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# Solvability of a System of Vector Equilibrium Problems Involving Topological Pseudomonotonicity

**Rong Hu**[1]

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**Abstract** In this paper, we study the solvability of a system of vector equilibrium problems. We extend the concept of topological pseudomonotonicity to a family of mappings. We prove existence results for the system of vector equilibrium problems under topological pseudomonotonicity conditions by the Kakutani–Fan–Glicksberg fixed point theorem. As applications, we obtain existence results for systems of equilibrium problems under topological pseudomonotonicity conditions.

**Keywords** Systems of vector equilibrium problems · Topological pseudomonotonicity · Kakutani–Fan–Glicksberg fixed point · Solvability

**Mathematics Subject Classification** 49J40 · 47H10

## 1 Introduction

The equilibrium problem considered by Blum and Oettli [8] has wide applications in the problems arising in game theory, economics, operations research, and engineering science. The equilibrium problem is very general in the sense that it includes the variational inequality problem, the Nash equilibrium problem, the minimization problem, the fixed point problem, and the complementarity problem as special cases (see, for instance, Blum and Oettli [8]). Due to its wide applications, the equilibrium problem

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✉ Rong Hu
ronghumath@aliyun.com

[1] Department of Applied Mathematics, Chengdu University of Information Technology, Chengdu, Sichuan, People's Republic of China

has been studied intensively(see e.g., [5,6,8,20,28]). The vector equilibrium, as a vector extension of the equilibrium problem, has also been studied by many authors (see e.g., [7,10,21,27]). The study of the vector equilibrium problem was initially motivated by the earlier works on the vector variational inequalities([13,14,23,34]). Solvability is one of the most interesting and important topics in the field of variational inequalities and equilibrium problems. A very large number of papers in the literature deal with the existence of solutions for variational inequalities and equilibrium problems. In order to establish existence results, a usual and useful assumption is generalized monotonicity (in the Karamardian's sense [29]) (see e.g., [5–8,13,14,20,23,29]). In recent years, some authors studied the existence of solutions for variational inequalities and equilibrium problems by using $(S_+)$-conditions instead of generalized monotonicity conditions (see e.g., [11,16,21,25]). Recently, Chadli et al. [10] extended the concept of topological pseudomonotonicity [9] to vector-valued bifunctions and derived some existence results for vector equilibrium problems by using topological pseudomonotonicity instead of generalized monotonicity and $(S_+)$-conditions.

On the other hand, various systems of variational inequality problems and systems of equilibrium problems have been introduced and studied by many authors. Kassay and Kolumbán [30] introduced a system of variational inequalities and established an existence theorem by using Ky Fan lemma. Kassay et al. [31] further introduced and studied Minty and Stampacchia variational inequality systems. Fang and Huang [22] established some existence results for systems of vector equilibriums by using the Kakutani–Fan–Glicksberg fixed point theorem [24]. Ansari and others [1–3] introduced and studied systems of vector equilibrium problems by using a maximal element theorem due to Deguire et al. [17]. For more works on this topic, we refer the readers to [12,18,19,32] and the references therein.

Motivated and inspired by the above works, in this paper, we study the solvability of a system of vector equilibrium problems with topological pseudomonotonicity. We extend the concept of topological pseudomonotonicity to a family of mappings. We prove existence results for the system of vector equilibrium problems under topological pseudomonotonicity conditions. As applications, we obtain existence results for systems of scalar equilibrium problems under topological pseudomonotonicity conditions. Our results generalize the results of [10] to the system of vector equilibrium problems. The rest of this paper is organized as follows: In Sect. 2, we give some concepts and notations. In Sect. 3, we introduce some concepts of topological pseudomonotonicity. Section 4 is devoted to the existence of solutions to the system of vector equilibrium problems.

## 2 Preliminaries and Notations

In this section, we recall some concepts and notations. Let $Z$ be a real Hausdorff topological vector space with an ordering cone $C$, that is, $C$ is a closed convex cone in $Z$ with $int\, C \neq \emptyset$ and $C \neq Z$, where $int\, C$ denotes the interior of $C$. Let $D$ be a nonempty subset of a real Hausdorff topological vector space $E$.

**Definition 2.1** [4] A mapping $F : D \to 2^Z$ (the family of all nonempty subsets of $Z$) is said to be

(1) upper semicontinuous at $x \in D$ if for any open set $V$ containing $F(x)$, there exists a neighborhood $U$ of $x$ such that $F(U) \subset V$;
(2) upper semicontinuous on $D$ if $F$ is upper semicontinuous at every $x \in D$; and
(3) closed if the graph $Graph\, F = \{(x, u) \in D \times X : u \in F(x)\}$ of $F$ is closed.

*Remark 2.1* If the image of $F$ is contained in a compact subset of $Z$, then $F : D \to 2^Z$ is upper semicontinuous if and only if $F$ is closed.

**Definition 2.2** [7,33] A mapping $f : D \to Z$ is said to be $C$-upper semicontinuous on $D$ if it satisfies one of the following three equivalent conditions:

(i) For any $a \in Z$, the set $\{x \in D : f(x) \in a - int\, C\}$ is open in $D$ (Bianchi et al. [7]).
(ii) For any $x_0 \in D$ and any $v \in int\, C$, there exists an open neighborhood $U$ of $x_0$ such that $f(x_0) \in f(x) + v - int\, C$ for all $x \in U$)(Tanaka [33]).
(iii) For any $x \in D$, for any $v \in int\, C$, and any net $\{x_\alpha\}_{\alpha \in \Lambda}$ in $D$ converging to $x \in D$, there exists $\alpha_0 \in \Lambda$ such that

$$\overline{\{f(x_\beta) : \beta \geq \alpha\}} \subset f(x) + v - int\, C, \forall \alpha \geq \alpha_0.$$

*Remark 2.2* (a) The equivalence of conditions (i)–(iii) is shown in Proposition 2.1 of Tanaka [33], Lemma 2.3 of Bianchi et al. [7], and Theorem 2.4 of Chadli et al. [10]. (b) $f$ is said to be $C$-lower semicontinuous if $-f$ is $C$-upper semicontinuous. (c) If $f$ is upper semicontinuous, then it is also $C$-upper semicontinuous. (d) If $Z = R$ and $C = R_+$, then Definition 2.2 reduces to the definition of usual upper semicontinuous functions.

**Definition 2.3** A mapping $f : D \to Z$ is said to be $C$-concave-like if for any $x_1, x_2 \in D$, $t \in [0, 1]$,

$$f(x_1) \in f(tx_1 + (1-t)x_2) - C \quad \text{or} \quad f(x_2) \in f(tx_1 + (1-t)x_2) - C.$$

*Remark 2.3* (1) $C$-concave-likeness is called $C$-quasiconcaveness in [15]. (2) When $Z = R$ and $C = R_+$, Definition 2.3 reduces to the definition of quasiconcave functions.

**Definition 2.4** [15] A mapping $h : D \times D \to Z$ is said to be

(I) $C$-quasiconvex-like if for any $x, y_1, y_2 \in D$, $t \in [0, 1]$,

$$h(x, ty_1 + (1-t)y_2) \in h(x, y_1) - C \quad \text{or} \quad h(x, ty_1 + (1-t)y_2) \in h(x, y_2) - C;$$

(II) vector 0-diagonally convex if for any finite set $\{y_1, y_2, \ldots, y_n\} \subset D$,

$$\sum_{j=1}^{n} t_j h(x, y_j) \notin -int\, C$$

whenever $x = \sum_{j=1}^{n} t_j y_j$ with $t_j \geq 0$ and $\sum_{j=1}^{n} t_j = 1$.

*Remark 2.4* When $Z = R$ and $C = R_+$, (II) of Definition 2.4 reduces to the definition of 0-diagonally convex functions due to Zhou and Chen [35].

In what follows, unless otherwise specified, we always suppose that $I$ is an index set, $K_i$ is a nonempty, closed, and convex subset of a real Hausdorff topological vector space $X_i$, and $C_i$ is an ordering cone of a real Hausdorff topological vector space $Z_i$ for each $i \in I$. Let $X = \prod_{i \in I} X_i$, $K = \prod_{i \in I} K_i$, $X_{\bar{i}} = \prod_{j \in I, j \neq i} X_j$, $K_{\bar{i}} = \prod_{j \in I, j \neq i} K_j$, and $F_i : K_{\bar{i}} \times K_i \times K_i \to Z_i$ be a mapping for each $i \in I$. The system of vector equilibrium problems is to find $x = (x_i)_{i \in I} \in K$ such that for each $i \in I$,

$$(SVEP) \qquad F_i(x_{\bar{i}}, x_i, y_i) \notin -int\, C_i, \quad \forall y_i \in K_i,$$

where $x_{\bar{i}} = (x_j)_{j \in I, j \neq i} \in K_{\bar{i}}$.

*Remark 2.5* :

(1) If for each $i \in I$, $Z_i = R$, $C_i = R_+$, and $F_i = \varphi_i$, where $\varphi_i : K_{\bar{i}} \times K_i \times K_i \to R$ is a function, then $(SVEP)$ reduces to the system of equilibrium problems formulated by finding $x = (x_i)_{i \in I} \in K$ such that for each $i \in I$,

$$(SEP) \qquad \varphi_i(x_{\bar{i}}, x_i, y_i) \geq 0, \quad \forall y_i \in K_i.$$

(2) If for each $i \in I$, $Z_i = R$, $C_i = R_+$, and $F_i(x_{\bar{i}}, x_i, y_i) = \langle T_i(x_{\bar{i}}, x_i), y_i - x_i \rangle$, where $T_i : K_{\bar{i}} \times K_i \to X_i^*$ and $X_i^*$ denotes the dual space of $X_i$, then $(SVEP)$ reduces to the system of variational inequality problems formulated by finding $x = (x_i)_{i \in I} \in K$ such that for each $i \in I$,

$$(SVIP) \qquad \langle T_i(x_{\bar{i}}, x_i), y_i - x_i \rangle \geq 0, \quad \forall y_i \in K_i.$$

(3) If for each $i \in I$, $F_i(x_{\bar{i}}, x_i, y_i) = \Phi_i(x_{\bar{i}}, y_i) - \Phi_i(x_{\bar{i}}, x_i)$, where $\Phi_i : K_{\bar{i}} \times K_i \to Z_i$, then $(SVEP)$ reduces to the vector Nash equilibrium problem formulated by finding $x = (x_i)_{i \in I} \in K$ such that for each $i \in I$,

$$(VNEP) \qquad \Phi_i(x_{\bar{i}}, y_i) - \Phi_i(x_{\bar{i}}, x_i) \notin -int\, C_i, \quad \forall y_i \in K_i.$$

(4) If for each $i \in I$, $Z_i = R$, $C_i = R_+$, and $\Phi_i = \phi_i$, where $\phi_i : K_{\bar{i}} \times K_i \to R$ is a function, then $(VNEP)$ reduces to the classical Nash equilibrium problem formulated by finding $x = (x_i)_{i \in I} \in K$ such that for each $i \in I$,

$$(NEP) \qquad \phi_i(x_{\bar{i}}, y_i) \geq \phi_i(x_{\bar{i}}, x_i), \quad \forall y_i \in K_i.$$

(5) If $I$ is a singleton, then $(SVEP)$ reduces to the known vector equilibrium problem $(VEP)$, which also includes as special cases the classical equilibrium problem and variational inequality problem.

## 3 Topological Pseudomonotonicity

In this section, we shall extend the concept of topological pseudomonotonicity to a family of mappings. First, recall some concepts and notations presented in [10,16,21].

Let $A$ be a nonempty subset of a real Hausdorff topological space $Z$ and $C \subset Z$ be an ordering cone. The superior of $A$ with respect to $C$ is defined by

$$Sup\ A = \{z \in \bar{A} : A \cap (z + int\ C) = \emptyset\}$$

and the inferior of $A$ with respect to $C$ is defined by

$$Inf\ A = \{z \in \bar{A} : A \cap (z - int\ C) = \emptyset\},$$

where $\bar{A}$ denotes the closure of $A$.

As pointed out in [10,16], the superior and inferior of a subset of $Z$ with respect to $C$ are, respectively, extensions of the usual supremum and infimum of a subset of $R$. If $A$ is a nonempty compact subset of $Z$, then both $Sup\ A$ and $Inf\ A$ are nonempty. Let $\{z_\alpha\}_{\alpha\in\Lambda}$ be a net in $Z$. The *limit superior* and *limit inferior* of $\{z_\alpha\}_{\alpha\in\Lambda}$ (with respect to $C$) are defined by

$$Limsup\, z_\alpha = Inf \bigcup_{\alpha\in\Lambda} Sup\ S_\alpha \qquad Liminf\, z_\alpha = Sup \bigcup_{\alpha\in\Lambda} Inf\ S_\alpha,$$

where $S_\alpha = \{z_\beta : \beta \succeq \alpha\}$. The *limit superior* and *limit inferior* of $\{z_\alpha\}_{\alpha\in\Lambda}$ (with respect to $C$) are also extensions of the usual limit superior and limit inferior of $\{z_\alpha\}$, respectively (see [10,16]).

In the sequel, we recall some concepts of topological pseudomonotonicity. Let $D$ be a nonempty closed subset of a real Hausdorff topological vector space of $E$.

**Definition 3.1** A mapping $T : D \to E^*$ is said to be topological pseudomonotone (or pseudomonotone in the sense of Brézis [9]) if for any net $\{x_\alpha\} \subset D$,

$$x_\alpha \rightharpoonup x \text{ and } \limsup\langle Tx_\alpha, x_\alpha - x\rangle \leq 0 \Rightarrow \langle Tx, x - y\rangle \leq \liminf\langle Tx_\alpha, x_\alpha - y\rangle, \forall y \in D,$$

where $\rightharpoonup$ means weak convergence.

The concept of topological pseudomonotonicity has been generalized to bifunctions.

**Definition 3.2** [4,26] A bifunction $f : D \times D \to R$ is said to be topologically pseudomonotone if for any net $\{x_\alpha\} \subset D$ contained a compact subset of $D$,

$$x_\alpha \to x \text{ and } \liminf_\alpha f(x_\alpha, x) \geq 0 \Rightarrow f(x, y) \geq \limsup_\alpha f(x_\alpha, y), \forall y \in D.$$

Chadli et al. [10] extended the notion of topological pseudomonotonicity to vector-valued bifunctions.

**Definition 3.3** [10] A vector-valued bifunction $f : D \times D \to Z$ is said to be topologically pseudomonotone if for any $v \in int\, C$ and any net $\{x_\alpha\}$ in $D$ satisfying

$$x_\alpha \to x \in D \text{ and } Liminf_\alpha f(x_\alpha, x) \cap (-int\, C) = \emptyset,$$

there is some index $\alpha_0$ such that

$$\overline{\{f(x_\beta, y) : \beta \geq \alpha\}} \subset f(x, y) + v - int\, C, \forall \alpha \geq \alpha_0 \text{ and } \forall y \in D.$$

*Remark 3.1* As pointed out by Chadli [10], Definition 3.3 generalizes Definitions 3.1 and 3.2 in a natural way.

Now, we extend the concept of topological pseudomonotonicity to a family of mappings.

**Definition 3.4** Let $I$ be an index set and $\Psi_i : K_{\bar{i}} \times K_i \times K_i \to Z_i$ be a mapping for all $i \in I$. We say that $\{\Psi_i\}_{i \in I}$ is topologically pseudomonotone if for any net $\{x^\alpha\} = \{(x_i)^\alpha_{i \in I}\} \subset K$ satisfying

$$x^\alpha \to x = (x_i)_{i \in I} \in K \text{ and } Liminf_\alpha \Psi_i(x^\alpha_{\bar{i}}, x^\alpha_i, x_i) \cap (-int\, C_i) = \emptyset, \forall i \in I,$$

and for any $i_0 \in I$ and any $v_{i_0} \in int\, C_{i_0}$, there exists $\alpha_0$ such that

$$\overline{\left\{\Psi_{i_0}\left(x^\beta_{\bar{i}_0}, x^\beta_{i_0}, y_{i_0}\right) : \beta \geq \alpha\right\}} \subset \Psi_{i_0}(x_{\bar{i}_0}, x_{i_0}, y_{i_0}) + v_{i_0} - int\, C_{i_0}, \forall \alpha \geq \alpha_0 \text{ and } \forall y_{i_0} \in K_{i_0}.$$

*Remark 3.2* (1) If for each $i \in I$, $Z_i = R$, $C_i = R_+$, then Definition 3.4 reduces to the definition of topological pseudomonotonicity for a family of functions $\{\varphi_i\}_{i \in I}$, i.e., $\{\varphi_i\}_{i \in I}$ is said to be topologically pseudomonotone if for any net $\{x^\alpha\} = \{(x_i)^\alpha_{i \in I}\} \subset K$ satisfying

$$x^\alpha \to x = (x_i)_{i \in I} \in K \text{ and } \liminf_\alpha \varphi_i(x^\alpha_{\bar{i}}, x^\alpha_i, x_i) \geq 0, \forall i \in I,$$

and for any $i_0 \in I$ and any $\epsilon_{i_0} > 0$, there exists $\alpha_0$ such that

$$\varphi_{i_0}\left(x^\alpha_{\bar{i}_0}, x^\alpha_{i_0}, y_{i_0}\right) < \varphi_{i_0}\left(x_{\bar{i}_0}, x_{i_0}, y_{i_0}\right) + \epsilon_{i_0}, \forall \alpha \geq \alpha_0 \text{ and } \forall y_{i_0} \in K_{i_0},$$

where $\varphi_i : K_{\bar{i}} \times K_i \times K_i \to R$ is a function for all $i \in I$.

(2) If for each $i \in I$, $Z_i = R$, $C_i = R_+$, and $\Psi_i(x_{\bar{i}}, x_i, y_i) = \langle T_i(x_{\bar{i}}, x_i), y_i - x_i \rangle$, where $T_i : K_{\bar{i}} \times K_i \to X_i^*$, then Definition 3.4 reduces to the definition of topological pseudomonotonicity for $\{T_i\}_{i \in I}$, i.e., $\{T_i\}_{i \in I}$ is said to be topologically pseudomonotone if for any net $\{x^\alpha\} = \{(x_i)^\alpha_{i \in I}\} \subset K$ satisfying

$$x^\alpha \to x = (x_i)_{i \in I} \in K \text{ and } \liminf_\alpha \langle T_i(x^\alpha_{\bar{i}}, x^\alpha_i), x_i - x^\alpha_i \rangle \geq 0, \forall i \in I,$$

for any $i_0 \in I$ and any $\epsilon_{i_0} > 0$, there exists $\alpha_0$ such that

$$\left\langle T_{i_0}\left(x^{\alpha}_{\bar{i}_0}, x^{\alpha}_{i_0}\right), y_{i_0} - x^{\alpha}_{i_0}\right\rangle < \left\langle T_{i_0}\left(x_{\bar{i}_0}, x_{i_0}\right), y_{i_0} - x_{i_0}\right\rangle + \epsilon_{i_0}, \forall \alpha \geq \alpha_0 \text{ and } y_{i_0} \in K_{i_0}.$$

(3) If $I$ is a singleton, then Definition 3.4 coincides with Definition 3.3.

## 4 Existence Results

In this section, we study the existence of solutions to $(SVEP)$ by using topological pseudomonotonicity. First we need the following lemmas.

**Lemma 4.1** *See* Lemma 3.6 *of* [10] *and* Lemma 3.1 *of* [21]. *Let $D$ be a nonempty, compact, and convex subset of a real Hausdorff topological vector space $E$ and $C$ be an ordering cone of a real Hausdorff topological space $Z$. Let $f : D \times D \to Z$ be a mapping satisfying the following conditions:*

*(1) for every $y \in D$, $f(\cdot, y)$ is $C$-upper semicontinuous;*
*(2) $f$ is vector 0-diagonally convex; and*
*(3) for every $y \in D$, $f(\cdot, y)$ is $C$-concave-like.*

*Then the problem formulated by finding $\bar{x} \in D$ such that*

$$f(\bar{x}, y) \notin -int\, C, \quad \forall y \in D$$

*admits a nonempty, compact, and convex solution set.*

**Lemma 4.2** *Let $D$ be a nonempty, compact, and convex subset of a real Hausdorff topological vector space $E$ and $C$ be an ordering cone of a real Hausdorff topological space $Z$. Let $f : D \times D \to Z$ be a mapping satisfying the following conditions:*

*(1) $f(x, x) \notin -int\, C$ for all $x \in D$;*
*(2) for every $y \in D$, $f(\cdot, y)$ is $C$-upper semicontinuous;*
*(3) $f$ is $C$-quasiconvex-like; and*
*(4) for every $y \in D$, $f(\cdot, y)$ is $C$-concave-like.*

*Then the problem formulated by finding $\bar{x} \in D$ such that*

$$f(\bar{x}, y) \notin -int\, C, \quad \forall y \in D$$

*admits a nonempty, compact, and convex solution set.*

*Proof* The conclusion follows from the same arguments of the proofs of Lemma 3.9 of [10] and Lemma 3.1 of [21]. □

**Theorem 4.1** *Let $F_i : K_{\bar{i}} \times K_i \times K_i \to Z_i$ be a mapping for all $i \in I$. Assume that*

*(1) for each $i \in I$ and for all $x = (x_i)_{i \in I} \in K$, $F_i(x_{\bar{i}}, \cdot, x_i)$ is $C_i$-upper semicontinuous on the convex hull of every nonempty finite subset of $K_i$;*
*(2) for each $i \in I$ and for all $x_{\bar{i}} \in K_{\bar{i}}$, $F_i(x_{\bar{i}}, \cdot, \cdot)$ is vector 0-diagonally convex;*

(3) *for each* $i \in I$ *and for all* $x = (x_i)_{i \in I} \in K$, $F_i(x_{\bar{i}}, \cdot, x_i)$ *is* $C_i$*-concave-like;*
(4) *for each* $i \in I$ *and for all* $x_i \in K_i$, $F_i(\cdot, \cdot, x_i)$ *is* $C_i$*-upper semicontinuous on the convex hull of every nonempty finite subset of* $K$*;*
(5) *for each* $i \in I$, *there is a nonempty compact set* $A_i \subset K_i$, *and there is a nonempty, compact, and convex set* $B_i \subset K_i$ *such that if* $x_i \in K_i \cap A_i^c$, *where* $A_i^c$ *denotes the complement of* $A_i$ *in* $X_i$, *then* $F_i(x_{\bar{i}}, x_i, y_i) \in -int\, C_i$ *for some* $y_i \in B_i$*; and*
(6) $\{F_i\}_{i \in I}$ *is topologically pseudomonotone.*

*Then* $(SVEP)$ *is solvable.*

*Proof* Set

$$\mathcal{M} = \left\{ M \subset K : M = \prod_{i \in I} M_i \text{ with } M_i \text{ being the convex hull of a finite subset of } K_i \text{ for all } i \in I \right\}.$$

For given $M \in \mathcal{M}$ and $z = (z_i)_{i \in I} \in K$, consider the following problems:

$$(AP)_M^i \qquad \text{find } x_i \in M_i \text{ such that } F_i(z_{\bar{i}}, x_i, y_i) \notin -int\, C_i, \forall y_i \in M_i.$$

It follows from conditions (1)–(3) and Lemma 4.1 that for each $i \in I$, $(AP)_M^i$ has a nonempty, compact, and convex solution set. For each $i \in I$, define a multivalued mapping $T_M^i : M_{\bar{i}} \to 2^{M_i}$ by

$$T_M^i(z_{\bar{i}}) = \{x_i \in M_i : F_i(z_{\bar{i}}, x_i, y_i) \notin -int\, C_i, \forall y_i \in M_i\}, \quad \forall z_{\bar{i}} \in M_{\bar{i}}.$$

Then for any $i \in I$ and any $z_{\bar{i}} \in M_{\bar{i}}$, $T_M^i(z_{\bar{i}})$ is nonempty, compact, and convex. Furthermore, for each $i \in I$, it is easy to verify that $T_M^i$ has a closed graph from condition (4). By Remark 2.1, $T_M^i$ is upper semicontinuous for all $i \in I$. Define $T_M : M \to 2^M$ by

$$T_M(z) = (T_M^i(z_{\bar{i}}))_{i \in I}, \quad \forall z = (z_i)_{i \in I} \in M.$$

By the above arguments, $T_M$ is upper semicontinuous with nonempty, compact, and convex values. By Kakutani–Fan–Glicksberg fixed point theorem (see [24]), $T_M$ has a fixed point $u$ on $M$, i.e., there exists $u = (u_i)_{i \in I} \in M$ such that for each $i \in I$,

$$F_i(u_{\bar{i}}, u_i, y_i) \notin -int\, C_i, \forall y_i \in M_i.$$

For any $M = \prod_{i \in I} M_i \in \mathcal{M}$, let

$$S_M = \{u = (u_i)_{i \in I} \in M : F_i(u_{\bar{i}}, u_i, y_i) \notin -int\, C_i, \forall y_i \in M_i, \forall i \in I\}$$

and

$$N_M = \left\{ u = (u_i)_{i \in I} \in A = \prod_{i \in I} A_i : F_i(u_{\bar{i}}, u_i, y_i) \notin -int\ C_i, \forall y_i \right.$$
$$\left. \in co(M_i \cup B_i), \forall i \in I \right\},$$

where $co$ denotes the convex hull operator. By the above arguments $S_M$ is nonempty. We also have $S_{\tilde{M}} \subset N_M$ by condition (5), where $\tilde{M} = \prod_{i \in I} \tilde{M}_i$ with $\tilde{M}_i = co(M_i \cup B_i)$. Thus $\overline{N}_M$ is nonempty compact for all $M \in \mathcal{M}$, where $\overline{N}_M$ is the closure of $\overline{N}_M$. Let $M^j = \prod_{i \in I} M_i^j \in \mathcal{M}$, $j = 1, 2, \ldots, n$ and $L = \prod_{i \in I} L_i$ with $L_i$ being the convex hull of $\cup_{j=1}^n M_i^j$ for all $i \in I$. It is easy to see that $N_L \subset \cap_{j=1}^n N_{M^j}$. Hence $\{\overline{N}_M : M \in \mathcal{M}\}$ has the finite intersection property. It follows that

$$\cap_{M \in \mathcal{M}} \overline{N}_M \neq \emptyset.$$

Let $u^* = (u_i^*)_{i \in I} \in \cap_{M \in \mathcal{M}} \overline{N}_M$. We assert that $u^*$ is a solution of $(SVEP)$. Assume by contradiction that there exist $i_0 \in I$ and $y_{i_0} \in K_{i_0}$ such that

$$F_{i_0}(u_{\bar{i}_0}^*, u_{i_0}^*, y_{i_0}) \in -int\ C_{i_0}.$$

Let $y = (y_i)_{i \in I} \in K$ and $\hat{M} = \prod_{i \in I} \hat{M}_i \in \mathcal{M}$ with $\hat{M}_i = co\{u_i^*, y_i\}$ for all $i \in I$. Since $u^* \in \overline{N}_{\hat{M}}$, there exists a net $\{u^\alpha\} = \{(u_i)_{i \in I}^\alpha\} \in N_{\hat{M}}$ such that $u^\alpha \to u^*$. It follows that for each $i \in I$,

$$F_i(u_{\bar{i}}^\alpha, u_i^\alpha, u_i^*) \notin -intC_i, \quad \forall \alpha.$$

Hence

$$Liminf\, F_i(u_{\bar{i}}^\alpha, u_i^\alpha, u_i^*) \cap (-int\ C_i) = \emptyset, \quad \forall i \in I.$$

By condition (6), for $i_0 \in I$ and $v_{i_0} = -F_{i_0}(u_{\bar{i}_0}^*, u_{i_0}^*, y_{i_0}) \in intC_{i_0}$, there exists $\alpha_0$ such that

$$\overline{\left\{ F_{i_0}\left(u_{\bar{i}_0}^\beta, u_{i_0}^\beta, y_{i_0}\right) : \beta \geq \alpha \right\}} \subset F_{i_0}(u_{\bar{i}_0}^*, u_{i_0}^*, y_{i_0}) - F_{i_0}(u_{\bar{i}_0}^*, u_{i_0}^*, y_{i_0}) - int\ C_{i_0}$$
$$= -int\ C_{i_0}, \forall \alpha \geq \alpha_0.$$

This is a contradiction since

$$F_{i_0}\left(u_{\bar{i}_0}^\alpha, u_{i_0}^\alpha, y_{i_0}\right) \notin -int\ C_{i_0}, \forall \alpha.$$

Thus $u^*$ is a solution of $(SVEP)$. □

By using Lemma 4.2 and similar proof as in Theorem 4.1, we obtain the following result.

**Theorem 4.2** *Let $F_i : K_{\bar{i}} \times K_i \times K_i \to Z_i$ be a mapping for all $i \in I$. Assume that*

*(1) for each $i \in I$ and for all $x = (x_i)_{i \in I} \in K$, $F_i(x_{\bar{i}}, x_i, x_i) \notin -int\, C_i$;*
*(2) for each $i \in I$ and for all $x = (x_i)_{i \in I} \in K$, $F_i(x_{\bar{i}}, \cdot, x_i)$ is $C_i$-upper semicontinuous on the convex hull of every nonempty finite subset of $K_i$;*
*(3) for each $i \in I$ and for all $x_{\bar{i}} \in K_{\bar{i}}$, $F_i(x_{\bar{i}}, \cdot, \cdot)$ is $C_i$-quasiconvex-like;*
*(4) for each $i \in I$ and for all $x = (x_i)_{i \in I} \in K$, $F_i(x_{\bar{i}}, \cdot, x_i)$ is $C_i$-concave-like;*
*(5) for each $i \in I$ and for all $x_i \in K_i$, $F_i(\cdot, \cdot, x_i)$ is $C_i$-upper semicontinuous on the convex hull of every nonempty finite subset of $K$;*
*(6) for each $i \in I$, there is a nonempty compact set $A_i \subset K_i$, and there is a nonempty, compact, and convex set $B_i \subset K_i$ such that if $x_i \in K_i \cap A_i^c$, then $F_i(x_{\bar{i}}, x_i, y_i) \in -int\, C_i$ for some $y_i \in B_i$; and*
*(7) $\{F_i\}_{i \in I}$ is topologically pseudomonotone.*

*Then $(SVEP)$ is solvable.*

**Corollary 4.1** *Let $\varphi_i : K_{\bar{i}} \times K_i \times K_i \to R$ be a function for all $i \in I$. Assume that*

*(1) for each $i \in I$ and for all $x = (x_i)_{i \in I} \in K$, $\varphi_i(x_{\bar{i}}, \cdot, x_i)$ is upper semicontinuous on the convex hull of every nonempty finite subset of $K_i$;*
*(2) for each $i \in I$ and for all $x_{\bar{i}} \in K_{\bar{i}}$, $\varphi_i(x_{\bar{i}}, \cdot, \cdot)$ is 0-diagonally convex;*
*(3) for each $i \in I$ and for all $x = (x_i)_{i \in I} \in K$, $\varphi_i(x_{\bar{i}}, \cdot, x_i)$ is quasiconcave;*
*(4) for each $i \in I$ and $x_i \in K_i$, $\varphi_i(\cdot, \cdot, x_i)$ is upper semicontinuous on the convex hull of every nonempty finite subset of $K$;*
*(5) for each $i \in I$, there is a nonempty compact set $A_i \subset K_i$, and there is a nonempty, compact, and convex set $B_i \subset K_i$ such that if $x_i \in K_i \cap A_i^c$, then $\varphi_i(x_{\bar{i}}, x_i, y_i) < 0$ for some $y_i \in B_i$; and*
*(6) $\{\varphi_i\}_{i \in I}$ is topologically pseudomonotone.*

*Then $(SEP)$ is solvable.*

*Proof* The conclusion follows directly from Theorem 4.1. □

**Corollary 4.2** *Let $\varphi_i : K_{\bar{i}} \times K_i \times K_i \to R$ be a function for all $i \in I$. Assume that*

*(1) for each $i \in I$ and for all $x = (x_i)_{i \in I} \in K$, $\varphi_i(x_{\bar{i}}, x_i, x_i) \geq 0$;*
*(2) for each $i \in I$ and for all $x = (x_i)_{i \in I} \in K$, $\varphi_i(x_{\bar{i}}, \cdot, x_i)$ is upper semicontinuous on the convex hull of every nonempty finite subset of $K_i$;*
*(3) for each $i \in I$ and for all $x = (x_i)_{i \in I} \in K$, $\varphi_i(x_{\bar{i}}, x_i, \cdot)$ is quasiconvex;*
*(4) for each $i \in I$ and for all $x = (x_i)_{i \in I} \in K$, $\varphi_i(x_{\bar{i}}, \cdot, x_i)$ is quasiconcave;*
*(5) for each $i \in I$ and for all $x_i \in K_i$, $\varphi_i(\cdot, \cdot, x_i)$ is upper semicontinuous on the convex hull of every nonempty finite subset of $K$;*
*(6) for each $i \in I$, there is a nonempty compact set $A_i \subset K_i$, and there is a nonempty, compact, and convex set $B_i \subset K_i$ such that if $x_i \in K_i \cap A_i^c$, then $\varphi_i(x_{\bar{i}}, x_i, y_i) < 0$ for some $y_i \in B_i$; and*
*(7) $\{\varphi_i\}_{i \in I}$ is topologically pseudomonotone.*

*Then $(SEP)$ is solvable.*

*Proof* The conclusion follows directly from Theorem 4.2. □

*Remark 4.1* The approach used in the proof of Theorem 4.1 is quite different from those in [1–3,18,19,22,32], where some existence results for $(SVEP)$ and $(SEP)$ with different assumptions were also established.

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