G_1^*) + M$, $d_{G_1}\big(\gamma(t_1^*), x_{j_1+1}\big) \geq r_1 - t_1 - 2M$ and $d_{G_1}\big(\gamma(t_2^*), x_{j_2}\big) \geq t_2 - r_2 - 2M$. Now, using Corollary 3.5,

$$\begin{aligned} d_{G_1}\big(g(t_1), g(t_2)\big) &\geq d_{G_1}\big(\gamma(t_1^*), \gamma(t_2^*)\big) - 8\delta(G_1^*) - 2M \\ &= d_{G_1}\big(\gamma(t_1^*), x_{j_1+1}\big) + d_{G_1}\big(x_{j_1+1}, x_{j_2}\big) + d_{G_1}\big(\gamma(t_2^*), x_{j_2}\big) \\ &\quad - 8\delta(G_1^*) - 2M \\ &\geq r_1 - t_1 - 2M + \frac{1}{3}(r_2 - r_1) + t_2 - r_2 - 2M - 8\delta(G_1^*) - 2M \\ &\geq \frac{1}{3}(t_2 - t_1) - 8\delta(G_1^*) - 6M, \end{aligned}$$

and we conclude that g is a $\big(3, 8\delta(G_1^*)+6M\big)$-quasi-geodesic in G_1. Since G_1^* is hyperbolic, the geodesic stability gives that $\mathcal{H}_{G_1}\big(g_j, [x_jx_{j+1}]\big) = \mathcal{H}_{T^{n+j}G_1^*}\big(g_j, [x_jx_{j+1}]\big) \leq H$ for $2 \leq j \leq k-2$, where H is a constant depending just on $\delta(G_1^*)$ and M. Hence, $\mathcal{H}_{G_1}(g, \gamma) \leq H$. □

Remark 5.11 The argument in the proof of Lemma 5.10 proves, in fact, a more general result. On the one hand, the conclusion holds (with the same constants) if one replaces g_j by $[x_jx_{j+1}]$ for any subset of $\{2 \leq j \leq k-2\}$. On the other hand, the conclusion also holds (with the same constants) for non-straight geodesics: it suffices to consider each connected subcurve of $\gamma \cap T^{n+j}G^*$ joining $T^{n+j}\gamma_0$ with $T^{n+j+1}\gamma_0$ instead of $[x_jx_{j+1}]$ (if a connected subcurve of $\gamma \cap T^{n+j}G^*$ joins two points in $T^{n+j}\gamma_0$ one can replace it, in order to obtain g, by the geodesic contained in $T^{n+j}\gamma_0$ with the same endpoints; in a similar way, if it joins two points in $T^{n+j+1}\gamma_0$ one can replace it by the geodesic contained in $T^{n+j+1}\gamma_0$ with the same endpoints).

Lemma 5.12 *Consider a periodic graph G and a parametrization η_0 of γ_0 satisfying both (5.1) and $\lim_{t\to-\infty} d_G\big(\eta_0(t), T\eta_0(t)\big) = \infty$. If G^* is hyperbolic, then there exists a constant t_0 with the following properties:*

(1) *If $x \in T^n\gamma_0$, $y \in T^{n+1}\gamma_0$ and $[xy]$ is a geodesic in T^nG^* joining them, then there exist p, s_x, s_y so that $p \in [xy]$ and $s_x, s_y \geq t_0+6\delta(G^*)$ with $d_G\big(p, T^n\eta_0(s_x)\big) \leq 2\delta(G^*)$ and $d_G\big(p, T^{n+1}\eta_0(s_y)\big) \leq 2\delta(G^*)$.*

(2) *Let* $\gamma = [xy]$ *be a geodesic in* G*, with* $x \in T^n(G^*)$*,* $y \in T^{n+k}(G^*)$ *and* $k \ge 3$*. Let* $x_j \in T^{n+j}\gamma_0 \cap \gamma$*,* $2 \le j \le k-1$*. Then* $x_j = T^{n+j}\eta_0(s_j)$ *with* $s_j \ge t_0$ *for* $2 \le j \le k-1$*.*

Proof (1) Given $x \in T^n\gamma_0$ and $y \in T^{n+1}\gamma_0$, since $\liminf_{t\to+\infty} d_G\big(\eta_0(t), T\eta_0(t)\big) = 0$, there exists t large enough such that the geodesic $[T^n\eta_0(t)T^{n+1}\eta_0(t)]$ in T^nG^* satisfies $d_G\big([xy], [T^n\eta_0(t)T^{n+1}\eta_0(t)]\big) > 2\delta(G^*)$. Consider the geodesic quadrilateral $Q := \{x, y, T^{n+1}\eta_0(t), T^n\eta_0(t)\}$ in T^nG^*, that is $2\delta(G^*)$-thin. Then for every $q \in [xy]$ one has $d_G\big(q, [xT^n\eta_0(t)] \cup [yT^{n+1}\eta_0(t)]\big) \le 2\delta(G^*)$. Hence, there exist a point $p \in [xy]$ such that $d_G\big(p, [xT^n\eta_0(t)]\big) \le 2\delta(G^*)$ and $d_G\big(p, [yT^{n+1}\eta_0(t)]\big) \le 2\delta(G^*)$. Choose s_x, s_y such that $d_G\big(p, T^n\eta_0(s_x)\big) \le 2\delta(G^*)$ and $d_G\big(p, T^{n+1}\eta_0(s_y)\big) \le 2\delta(G^*)$. Then $d_G\big(T^n\eta_0(s_x), T^{n+1}\eta_0(s_y)\big) \le 4\delta(G^*)$ and by Corollary 3.4, $d_G\big(T^n\eta_0(s_x), T^{n+1}\eta_0(s_x)\big) \le 2d_G\big(T^n\eta_0(s_x), T^{n+1}\gamma_0\big) \le 2d_G\big(T^n\eta_0(s_x), T^{n+1}\eta_0(s_y)\big) \le 8\delta(G^*)$.

A symmetric argument gives $d_G\big(T^n\eta_0(s_y), T^{n+1}\eta_0(s_y)\big) \le 8\delta(G^*)$. Since $\lim_{t\to-\infty} d_G\big(\eta_0(t), T\eta_0(t)\big) = \infty$, there exists a constant t_0 such that $d_G\big(\eta_0(t), T\eta_0(t)\big) > 8\delta(G^*)$ for every $t < t_0 + 6\delta(G^*)$; hence, $s_x, s_y \ge t_0 + 6\delta(G^*)$.

(2) Fix $x_j = T^{n+j}\eta_0(s_j)$ with $2 \le j \le k-1$. By (1), there exist $p \in [x_{j-1}x_j] \cap T^{n+j-1}G^*$, $p' \in [x_jx_{j+1}] \cap T^{n+j}G^*$ and $s, s' \ge t_0 + 6\delta(G^*)$ such that $d_G\big(p, T^{n+j}\eta_0(s)\big) \le 2\delta(G^*)$ and $d_G(p', T^{n+j}\eta_0(s')) \le 2\delta(G^*)$.

By symmetry, assume that $s \ge s'$. Assume also that $s_j < s'$, since otherwise $s_j \ge s' \ge t_0 + 6\delta(G^*)$. Thus

$$\begin{aligned} d_G(p, p') &\le d_G\big(p, T^{n+j}\eta_0(s)\big) + d_G\big(T^{n+j}\eta_0(s), T^{n+j}\eta_0(s')\big) \\ &\quad + d_G\big(T^{n+j}\eta_0(s'), p'\big) \\ &\le 4\delta(G^*) + d_G\big(T^{n+j}\eta_0(s), T^{n+j}\eta_0(s')\big), \end{aligned}$$

$$\begin{aligned} &d_G\big(x_j, T^{n+j}\eta_0(s')\big) + d_G\big(T^{n+j}\eta_0(s'), T^{n+j}\eta_0(s)\big) \\ &\quad = d_G\big(x_j, T^{n+j}\eta_0(s)\big) \\ &\quad \le d_G\big(x_j, p\big) + d_G\big(p, T^{n+j}\eta_0(s)\big) \\ &\quad \le d_G\big(x_j, p\big) + 2\delta(G^*) \le d_G\big(p', p\big) + 2\delta(G^*) \\ &\quad \le 6\delta(G^*) + d_G\big(T^{n+j}\eta_0(s), T^{n+j}\eta_0(s')\big), \end{aligned}$$

and thus $d_G\big(x_j, T^{n+j}\eta_0(s')\big) \le 6\delta(G^*)$. Since $6\delta(G^*) \ge d_G\big(x_j, T^{n+j}\eta_0(s')\big) = s' - s_j \ge t_0 + 6\delta(G^*) - s_j$, one gets $s_j \ge t_0$. □

Lemma 5.13 *Let* G *be a periodic graph with quasi-exponential decay and* G^* *hyperbolic. Then there exists a constant* N *such that* $\mathcal{H}_G(g_1, g_2) \le N$ *for every geodesics* g_1, g_2 *in* G *with the same endpoints and* $g_1 \subset \gamma_0$*.*

Proof Consider first the case $g_2 \subset \cup_{j\ge0} T^jG^*$. Define $n_2 := \max\{j \in \mathbb{Z} : g_2 \cap T^jG^* \ne \emptyset\}$. Let $\{g_j^1, \dots, g_j^{r_j}\}$ be the connected components of $g_2 \cap T^jG^*$ and $\mathcal{G} := \{g_j^i \,|\, 1 \le i \le r_j, 0 \le j \le n_2\}$.

If $n_2 = 0$, then $\mathcal{H}_G(g_1, g_2) \le H(\delta(G^*), 1, 0)$, where H is the function of the geodesic stability (see the paragraph after Lemma A).

If $n_2 > 0$, for each $g^i_{n_2}$, define $\gamma^i_{n_2}$ as follows: if $g^i_{n_2}$ joins $T^{n_2}\eta_0(s^i)$ and $T^{n_2}\eta_0(t^i)$ with $s^i \le t^i$, then $\gamma^i_{n_2} := T^{n_2}\eta_0([s^i, t^i])$. Let g'_2 be the geodesic in $\cup_{0\le j\le n_2-1} T^j G^*$ obtained from g_2 by replacing each $g^i_{n_2}$ by $\gamma^i_{n_2}$; then $\mathcal{H}_G(g_2, g'_2) \le H(\delta(G^*), 1, 0)$. In a similar way one can find a geodesic g''_2 contained in $\cup_{0\le j\le n_2-2} T^j G^*$ with $\mathcal{H}_G(g_2, g''_2) \le 2H(\delta(G^*), 1, 0)$ (if $n_2 \ge 2$). Hence, if $n_2 \le 2$, then $\mathcal{H}_G(g_1, g_2) \le 3H(\delta(G^*), 1, 0)$. Assume now that $n_2 \ge 3$.

For each $g^i_j \in \mathcal{G}$ with $1 \le j \le n_2 - 2$, define γ^i_j as follows: if g^i_j joins $T^j\eta_0(s^i_j)$ and $T^{j+1}\eta_0(t^i_j)$ with $s^i_j \le t^i_j$, then $\gamma^i_j := T^j\eta_0([s^i_j, t^i_j]) \cup U_{j,t^i_j}$; if $s^i_j > t^i_j$, then $\gamma^i_j := T^j\eta_0([t^i_j, s^i_j]) \cup U_{j,s^i_j}$; if g^i_j joins $T^j\eta_0(s^i_j)$ and $T^j\eta_0(t^i_j)$ with $s^i_j \le t^i_j$, then $\gamma^i_j := T^j\eta_0([s^i_j, t^i_j])$; if g^i_j joins $T^{j+1}\eta_0(s^i_j)$ and $T^{j+1}\eta_0(t^i_j)$ with $s^i_j \le t^i_j$, then $\gamma^i_j := T^{j+1}\eta_0([s^i_j, t^i_j])$. Define I as the set of indices $1 \le i \le r_0$ such that g^i_0 joins $T\eta_0(s^i_0)$ and $T\eta_0(t^i_0)$ with $s^i_0 \le t^i_0$; define $\gamma^i_0 := T\eta_0([s^i_0, t^i_0])$ for every $i \in I$. By Lemma 5.5, the relation (5.1) holds and then, by Lemma 5.12, $s^i_j, t^i_j \ge t_0$, where t_0 is the constant in Lemma 5.12, and therefore $\gamma^i_j \subset G_1$. By Remark 5.11, $\mathcal{H}_{G_1}(g^i_j, \gamma^i_j) \le H_0$, where H_0 is a constant depending just on $\delta(G^*_1)$ and on $\sup_{t\ge t_0} d_G(\eta_0(t), T\eta_0(t))$.

Define $\gamma_2 := \big(g''_2 \setminus \big(\big(\cup_{j=1}^{n_2-2} \cup_{i=1}^{j_r} g^i_j\big) \cup \big(\cup_{i\in I}\, g^i_0\big)\big)\big) \cup \big(\cup_{j=1}^{n_2-2} \cup_{i=1}^{j_r} \gamma^i_j\big) \cup \big(\cup_{i\in I}\, \gamma^i_0\big)$. Therefore, $\mathcal{H}_{G_1}(g_2, \gamma_2) \le H_1 := H_0 + 2H(\delta(G^*_1), 1, 0)$.

By Remark 5.11, γ_2 is an (α, β)-quasigeodesic in G_1 (with its arc-length parametrization), where α, β are the constants in Lemma 5.10. Let $\gamma'_2 := \gamma_2 \cap \big(\cup_{j=1}^{n_2-2} T^j G^*\big) \subset G_2$. Note that γ'_2 is connected and joins two points in $T\gamma_0$. Since $d_{G_1} \le d_{G_2}$ on G_2, γ'_2 is also an (α, β)-quasigeodesic in G_2.

By Lemma 5.5, $\sup\big\{t_2 - t_1 \,:\, \Phi(t_1) = \Phi(t_2),\ t_2 \ge t_1 \ge t_0\big\} < \infty$ and (5.1) holds. Hence, by Lemma 5.3, there exists a quasi-isometry $f^{-1} : G_2 \to G_3$ and there also exist constants α', β', which just depend on G, such that $f^{-1}(\gamma'_2)$ is an (α', β')-quasigeodesic in G_3. Note that G_3 is hyperbolic by Lemma 5.7; therefore, if $\gamma'_3 \subset T\gamma_0$ is the geodesic joining the endpoints of $f^{-1}(\gamma'_2)$ in G_3, then $\mathcal{H}_{G_3}\big(\gamma'_3, f^{-1}(\gamma'_2)\big) \le H_3 := H\big(\delta(G_3), \alpha', \beta'\big)$. Since f is the identity map on $\cup_{n\in\mathbb{Z}} T^n\eta_0([t_0, \infty))$, $f(\gamma'_3) \subset T\gamma_0$ is a geodesic in G_2 joining the endpoints of γ'_2; since f is a quasi-isometry, there exists a constant H_4, which just depend on G, such that $\mathcal{H}_{G_2}(f(\gamma'_3), \gamma'_2) \le H_4$. Since $d_{G_1} \le d_{G_2}$ on G_2, $\mathcal{H}_{G_1}(f(\gamma'_3), \gamma'_2) \le H_4$. Define $\gamma_3 := (\gamma_2 \setminus \gamma'_2) \cup f(\gamma'_3) \subset G$; then $\mathcal{H}_{G_1}(\gamma_3, \gamma_2) = \mathcal{H}_{G_1}(f(\gamma'_3), \gamma'_2) \le H_4$ and $\mathcal{H}_G(g_2, \gamma_3) = \mathcal{H}_{G_1}(g_2, \gamma_3) \le H_1 + H_4$. Since γ_3 is a geodesic in G^* with the same endpoints that g_1, one gets $\mathcal{H}_G(\gamma_3, g_1) \le H(\delta(G^*), 1, 0)$ and $\mathcal{H}_G(g_1, g_2) \le H_1 + H_4 + H(\delta(G^*), 1, 0)$.

Hence, if $g_2 \subset \cup_{j\ge 0} T^j G^*$ the lemma holds with $N = H_1 + H_4 + H(\delta(G^*), 1, 0)$. If $g_2 \subset \cup_{j<0} T^j G^*$, the same result holds by symmetry. The general case follows by applying these two cases to the connected components $g_{2,1}, \ldots, g_{2,m}$ of $g_2 \cap \cup_{j\ge 0} T^j G^*$ and to the closure of the connected components of $g_2 \setminus \cup_{j=1}^m g_{2,j}$. □

Corollary 5.14 *Let G be a periodic graph with quasi-exponential decay and G^* hyperbolic. Then for each geodesic γ in G there exists a straight geodesic γ' with the same endpoints and $\mathcal{H}_G(\gamma, \gamma') \le N$, where N is the constant in Lemma 5.13.*

Proof Fix a geodesic $\gamma : [a,b] \to G$ with $\gamma(a) \in T^{n_1}G^*$, $\gamma(b) \in T^{n_2}G^*$ and $n_1 \le n_2$. Assume that $\gamma \cap T^{n_1}\gamma_0 \ne \emptyset$ (otherwise, we consider $T^{n_1+1}\gamma_0$ instead of $T^{n_1}\gamma_0$) and that $\gamma \cap T^{n_2+1}\gamma_0 \ne \emptyset$ (otherwise, we consider $T^{n_2}\gamma_0$ instead of $T^{n_2+1}\gamma_0$). Define inductively s_j, t_j ($n_1 \le j \le n_2+1$) as follows: $s_{n_1} := \min\{t \in [a,b] : \gamma(t) \in T^{n_1}\gamma_0\}$, $t_{n_1} := \max\{t \in [a,b] : \gamma(t) \in T^{n_1}\gamma_0\}$, $s_j := \min\{t \in (t_{j-1}, b] : \gamma(t) \in T^j\gamma_0\}$, $t_j := \max\{t \in (t_{j-1}, b] : \gamma(t) \in T^j\gamma_0\}$. We define also $\gamma^j := [\gamma(s_j)\gamma(t_j)] \subset T^j\gamma_0$ for $n_1 \le j \le n_2+1$.

By Lemma 5.13, $\mathcal{H}_G(\gamma([s_j,t_j]), \gamma^j) \le N$. Then $\gamma' := \big(\gamma \setminus \cup_{j=n_1}^{n_2+1} \gamma([s_j,t_j])\big) \cup \big(\cup_{j=n_1}^{n_2+1} \gamma^j\big)$ is a straight geodesic in G and that $\mathcal{H}_G(\gamma,\gamma') \le N$. □

Finally, let us show the proof of the second part of Theorem 1.1.

Proof (*Second part of Theorem* 1.1). Assume that G is hyperbolic. Lemma B implies that G^* is also hyperbolic.

Since $\inf_{z\in\gamma_0} d_G(z, Tz) = 0$, without loss of generality one can consider only arc-length parametrizations η_0 of γ_0 for which $\liminf_{t\to+\infty} d_G\big(\eta_0(t), T\eta_0(t)\big) = 0$. Fix one of these. It will be shown that $\lim_{t\to-\infty} F(t) = \infty$, where $F(t) := d_G(\eta_0(t), T\eta_0(t))$. Indeed,

(a) Assume that $\liminf_{t\to-\infty} F(t) = 0$. Then there exists a sequence of positive numbers $\{c_k\}$ converging to 0 and two sequences $\{s_{1,k}\}, \{s_{2,k}\} \subset \mathbb{R}$ such that $\lim_{k\to\infty} s_{2,k} = \infty$, $\lim_{k\to\infty} s_{1,k} = -\infty$, $F(s_{1,k}) = F(s_{2,k}) = c_k$, and $F(t) \ge c_k$ for every $t \in [s_{1,k}, s_{2,k}]$ and every k. Therefore, Lemmas 2.1 and 5.9 imply that G is not hyperbolic.

(b) If $0 < \liminf_{t\to-\infty} F(t)$ and $\limsup_{t\to-\infty} F(t) < \infty$, one can also easily construct quasi-geodesic quadrilaterals Q with $\delta(Q)$ arbitrarily large, and thus G is not hyperbolic (by lemmas 2.1 and 5.9). (The Cayley graph of $\mathbb{Z}^2$, for which $1 \le F(t) \le \frac{3}{2}$, is a basic example of this situation.)

(c) Assume that $\liminf_{t\to-\infty} F(t) < \infty$ and $\limsup_{t\to-\infty} F(t) = \infty$. Note that F is a Lipschitz function; in fact, $|F(t_1) - F(t_2)| \le 2|t_1 - t_2|$. Fix a constant $c > \liminf_{t\to-\infty} F(t)$. There exist two sequences $\{s_{1,k}\}, \{s_{2,k}\} \subset \mathbb{R}^-$ such that $F(s_{1,k}) = F(s_{2,k}) = c$, $F(t) \ge c$ for every $t \in [s_{1,k}, s_{2,k}]$ and $F(t_k) \ge k$ for some $t_k \in [s_{1,k}, s_{2,k}]$, for every k. Since F is 2-Lipschitz, $s_{2,k} - s_{1,k} \ge k - c$ for every k, and then $\lim_{k\to\infty}(s_{2,k} - s_{1,k}) = \infty$. Therefore, Lemmas 2.1 and 5.9 give that G is not hyperbolic.

Thus, $\lim_{t\to-\infty} F(t) = \infty$.

The argument in (c) also gives $\limsup_{t\to+\infty} F(t) < \infty$ since $\liminf_{t\to+\infty} F(t) = 0$; then (5.1) holds.

Assume that G has not quasi-exponential decay, so $\sup_{s_2\ge s_1\ge 0}(s_2-s_1)\Phi(s_2)/\Phi(s_1) = \infty$. By Lemma 5.6, $\mathfrak{L}(G_3) = \infty$ and G_3 is not hyperbolic and, by Lemma 5.4, since G is hyperbolic, $\sup\big\{t_2 - t_1 : \Phi(t_1) = \Phi(t_2),\ t_2 \ge t_1 \ge 0\big\} = \infty$. Consider $t_2 > t_1 > 0$ with $\Phi(t_1) = \Phi(t_2) < \Phi(0)$ which are maximal in the following sense: $\Phi(t_1 - \varepsilon) > \Phi(t_1)$ and $\Phi(t_2) > \Phi(t_2 + \varepsilon)$ for every $\varepsilon > 0$. Therefore, $\Phi(t_1) = F(t_1) = \Phi(t_2) = F(t_2)$ and $F(t) \ge F(t_1) = F(t_2)$ for every $t \in [t_1, t_2]$. Lemma 5.9 (taking $c_1 = c_2 = c^* = c = F(t_1) < \Phi(0)$) provides a $(3, 2\Phi(0))$-quasigeodesic quadrilateral Q with $\delta(Q) \ge (t_2 - t_1)/6 - \Phi(0)/4$. Hence, Lemma

2.1 shows that G is not hyperbolic. This is a contradiction. Therefore, G has quasi-exponential decay.

Let us show the other direction by assuming that G^* is hyperbolic and G has quasi-exponential decay. By Lemma 5.5, $\sup\{t_2 - t_1 : \Phi(t_1) = \Phi(t_2),\ t_2 \geq t_1 \geq t_0\} < \infty$ for any fixed t_0, and (5.1) holds.

Fix any geodesic triangle $\mathcal{T}_0 := \{z_1, z_2, z_3\}$ in G, with $z_i \in T^{n_i} G^*$ for $1 \leq i \leq 3$ and $n_1 \leq n_2 \leq n_3$. One just needs to deal with the case $n_1 + 4 \leq n_2 \leq n_3 - 4$; the other cases are similar and simpler.

By Corollary 5.14, without loss of generality, assume that the geodesics of $\mathcal{T}_0$ are straight.

By Lemma 5.12 there exists a constant t_0 such that if $x \in \mathcal{T}_0 \cap T^n \gamma_0$ with either $n_1 + 2 \leq n \leq n_2 - 1$ or $n_2 + 2 \leq n \leq n_3 - 1$, then $(T^n \eta_0)^{-1}(x) \geq t_0$. Consider the geodesic metric spaces G_1 and G_2 defined after (5.1) (with this constant t_0) and recall $G_1 = G \cup G_2$; since G is an isometric subspace of G_1, $\mathcal{T}_0$ is also a geodesic triangle in G_1.

Since $(T^n \eta_0)^{-1}(x) \geq t_0$ if $x \in \mathcal{T}_0 \cap T^n \gamma_0$ with either $n_1 + 2 \leq n \leq n_2 - 1$ or $n_2 + 2 \leq n \leq n_3 - 1$, and the geodesics of $\mathcal{T}_0$ are straight, by Lemma 5.10, there exist (α, β)-quasigeodesics g_{12}, g_{13} and g_{23} in G_1 such that g_{ij} joins z_i and z_j, and $\mathcal{H}_{G_1}(g_{ij}, [z_i z_j]) \leq H$, where H only depends on $\delta(G_1^*)$ and $\nu := \sup_{t \geq t_0} d_G(\eta_0(t), T\eta_0(t))$, $\alpha = 3$ and $\beta = 8\delta(G_1^*) + 6\nu$ (recall that G_1^* is hyperbolic by Lemma 5.2). Furthermore, $g_{12} = [z_1 z_2]$ in $T^{n_1} G_1^* \cup T^{n_1+1} G_1^* \cup T^{n_2-1} G_1^* \cup T^{n_2} G_1^*$, $g_{23} = [z_2 z_3]$ in $T^{n_2} G_1^* \cup T^{n_2+1} G_1^* \cup T^{n_3-1} G_1^* \cup T^{n_3} G_1^*$, $g_{13} = [z_1 z_3]$ in $T^{n_1} G_1^* \cup T^{n_1+1} G_1^* \cup T^{n_2-1} G_1^* \cup T^{n_2} G_1^* \cup T^{n_2+1} G_1^* \cup T^{n_3-1} G_1^* \cup T^{n_3} G_1^*$, $g_{12} \cap (\cup_{n_1+1<n<n_2-1} T^n G_1^*) \subset G_2$, $g_{23} \cap (\cup_{n_2+1<n<n_3-1} T^n G_1^*) \subset G_2$, $g_{13} \cap \{(\cup_{n_1+1<n<n_2-1} T^n G_1^*) \cup (\cup_{n_2+1<n<n_3-1} T^n G_1^*)\} \subset G_2$. Then $\mathcal{T}_1 := \{g_{12}, g_{13}, g_{23}\}$ is an (α, β)-quasi-geodesic triangle in G_1.

Define $G_2(\mathcal{T}_1)$ and $G_3(\mathcal{T}_1)$ as the geodesic metric spaces given by

$$\begin{aligned}
G_2(\mathcal{T}_1) := {} & T^{n_1} G_1^* \cup T^{n_1+1} G_1^* \cup \left(\cup_{n_1+1<n<n_2-1, t \geq t_0} U_{n,t} \right) \\
& \cup T^{n_2-1} G_1^* \cup T^{n_2} G_1^* \cup T^{n_2+1} G_1^* \\
& \cup \left(\cup_{n_2+1<n<n_3-1, t \geq t_0} U_{n,t} \right) \cup T^{n_3-1} G_1^* \cup T^{n_3} G_1^*, \\
G_3(\mathcal{T}_1) := {} & T^{n_1} G_1^* \cup T^{n_1+1} G_1^* \cup \left(\cup_{n_1+1<n<n_2-1, t \geq t_0} V_{n,t} \right) \\
& \cup T^{n_2-1} G_1^* \cup T^{n_2} G_1^* \cup T^{n_2+1} G_1^* \\
& \cup \left(\cup_{n_2+1<n<n_3-1, t \geq t_0} V_{n,t} \right) \cup T^{n_3-1} G_1^* \cup T^{n_3} G_1^*.
\end{aligned}$$

Note that $G_2(\mathcal{T}_1)$ is contained in G_1.

By Corollary 5.8 there exists a constant δ, which does not depend on $n_1, n_2, n_3, \mathcal{T}_0$, such that the subspaces $\cup_{n_1+1<n<n_2-1, t \geq t_0} V_{n,t}$ and $\cup_{n_2+1<n<n_3-1, t \geq t_0} V_{n,t}$ are δ-hyperbolic.

Since G^* is hyperbolic, by Lemma 5.2 there exists a constant δ^*, which does not depend on $n_1, n_2, n_3, \mathcal{T}_0$, such that G_1^* is δ^*-hyperbolic. By Lemma B, $T^{n_1} G_1^* \cup T^{n_1+1} G_1^*$, $T^{n_2-1} G_1^* \cup T^{n_2} G_1^* \cup T^{n_2+1} G_1^*$ and $T^{n_3-1} G_1^* \cup T^{n_3} G_1^*$ are $(120)^2 \delta^*$-hyperbolic. Hence, by Lemma B, $G_3(\mathcal{T}_1)$ is $(120)^4 \max\{\delta, (120)^2 \delta^*\}$-hyperbolic.

As in the proof of Lemma 5.3, one can check that $G_3(\mathcal{T}_1)$ and $G_2(\mathcal{T}_1)$ are quasi-isometric (with constants which just depend on G^*); thus, by invariance of hyperbolicity, there exists a constant δ_2 which does not depend on $n_1, n_2, n_3, \mathcal{T}_0$, such that $G_2(\mathcal{T}_1)$ is δ_2-hyperbolic. Since $\mathcal{T}_1$ is also an (α, β)-quasi-geodesic triangle in $G_2(\mathcal{T}_1) \subset G_1$, $\mathcal{T}_1$ is δ_2'-thin, where δ_2' is a constant that does not depend on $n_1, n_2, n_3, \mathcal{T}_0$. Since $d_{G_1} \le d_{G_2(\mathcal{T}_1)}$, we have that $\mathcal{T}_1$ is also δ_2'-thin in G_1. Since $\mathcal{H}_{G_1}(g_{ij}, [z_i z_j]) \le H$, the triangle $\mathcal{T}_0$ is $(\delta_2' + 2H)$-thin in G_1. Since $\mathcal{T}_0 \subset G$ and G is an isometric subspace of G_1, the geodesic triangle $\mathcal{T}_0$ is also $(\delta_2' + 2H)$-thin in G. □

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