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C-Normal and Hypercyclically Embedded Subgroups of Finite Groups

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Abstract Let p be a prime, E be a normal subgroup of a finite group G. In this paper, we will investigate the way E embedded in G under the assumption that some p-subgroups of E are c-normal in G. We pay more attention to the p-subgroups of E with given order p^d. We generalized several recent results of other scholars.

Keywords C-normal · Hypercenter · p-nilpotent · Supersolvable

Mathematics Subject Classification 20D10

1 Introduction

All groups considered in this paper are finite. We use conventional notions and notation, as in [5]. G always denotes a finite group, $|G|$ is the order of G, $\pi(G)$ denotes the set of all primes dividing $|G|$, and G_p is a Sylow p-subgroup of G for a prime $p \in \pi(G)$.

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In [13], Wang defined the c-normality of a subgroup as follows and prove that a finite group G is solvable if and only if every maximal subgroup of G is c-normal in G.

Definition 1.1 ([13, Definition 1.1]) Let H be a subgroup of a finite group G. We say that H is c-normal in G if there exists a normal subgroup N of G such that $HN = G$ and $H \cap N \leq H_G$.

The basic properties of c-normality are as follows.

Lemma 1.2 (see [13, Lemma 2.1] and [9, Lemma 2.4]) *Let G be a group. Then*

(1) *If H is normal in G, then H is c-normal in G.*
(2) *If H is c-normal in G and $H \leq K \leq G$, then H is c-normal in K.*
(3) *Suppose that K is a normal subgroup of G and that H is c-normal in G. Then HK/K is c-normal in G/K when $K \leq H$ or $(|H|, |K|) = 1$.*

Let G be a finite group. Several authors successfully use the c-normal property of some subgroups of G of prime power order to determine the structure of G (see [1,2,8–12]). Many results in these previous papers have the following form: Let E be a normal subgroup of G and $\mathcal{F}$ be a saturated formation containing the class of all supersolvable groups. Suppose that G/E is in $\mathcal{F}$. If for each prime divisor p of $|E|$, some p-subgroups of E are c-normal G, then $G \in \mathcal{F}$. Actually, in a more general case, if we can get a criterion that E lies in the $\mathcal{F}$-hypercenter, then $G/E \in \mathcal{F}$ implies that $G \in \mathcal{F}$. In order to get good results, many authors have to impose the c-normal hypotheses on all the prime divisors or the minimal or maximal divisor p of $|G|$ rather than any prime divisor. In this paper, we try to get more general results.

Now let p be a fixed prime. In this paper, we focus on how a normal subgroup E embedded in G provided every p-subgroup of E with some fix order is c-normal in G. For this purpose, we introduce the concept of p-hypercentrally embedded:

Definition 1.3 Let G be a finite group. A normal subgroup E of G is said to be p-hypercentrally embedded in G if every p-chief factor of G below E is cyclic.

The main result of this paper is the following theorem.

Theorem A *Let p be a fixed prime and E be a normal subgroup of a finite group G. Suppose that E_p is a Sylow p-subgroup of E and p^d is a prime power such that $1 < p^d \leq \max(|E_p|/p, p)$. If all the subgroups of E_p with order p^d and $2p^d$ (if a quaternion group is involved in E_p) are c-normal in G, then E is p-hypercyclically embedded in G.*

2 Proof of the Results

First, we need some results on c-supplemented subgroups of finite groups. Following [3], a group H is said to be c-supplemented in G if there exists a subgroup K of G such that $G = HK$ and $H \cap K \leq H_G$. It is clear from the definition that if a subgroup H of G is c-normal in G, then H is c-supplemented in G.

Lemma 2.1 *If N is a minimal abelian normal subgroup of G, then any proper subgroup of N is not c-supplemented in G.*

Proof Suppose that this Lemma is not true and let H be a proper subgroup of N such that H is c-supplemented in G. Obviously, $H_G = 1$ since $H_G < N$ and N is a minimal normal subgroup of G. By the definition of c-supplemented subgroups, there exists a subgroup M of G such that $G = HM$ with $H \cap M \leq H_G = 1$. Hence $NM \geq HM = G$. Since N is abelian, we know that $N \cap M \trianglelefteq G$. Hence $N \cap M = 1$. Therefore, we have $|G| = |NM| = |N||M| > |H||M| = |HM|$, a contradiction. □

For a saturated formation $\mathcal{F}$, the $\mathcal{F}$-hypercenter of a group G is denoted by $Z_{\mathcal{F}}(G)$ (see [5, p. 389, Notation and Definitions 6.8(b)]). Let $\mathcal{U}$ denote the class of all supersolvable groups and let $\mathcal{N}$ denote the class of all nilpotent groups. Suppose that A is a normal subgroup of G. It is clear that $A \leq Z_{\mathcal{U}}(G)$ if and only if every chief factor of G below A is cyclic and that $A \leq Z_{\mathcal{N}}(G)$ if and only if every chief factor of G below A is central. In [1], Asaad gave the following result: Let P be a nontrivial normal p-subgroup, where p is an odd prime. If every minimal subgroup of P is c-supplemented in G, then $P \leq Z_{\mathcal{U}}(G)$. It is helpful to give a result for $p = 2$. In fact, we have the following proposition:

Proposition 2.2 *Let P be a normal 2-subgroup of G. If all minimal subgroups of P and all cyclic subgroups of P with order 4 (if a quaternion group is involved in P) are c-supplemented in G, then $P \leq Z_{\mathcal{N}}(G)$.*

Proof Let Q be a Sylow q-subgroup of G, where q is prime different from p. We claim that PQ is nilpotent. Suppose that the claim is not true and let H be a minimal non-nilpotent subgroup of PQ. Then $H = [H_2]H_q$, where $H_q \in Syl_q(H)$ and H_2 is a normal Sylow 2-subgroup of H. By Itô's result (see [4, Chap. 3, 5.2]), we have that $\exp(H_2) \leq 4$ and that H_q acts irreducibly on $H_2/\Phi(H_2)$. It is easy to see that $|H_2/\Phi(H_2)| \geq 4$. Clearly, H_q acts nontrivially on H_2 but acts trivially on any proper H_q-invariant subgroup of H_2. It follows by the reduction theorem of Hall and Higmman (see [7, Chap. 5 Theorem 3.7]) that $H_2' = \Phi(H_2)$.

Case 1. Suppose first that a quaternion group is involved in H_2. Then $\exp(H_2) = 4$, and we may take a subgroup $\langle x \rangle \leq H_2$ of G of order 4 and $\langle x \rangle \nleq \Phi(H_2)$. By hypotheses $\langle x \rangle$ is c-supplemented in G, and thus $\langle x \rangle \Phi(H_2)/\Phi(H_2) \neq 1$ is c-supplemented in $H/\Phi(H)$, which contradicts Lemma 2.1.

Case 2. Suppose that H_2 is quaternion free. Assume that H_2 is abelian. Then $H_2' = \Phi(H_2) = 1$ and thus H_2 is a minimal normal subgroup of H. It follows from Lemma 2.1 that $H_2 = 1$, a contradiction. Assume that H_2 is not abelian. Applying [6, Theorem 2.7], H_q acts on $H_2/\Phi(H_2)$ with at least one fixed point. This implies that $|H_2/\Phi(H_2)| = 2$, a contradiction.

The above proof shows that PQ is nilpotent as claimed. In particular, P centralizes all odd elements of G. Thus for any G-chief factor H/K of P, $G/C_G(H/K)$ is a 2-group. By [5, A, Lemma 13.6], we have $O_2(G/C_G(H/K)) = 1$. It follows that $G/C_G(H/K) = 1$ and thus $P \leq Z_{\mathcal{N}}(G)$. □

As an application of Proposition 2.2, we have

Corollary 2.3 *If all minimal subgroups of G_2 and all cyclic subgroups of G_2 with order* 4 *(if a quaternion group is involved in G_2) are c-supplemented in G, then G is* 2*-nilpotent.*

Proof Suppose that this corollary is not true and let G be a counterexample with minimal order. Obviously, the hypotheses are inhered by all subgroups of G, hence G is a minimal non- 2-nilpotent group. It follows that G_2 is a normal subgroup in G. Applying Proposition 2.2 to G_2, we get a contradiction. □

By combining [1, Theorem 1.1] and Proposition 2.2, we have

Lemma 2.4 *Let P be a normal p-subgroup of G. If all cyclic subgroups of P with order p or* 4 *(if a quaternion group is involved in P) are c-supplemented in G, then $P \leq Z_{\mathcal{U}}(G)$.*

Next, we will show that if some class of p-subgroups of G is c-normal in G, then G is p-solvable.

Lemma 2.5 *If G_p is c-normal in G then G is p-solvable.*

Proof Suppose that this Lemma is not true and consider G to be a counterexample with minimal order. By Lemma 1.2 (3), the hypothesis holds for both $G/\mathrm{O}_p(G)$ and $G/\mathrm{O}_{p'}(G)$, thus the minimal choice of G implies that $\mathrm{O}_p(G) = \mathrm{O}_{p'}(G) = 1$. By the definition of c-normality, there exists a normal subgroup H of G such that $G = G_pH$ and $H \cap G_p \leq (G_p)_G$. But $(G_p)_G = \mathrm{O}_p(G) = 1$, hence H is a normal p'-subgroup of G. The fact that $\mathrm{O}_{p'}(G) = 1$ then indicates that $H = 1$ and thus $G = G_p$, a contradiction. □

Lemma 2.6 *Let G be a finite group and p^d be a prime power such that $3 \leq p^d \leq |G|_p$. If all subgroups of G with order p^d are c-normal in G, then G is p-solvable.*

Proof By Lemma 2.4, we may assume that $p^d < |G|_p$. Let D be any subgroup of order p^d. By the hypotheses, there exists a normal subgroup H of G such that $G = DH$ and $D \bigcap H \leq D_G$. Assume that $H < G$. Since G/H is a p-group, we may take a normal subgroup M of G such that $M \geq H$ and $|G : M| = p$. As $p^d < |G|_p$, $p^d \leq |M|_p$.

Clearly, all subgroups of M with order p^d are c-normal in M. It follows by induction that M is p-solvable, and so is G. Hence we may assume that $H = G$ and then $D = D_G$ is normal in G. Assume that D is not a minimal normal subgroup of G. Let V be a minimal G-invariant subgroup of D and $|V| = p^e$. Then $p \leq |V| < p^d$ and all subgroups of G/V with order p^{d-e} are c-normal in G/V. By Induction G/V is p-solvable, and so is G. Hence we may assume that D is minimal normal in G whenever D is a subgroup of order p^d.

Note that if all subgroups of order p^d are contained in $Z(G)$, then G is p-nilpotent by a well-known result of Itô (see [4, IV, 5.3]). Hence we may assume that there is a subgroup U of order p^d such that $U \nleq Z(G)$. Suppose that $|U| = p^d \geq p^2$. Let K be a subgroup of order p^{d+1} such that $U < K$. Clearly U is not cyclic, and hence there is a maximal subgroup U_1 of K such that $U_1 \neq U$. Since U_1 is normal as assumed, we get that $U \cap U_1$ is a nontrivial G-invariant subgroup of U, and this contradicts the minimal normality of U. Hence $|U| = p$. Observe that $G/C_G(U)$ is a p'-group and that $C_G(U) < G$. It follows by induction that $C_G(U)$ is p-solvable, and so is G. □

Now, we will study the properties of p-hypercyclically embedding. Clearly, if a normal subgroup E is p-hypercyclically embedded in G, then E is p-solvable and every normal subgroup of G contained in E is also p-hypercyclically embedded in G. The following lemma shows that for a p-solvable normal subgroup E, we can deduce that E is p-hypercyclically embedded in G if the maximal p-nilpotent normal subgroup of E (denoted by $F_p(E)$) is p-hypercyclically embedded in G.

Lemma 2.7 *A p-solvable normal subgroup E is p-hypercyclically embedded in G if and only if $F_p(E)$ is p-hypercyclically embedded in G.*

Proof We only need to prove the sufficiency. Suppose that the assertion is false and let (G, E) be a counterexample with $|G| + |E|$ minimal. We claim that $O_{p'}(E) = 1$. Indeed, since $F_p(E/O_{p'}(E)) = F_p(E)/O_{p'}(E)$, it is easy to verify that the hypotheses hold for $(G/O_{p'}(E), E/O_{p'}(E))$. If $O_{p'}(E) \neq 1$, then the minimal choice of (G, E) implies that $E/O_{p'}(E)$ is p-hypercyclically embedded in $G/O_{p'}(E)$. Clearly $O_{p'}(E)$ is p-hypercyclically embedded in G. Therefore, E is p-hypercyclically embedded in G, a contradiction.

Let N be a minimal normal subgroup of G contained in E. N is an abelian normal p-subgroup since E is p-solvable and $O_{p'}(E) = 1$. Consider the group $C_E(N)/N$. Let $L/N = O_{p'}(C_E(N)/N)$ and K be a Hall p'-subgroup of L. Then $L = KN$. Since $K \leq L \leq C_E(N)$, we have $K = O_{p'}(L) \leq O_{p'}(G) = 1$. Consequently, $O_{p'}(C_E(N)/N) = 1$ and we have $F_p(C_E(N)/N) = O_p(C_E(N)/N) \leq O_p(E)/N = F_p(E)/N$. As a result, we know that the hypotheses hold for $(G/N, C_E(N)/N)$ and the minimal choice of (G, E) yields that $C_E(N)/N$ is p-hypercyclically embedded in G/N. But $N \leq F_p(G)$ and thus N is also p-hypercyclically embedded in G. It follows that $C_E(N)$ is p-hypercyclically embedded in G.

Since N is a normal p-subgroup that is p-hypercyclically embedded in G, $|N| = p$. It yields that $G/C_G(N)$ is a cyclic group. As a result, $EC_G(N)/C_G(N)$ is p-hypercyclically embedded in $G/C_G(N)$. Note that $E/C_E(N) = E/(E \cap C_G(N))$ is G-isomorphic to $EC_G(N)/C_G(N)$ and $E/C_E(N)$ is p-hypercyclically embedded in $G/C_E(N)$. But $C_E(N)$ is p-hypercyclically embedded in G and thus E is or p-hypercyclically embedded in G, a final contradiction. □

Denote $\mathcal{A}(p-1)$ as the formation of all abelian groups of exponent divisible by $p-1$. The following proposition is well known:

Lemma 2.8 ([15, Theorem 1.4]) *Let H/K be a chief factor of G and p be a prime divisor of $|H/K|$. Then $|H/K| = p$ if and only if $G/C_G(H/K) \in \mathcal{A}(p-1)$.*

Let f be a formation function and E be a normal subgroup of G. We say that G acts f-centrally on E if $G/C_G(H/K) \in f(p)$ for every chief factor H/K of G below E and every prime p dividing $|H/K|$ ([5], p. 387, Definitions 6.2). Fixing a prime p, define a formation function g_p as follows:

$$g_p(q) = \begin{cases} \mathcal{A}(p-1) & (\text{if } q = p) \\ \text{all finite group} & (\text{if } q \neq p) \end{cases}$$

From Lemma 2.8, we can see that E is p-hypercyclically embedded in G if and only if G acts g_p-centrally on E. By applying [5, p. 388, Theorem 6. 7], we get the following useful results:

Lemma 2.9 *A normal subgroup E of G is p-hypercyclically embedded in G if and only if $E/\Phi(E)$ is p-hypercyclically embedded in $G/\Phi(E)$.*

Then the following lemma is evident.

Lemma 2.10 *Let K and L be two normal subgroups of G contained in E. If E/K is p-hypercyclically embedded in G/K and E/L is p-hypercyclically embedded in G/L, then $E/(L \cap K)$ is p-hypercyclically embedded in $G/(L \cap K)$.*

The following proposition indicates that Theorem A holds when $p^d = p$.

Proposition 2.11 *Let E be a normal subgroup of G. If all cyclic subgroups of E_p with order p and 4 (if a quaternion group is involved in E_p) are c-normal in G, then E is p-hypercyclically embedded in G.*

Proof Note that E is p-solvable by Lemma 2.6. Suppose that $O_{p'}(E) > 1$. Since the hypotheses hold for $G/O_{p'}(E)$, we conclude by induction that $E/O_{p'}(E)$ is p-hypercyclically embedded in $G/O_{p'}(E)$ and thus E is p-hypercyclically embedded in G. Suppose that $O_{p'}(E) = 1$. By Lemma 2.4, $O_p(E) \leq Z_{\mathcal{U}}(G)$. As $O_{p'}(E) = 1$, $F_p(E) = O_p(E)$. It follows that E is p-hypercyclically embedded in G by Lemma 2.7. □

With the aid of the preceding results, we can now prove Theorem A.

Proof of Theorem A. Suppose that Theorem A is not true and let (G, E) be a counterexample such that $|G|+|E|$ is minimal. Then the minimal choice of (G, E) implies that $O_{p'}(E) = 1$. If $|E_p| = p$, then E_p itself is c-normal in G and by Lemma 1.2, E_p is also c-normal in E. By Lemma 2.5, we know that E is p-solvable and consequently E is p-hypercyclically embedded in G since $|E_p| = p$. Therefore, we may assume that $|E_p| > p$ and $1 < p^d < |E_p|$. By Proposition 2.11, we may further assume that $p^d > p$. By Lemma 2.6, E is p-solvable. We derive a contradiction through the following steps.

(1) If N is a minimal G-invariant subgroup of E, then $|N| > p$.

Suppose that $|N| = p$, then $p^d > |N|$ by the assumption that $p^d > p$. Hence $(G/N, E/N)$ also satisfies the hypotheses of this Theorem and E/N is p-hypercyclically embedded in G/N by the choice of (G, E). Since $|N| = p$, E is p-hypercyclically embedded in G, a contradiction. Hence $|N| > p$.

(2) If N is a minimal G-invariant subgroup of E, then $p^d > |N|$.

By Lemma 2.1 we have $p^d \geq |N|$. Suppose that $p^d = |N|$. Since $p^d < |E_p|$ by our assumption, E_p has a subgroup H such that N is a maximal subgroup of H. By (1), N is not cyclic and so is H. Hence we can choose a maximal subgroup K of H other than N. Obviously we have $H = NK$. If $N \cap K = 1$, then $|N| = |H|/|K| = p$, a contradiction. Thus $N \cap K \neq 1$ and $|K : K \cap N| = |KN : N| = |H : N| = p$. Since $K_G \cap N \leq K \cap N < N$, we have $K_G \cap N = 1$. Assume that $K_G > 1$.

Then $|K_G| = |K_G N/N| = |H/N| = p$ and this contradicts (1). Therefore we have $K_G = 1$. Since $|K| = |N| = p^d$, K is c-normal in G by the hypotheses of this theorem. So there exists a proper normal subgroup L of G such that $G = KL$ and $K \cap L \leq K_G = 1$. Since $K \cap N \neq 1$ and $K \cap L = 1$, we have $N \neq L$ and thus $N \cap L = 1$. Consequently, $|NL| = |N||L| = |K||L| = |KL| = |G|$ and thus $G = NL$. Let M be a maximal subgroup of G containing L, then $|G : M| = p$ since G/L is a p-group. Obviously $G = NM$ and $N \cap M = 1$. But then $|N| = |G : M| = p$, a contradiction.

(3) $\Phi(E) = 1$ and E contains a unique minimal G-invariant subgroup, say N.

Let L be any minimal G-invariant subgroup of E. Since $p^d > |L|$ by (2), it is easy to verify that $(G/L, E/L)$ satisfies the hypotheses of this theorem. Thus the minimal choice of (G, E) implies that E/L is p-hypercyclically embedded in G/L. By Lemma 2.9, we have $\Phi(E) = 1$. By Lemma 2.10, we have that E contains a unique minimal G-invariant subgroup, say N.

(4) Final contradiction.

Since $\Phi(E) = 1$, E is split over N (see [5, Chap. A, 9.10]). Hence $E = [N]Y$ for some subgroup Y of E, and so $E_p = [N]Y_p$ for some $Y_p \in Syl_p(Y)$. Let U be a maximal subgroup of N such that U is E_p-invariant. Since $d_p < |E_p|$, we may take a subgroup $D = UV$ such that $V \leq Y_p$ and $|D| = p^d$. Assume that $D_G > 1$. Then $D \geq D_G \geq N$ because N is the unique minimal G-invariant subgroup of E, a contradiction. Hence $D_G = 1$. Now, the hypotheses imply that $G = D[L]$ for some G-invariant subgroup L. Note that $E \cap L = 1$ contrary to $|D| = |E|$. Thus $E \cap L$ is a nontrivial normal subgroup of G, so $U < N \leq E \cap L \leq L$, but then $1 < U \leq D \cap L = 1$, a contradiction. □

Remark The conclusion of Theorem A does not hold if we replace "c-normal" with "c-supplemented" in the hypothesis. One can take A_5 for example. Obviously, every subgroup of A_5 with order 5 is c-supplemented in A_5, but A_5 is not 5-hypercyclically embedded in itself.

Corollary 2.12 *Let p be a fixed prime and G_p be a Sylow p-subgroup of a finite group G. Suppose that p^d is a prime power such that $1 < p^d \leq \max(|G_p|/p, p)$. If all the subgroups of G_p with order p^d and $2p^d$ (if a quaternion group is involved in G_p) are c-normal in G, then G is p-supersolvable.*

Corollary 2.13 *Let p be a fixed prime and E be a normal subgroup of a finite group G. Suppose that E_p is a Sylow p-subgroup of E and p^d is a prime power such that $1 < p^d \leq \max(|E_p|/p, p)$. If G/E is p-supersolvable and all the subgroups of E_p with order p^d and $2p^d$ (if a quaternion group is involved in E_p) are c-normal in G, then G is p-supersolvable.*

Corollary 2.14 *Let p be a fixed prime and E be a normal subgroup of a finite group G. Suppose that E_p is a Sylow p-subgroup of E and p^d is a prime power such that $1 < d \leq \max(|E_p|/p, p)$. Suppose that $N_G(E_p)$ is p-nilpotent. If either E_p is abelian or every subgroup of E_p with order p^d and $2p^d$ (if a quaternion group is involved in E_p) is c-normal in E, then G is p-nilpotent.*

Proof If E_p is abelian, then E is p-nilpotent by Burnside's theorem. If E_p is not abelian, then E is p-supersolvable by Theorem A. In both cases, we have that $E_p \mathrm{O}_{p'}(E)$ is a normal subgroup of G. By Frattini argument, $G = N_G(E_p)\mathrm{O}_{p'}(E)$. Note that $N_G(E_p)$ is p-nilpotent by hypotheses, and we have that G is p-nilpotent, as wanted. □

Corollary 2.15 *Let p be a fixed prime and G_p be a Sylow p-subgroup of a finite group G. Suppose that p^d is a prime power such that $1 < p^d \leq \max(|G_p|/p, p)$. Suppose that $N_G(G_p)$ is p-nilpotent. If all the subgroups of G_p with order p^d and $2p^d$ (if a quaternion group is involved in G_p) are c-normal in G, then G is p-nilpotent.*

3 Some Applications

In this section, we give some applications to show that we can apply our results to get some known results.

Corollary 3.1 ([2, Theorem 3.4]) *Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$. If all minimal subgroups and all cyclic subgroups with order 4 of $G^{\mathcal{F}}$ are c-normal in G, then $G \in \mathcal{F}$.*

Proof From Theorem A, we know that $G^{\mathcal{F}}$ is p-hypercentrally embedded in G for all $p \in \pi(G^{\mathcal{F}})$ and thus $G^{\mathcal{F}} \leq Z_{\mathcal{U}}(G)$. Since $\mathcal{F}$ is a saturated formation containing $\mathcal{U}$, we have that $Z_{\mathcal{U}}(G) \leq Z_{\mathcal{F}}(G)$. Consequently, $G \in \mathcal{F}$ because $G/G^{\mathcal{F}} \in \mathcal{F}$ and $G^{\mathcal{F}} \leq Z_{\mathcal{U}}(G) \leq Z_{\mathcal{F}}(G)$. □

The following lemma is evident.

Lemma 3.2 *Let G be a group and p be a prime such that $(p-1, |G|) = 1$. Then G is p-nilpotent if and only if G is p-supersolvable.*

Corollary 3.3 ([12, Theorem 0.1]) *Let E be a normal subgroup of a group G of odd order such that G/E is supersolvable. Suppose that every non-cyclic Sylow subgroup P of E has a subgroup D such that $1 < |D| < |P|$ and all subgroups H of P with order $|H| = |D|$ are c-normal in G. Then G is supersolvable.*

Proof Let p be the minimal prime divisor of $|E|$. If E_p is cyclic, then E is p-nilpotent by [14, Lemma 2.8]. If E_p is not cyclic, then by Theorem A, E is p-supersolvable and thus p-nilpotent by Lemma 3.2. By repeating this argument, we know that E has a Sylow-tower and therefore E is solvable. Let p be any prime divisor of $|E|$. If E_p is cyclic, then E is p-hypercentrally embedded in G since now E is p-solvable. If E_p is not cyclic, then E is also p-hypercentrally embedded in G by Theorem A. Therefore we have $E \leq Z_{\mathcal{U}}(G)$. It follows that G is supersolvable since G/E is supersolvable and $E \leq Z_{\mathcal{U}}(G)$. □

Corollary 3.4 ([9, Theorem 3.1]) *Let p be an odd prime dividing the order of a group G and P be a Sylow p-subgroup of G. If $N_G(P)$ is p-nilpotent and every maximal subgroup of P is c-normal in G, then G is p-nilpotent.*

Corollary 3.5 ([9, Theorem 3.4]) *Let p be the smallest prime dividing the order of a group G and P be a Sylow p-subgroup of G. If every maximal subgroup of P is c-normal in G, then G is p-nilpotent.*

Proof If $|P| = p$, then G is p-nilpotent by [14, Lemma 2.8]. If $|P| > p$, then by Corollary 2.12 G is p-supersolvable. Hence G is p-nilpotent by Lemma 3.2. □

Corollary 3.6 ([9, Theorem 3.6]) *Let p be the smallest prime dividing the order of group G and P be a Sylow p-subgroup of G. If every minimal subgroup of $P \cap G'$ is c-normal in G and when $p = 2$, either every cyclic subgroup of $P \cap G'$ with order 4 is also c-normal in or P is quaternion free, then G is p-nilpotent.*

Proof By Theorem A, G' is p-hypercyclically embedded in G. Since G/G' is abelian, G is p-supersolvable. It then follows from Lemma 3.2 that G is p-nilpotent. □

Corollary 3.7 ([9, Corollary 3.9]) *Let p be an odd prime dividing the order of a group G and P be a Sylow p-subgroup of G. If every minimal subgroup of $P \cap G'$ is c-normal in G, then G is p-supersolvable.*

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