CrossMark

A Matrix-Theoretic Perspective on Some Identities Involving Well-Known Sequences

A. R. Moghaddamfar[1] · **S. Navid Salehy[2]** ·
S. Nima Salehy[2]

Abstract We first consider the companion matrix associated with the characteristic polynomial of a linear recurrence relation, and we investigate its powers. Next, we introduce a new matrix associated with a given linear recurrence sequence, and we get a factorization of this matrix. Finally, we give several applications of our results. Actually, we obtain some identities concerning Fibonacci, Lucas, Pell, and Jacobsthal numbers using matrix theory.

Keywords Recurrence relation · Matrix factorization · Fibonacci sequence · Lucas sequence · Pell sequence · Jacobsthal sequence

---

Communicated by Emrah Kilic.

In memory of Professor Michael Neumann.

✉ A. R. Moghaddamfar
moghadam@ipm.ir; moghadam@kntu.ac.ir

S. Navid Salehy
navidsalehy@math.fsu.edu

S. Nima Salehy
nimasalehy@math.fsu.edu

[1] Department of Mathematics, K. N. Toosi University of Technology,
P. O. Box 16315-1618, Tehran, Iran

[2] Department of Mathematics, Florida State University, Tallahassee, FL 32306, USA

🖉 Springer

1 Introduction and Notation

Recurrence relations occur in a variety of mathematical contexts, and they play an important role in the field of complexity. They are actually one of the basic mathematical tools of computation. In particular, they can be used to represent mathematical sequences that cannot be easily represented non-recursively. Usually, a sequence generated by a recurrence relation is called a *recurrence sequence*. Some well-known recurrence sequences are the Fibonacci sequence, Lucas sequence, Pell sequence and Jacobsthal sequence.

In general, a sequence $\lambda = (\lambda_0, \lambda_1, \ldots)$ is said to satisfy a *linear recurrence relation* of order $k \geqslant 1$ with initial conditions $\lambda_0, \lambda_1, \ldots, \lambda_{k-1}$, if there exist constants $c_0, c_1, \ldots, c_{k-1}$, with $c_0 \neq 0$, such that

$$\lambda_i = c_0\lambda_{i-k} + c_1\lambda_{i-k+1} + \cdots + c_{k-1}\lambda_{i-1}, \quad (i \geqslant k). \tag{1}$$

A *linear recurrence sequence* is a sequence satisfying some linear recurrence as above. Occasionally, the indexing of the terms will start with something other than 0. In particular, we note that the sequence λ can be easily extended to negative indices using the rearranged recurrence relation

$$\lambda_{i-k} = c_0^{-1}(\lambda_i - c_1\lambda_{i-k+1} - \cdots - c_{k-1}\lambda_{i-1}), \quad (i < k). \tag{2}$$

This lets one define $\lambda_{-1}, \lambda_{-2}, \ldots$, to obtain a doubly infinite sequence

$$\ldots, \lambda_{-2}, \lambda_{-1}, \lambda_0, \lambda_1, \lambda_2, \ldots$$

that now satisfies the same linear recurrence for all integer indices i, positive or negative.

Some well-known linear recurrence sequences are special cases of (1), for instance, we have

k	λ_0	λ_1	c_0	c_1	λ	Name of sequence	OEIS
2	0	1	1	1	F	Fibonacci sequence	$A000045$ in [7]
2	2	1	1	1	L	Lucas sequence	$A000032$ in [7]
2	0	1	1	2	P	Pell sequence	$A000129$ in [7]
2	0	1	2	1	J	Jacobsthal sequence	$A001045$ in [7]

The *characteristic polynomial* of the linear recurrence relation (1) is defined to be the polynomial

$$p_\lambda(x) = x^k - c_{k-1}x^{k-1} - \cdots - c_2x^2 - c_1x - c_0. \tag{3}$$

It is well-known by Cayley-Hamilton theorem that the square matrix

$$A = \begin{bmatrix} 0 & 0 & 0 & 0 & \ldots & c_0 \\ 1 & 0 & 0 & 0 & \ldots & c_1 \\ 0 & 1 & 0 & 0 & \ldots & c_2 \\ \vdots & \vdots & \ddots & \vdots & \ldots & \vdots \\ 0 & 0 & 0 & 1 & \ldots & c_{k-2} \\ 0 & 0 & 0 & \ldots & 1 & c_{k-1} \end{bmatrix}$$

satisfies the equation $p_\lambda(A) = 0$; that is

$$A^k - c_{k-1}A^{k-1} - \cdots - c_2A^2 - c_1A - c_0I = 0,$$

where I is the $k \times k$ identity matrix (see [4]). The matrix A is called the *companion matrix* of the monic polynomial $p_\lambda(x)$, which will remain fixed throughout the paper.

In what follows, a new approach will be presented to find again this *matrix* and *its powers*. To illustrate this, we introduce k new linear recurrence sequences of order k as follows: for any integer $s \in \{0, 1, 2, \ldots, k-1\}$, we define the linear recurrence sequence

$$\omega^{(s)} = \left(\omega_0^{(s)}, \omega_1^{(s)}, \omega_2^{(s)}, \ldots\right)$$

of order k, by

$$\omega_i^{(s)} = \begin{cases} \delta_{i,s} & i < k, \\ \displaystyle\sum_{l=0}^{k-1} c_l \omega_{i-k+l}^{(s)} & i \geqslant k, \end{cases} \tag{4}$$

where $\delta_{i,s}$ is the *Kronecker delta*.

These sequences and a few first terms of them are shown below:

$s \backslash i$	$\omega_0^{(s)}$	$\omega_1^{(s)}$	$\omega_2^{(s)}$	$\ldots$	$\omega_{k-1}^{(s)}$	$\omega_k^{(s)}$	$\omega_{k+1}^{(s)}$	$\omega_{k+2}^{(s)}$	$\ldots$
$\omega^{(0)}$	1,	0,	0,	$\ldots,$ 0,		$c_0,$	$c_0 c_{k-1},$	$c_0 c_{k-2} + c_0 c_{k-1}^2,$	$\ldots$
$\omega^{(1)}$	0,	1,	0,	$\ldots,$ 0,		$c_1,$	$c_0 + c_1 c_{k-1},$	$c_1 c_{k-2} + c_0 c_{k-1} + c_1 c_{k-1}^2,$	$\ldots$
$\omega^{(2)}$	0,	0,	1,	$\ldots,$ 0,		$c_2,$	$c_1 + c_2 c_{k-1},$	$c_0 + c_2 c_{k-2} + c_1 c_{k-1} + c_2 c_{k-1}^2,$	$\ldots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ldots$
$\omega^{(k-1)}$	0,	0,	0,	$\ldots,$ 1,		$c_{k-1},$	$c_{k-2} + c_{k-1}^2,$	$c_{k-3} + 2c_{k-1}c_{k-2} + c_{k-1}^3,$	$\ldots$

Again, the sequences $\omega^{(s)}$ can be extended to negative indices using the rearranged recurrence relation

$$\omega_{i-k}^{(s)} = c_0^{-1}\left(\omega_i^{(s)} - \sum_{l=1}^{k-1} c_l \omega_{i-k+l}^{(s)}\right), \quad (i < k). \tag{5}$$

Note that each sequence $\omega^{(s)}$, $0 \leqslant s \leqslant k - 1$, is a special case of the sequence λ for which the initial conditions $(\lambda_0, \lambda_1, \ldots, \lambda_{k-1})$ are the row vector with all 0s except for a 1 in the sth position.

An Example. Suppose that $k = 2$, $\lambda_0 = 2$ and $\lambda_1 = c_0 = c_1 = 1$. Then, we have

Sequence\n	0	1	2	3	4	5	6	7	$\ldots$	i	$\ldots$
λ	2	1	3	4	7	11	18	29	$\ldots$	L_i	$\ldots$
$\omega^{(0)}$	1	**0**	**1**	1	2	3	5	8	$\ldots$	F_{i-1}	$\ldots$
$\omega^{(1)}$	0	**1**	**1**	2	3	5	8	13	$\ldots$	F_i	$\ldots$

In general, for any integer m, we define

$$A_m = \left[\omega^{(i)}_{m+j} \right]_{0 \leqslant i,\, j \leqslant k-1} = \begin{bmatrix} \omega^{(0)}_m & \omega^{(0)}_{m+1} & \omega^{(0)}_{m+2} & \cdots & \omega^{(0)}_{m+k-1} \\ \omega^{(1)}_m & \omega^{(1)}_{m+1} & \omega^{(1)}_{m+2} & \cdots & \omega^{(1)}_{m+k-1} \\ \omega^{(2)}_m & \omega^{(2)}_{m+1} & \omega^{(2)}_{m+2} & \cdots & \omega^{(2)}_{m+k-1} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ \omega^{(k-1)}_m & \omega^{(k-1)}_{m+1} & \omega^{(k-1)}_{m+2} & \cdots & \omega^{(k-1)}_{m+k-1} \end{bmatrix}.$$

We notice that the companion matrix A can be seen here again (shown in bold type above). Actually, if we take $m = 1$, then we have

$$A = A_1 = \begin{bmatrix} \omega^{(0)}_1 & \omega^{(0)}_2 & \omega^{(0)}_3 & \cdots & \omega^{(0)}_k \\ \omega^{(1)}_1 & \omega^{(1)}_2 & \omega^{(1)}_3 & \cdots & \omega^{(1)}_k \\ \omega^{(2)}_1 & \omega^{(2)}_2 & \omega^{(2)}_3 & \cdots & \omega^{(2)}_k \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \omega^{(k-2)}_1 & \omega^{(k-2)}_2 & \omega^{(k-2)}_3 & \cdots & \omega^{(k-2)}_k \\ \omega^{(k-1)}_1 & \omega^{(k-1)}_2 & \omega^{(k-1)}_3 & \cdots & \omega^{(k-1)}_k \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 & 0 & \cdots & c_0 \\ 1 & 0 & 0 & 0 & \cdots & c_1 \\ 0 & 1 & 0 & 0 & \cdots & c_2 \\ \vdots & \vdots & \ddots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & 1 & \cdots & c_{k-2} \\ 0 & 0 & 0 & \cdots & 1 & c_{k-1} \end{bmatrix}.$$

For any integer m, it is an easy observation that $A^m = A_m$ (see Theorem 1). Now, using the fact that $\det A^m = (\det A)^m$, we immediately get

$$\det A_m = \det(A^m) = (\det A)^m = (-1)^{m(k-1)} c_0^m,$$

(see also [5, Corollary 2.1]).

In the sequel, we shall introduce some more definitions and notation. Given integers x, y_i, and z_i, $1 \leqslant i \leqslant k-1$, we define the $k \times k$ matrices $R_k(x, y_1, \ldots, y_{k-1}, z_1, \ldots, z_{k-1})$ and $S_k(y_1, \ldots, y_{k-1})$ as follows:

$$R_k(x, y_1, \ldots, y_{k-1}, z_1, \ldots, z_{k-1}) := [\lambda_{x+iy_i+jz_j}]_{0 \leqslant i, j \leqslant k-1},$$

$$S_k(y_1, \ldots, y_{k-1}) := \left[\omega_{jy_j}^{(i)} \right]_{0 \leqslant i, j \leqslant k-1}.$$

The associated matrices $R_4(x, y_1, y_2, y_3, z_1, z_2, z_3)$ and $S_4(y_1, y_2, y_3)$, for example, are given by

$$R_4(x, y_1, y_2, y_3, z_1, z_2, z_3) = \begin{bmatrix} \lambda_x & \lambda_{x+z_1} & \lambda_{x+2z_2} & \lambda_{x+3z_3} \\ \lambda_{x+y_1} & \lambda_{x+y_1+z_1} & \lambda_{x+y_1+2z_2} & \lambda_{x+y_1+3z_3} \\ \lambda_{x+2y_2} & \lambda_{x+2y_2+z_1} & \lambda_{x+2y_2+2z_2} & \lambda_{x+2y_2+3z_3} \\ \lambda_{x+3y_3} & \lambda_{x+3y_3+z_1} & \lambda_{x+3y_3+2z_2} & \lambda_{x+3y_3+3z_3} \end{bmatrix},$$

and

$$S_4(y_1, y_2, y_3) = \begin{bmatrix} \omega_0^{(0)} & \omega_{y_1}^{(0)} & \omega_{2y_2}^{(0)} & \omega_{3y_3}^{(0)} \\ \omega_0^{(1)} & \omega_{y_1}^{(1)} & \omega_{2y_2}^{(1)} & \omega_{3y_3}^{(1)} \\ \omega_0^{(2)} & \omega_{y_1}^{(2)} & \omega_{2y_2}^{(2)} & \omega_{3y_3}^{(2)} \\ \omega_0^{(3)} & \omega_{y_1}^{(3)} & \omega_{2y_2}^{(3)} & \omega_{3y_3}^{(3)} \end{bmatrix}.$$

In the case when $x = 0$ and $y_i = z_i = 1$, $1 \leqslant i \leqslant k-1$, we denote the matrix $R_k(0, 1, \ldots, 1, 1, \ldots, 1)$ by R_k. In this paper, we will present a matrix factorization for $R_k(x, y_1, \ldots, y_{k-1}, z_1, \ldots, z_{k-1})$ (see Theorem 2). Finally, some applications of the main results are given.

The outline of the paper is as follows. Basic definitions and notation are summarized in Sect. 1. In Sect. 2, we derive some preparatory results. Section 3 contains three theorems and some applications of our findings to obtain many identities involving Fibonacci, Lucas, Pell, and Jacobsthal numbers.

2 Preliminaries

In the following lemma we give a relationship between the terms of sequences λ and $\omega^{(s)}$.

Lemma 1 *For every integer m and n, we have*

$$\sum_{l=0}^{k-1} \lambda_{m+l} \omega_n^{(l)} = \lambda_{m+n}.$$

In particular, if in addition $0 \leqslant s \leqslant k - 1$ is an integer, we get

$$\sum_{l=0}^{k-1} \omega_{m+l}^{(s)} \omega_n^{(l)} = \omega_{m+n}^{(s)}.$$

Proof First, we assume that $m + n \geqslant 0$. We will proceed by induction with respect to $m + n$. The case when $0 \leqslant m + n \leqslant k - 1$ is trivial, so we may assume that $m + n \geqslant k$. Therefore, we have

$$\lambda_{m+n} = \sum_{l=0}^{k-1} c_l \lambda_{m+n-k+l} \quad \text{(by Eq. (1))}$$

$$= \sum_{l=0}^{k-1} c_l \sum_{s=0}^{k-1} \lambda_{m+s} \omega_{n-k+l}^{(s)} \quad \text{(using the induction hypothesis)}$$

$$= \sum_{s=0}^{k-1} \lambda_{m+s} \sum_{l=0}^{k-1} c_l \omega_{n-k+l}^{(s)}$$

$$= \sum_{s=0}^{k-1} \lambda_{m+s} \omega_n^{(s)} \quad \text{(by Eq. (4))}$$

as required. Now, let $m + n < 0$. Again, by induction on $m + n$, we obtain

$$\lambda_{m+n} = c_0^{-1} \left(\lambda_{m+n+k} - \sum_{l=1}^{k-1} c_l \lambda_{m+n+l} \right) \quad \text{(by Eq. (2))}$$

$$= c_0^{-1} \left(\sum_{s=0}^{k-1} \lambda_{m+s} \omega_{n+k}^{(s)} - \sum_{l=1}^{k-1} c_l \sum_{s=0}^{k-1} \lambda_{m+s} \omega_{n+l}^{(s)} \right) \quad \text{(using the induction hypothesis)}$$

$$= \sum_{s=0}^{k-1} \lambda_{m+s} c_0^{-1} \left(\omega_{n+k}^{(s)} - \sum_{l=1}^{k-1} c_l \omega_{n+l}^{(s)} \right)$$

$$= \sum_{s=0}^{k-1} \lambda_{m+s} \omega_n^{(s)} \quad \text{(by Eq. (5))}$$

which completes the proof of the first result. The second result is a special case of the first one.   □

The following lemma gives some relationships between sequences $\omega^{(s)}$, $0 \leqslant s \leqslant k - 1$.

Lemma 2 *The following statements hold for any integer i.*

(i) $\omega_{i+1}^{(0)} = c_0 \omega_i^{(k-1)}$.

(ii) $\omega_{i+1}^{(s+1)} - \omega_i^{(s)} = c_{s+1} \omega_i^{(k-1)} \qquad (0 \leqslant s \leqslant k - 2)$.

(iii) $c_0 \omega_{i-1}^{(k-1)} + c_1 \omega_i^{(k-1)} = \omega_{i+1}^{(1)}$.

Proof (i) We apply induction on i. For $0 \leqslant i \leqslant k - 1$, the proof is obvious, because, if $0 \leqslant i \leqslant k - 2$, $\omega_{i+1}^{(0)} = c_0 \omega_i^{(k-1)} = 0$, and if $i = k - 1$, we have

$$\omega_k^{(0)} = \sum_{l=0}^{k-1} c_l \omega_l^{(0)} = c_0 = c_0 \omega_{k-1}^{(k-1)}.$$

Let us for the moment assume that $i \geqslant k$. Using Eq. (4) and the induction hypothesis, we have

$$\omega_{i+1}^{(0)} = \sum_{l=0}^{k-1} c_l \omega_{i-k+l+1}^{(0)} = \sum_{l=0}^{k-1} c_l \left(c_0 \omega_{i-k+l}^{(k-1)} \right) = c_0 \sum_{l=0}^{k-1} c_l \omega_{i-k+l}^{(k-1)} = c_0 \omega_i^{(k-1)}.$$

Now, let $i < 0$. We obtain

$$\omega_{i+1}^{(0)} = c_0^{-1} \left(\omega_{i+k+1}^{(0)} - \sum_{l=1}^{k-1} c_l \omega_{i+l+1}^{(0)} \right) \quad \text{(by Eq. (5))}$$

$$= c_0^{-1} \left(c_0 \omega_{i+k}^{(k-1)} - \sum_{l=1}^{k-1} c_l c_0 \omega_{i+l}^{(k-1)} \right) \quad \text{(using the induction hypothesis)}$$

$$= \omega_{i+k}^{(k-1)} - \sum_{l=1}^{k-1} c_l \omega_{i+l}^{(k-1)}$$

$$= c_0 \omega_i^{(k-1)} \quad \text{(by Eq. (5))}$$

which completes the proof of this part.

(ii) Suppose that $0 \leqslant s \leqslant k - 2$. Once again, we prove the result by induction on i. Assume first that $0 \leqslant i \leqslant k - 1$. In this case, we get

$$\omega_i^{(s)} + c_{s+1} \omega_i^{(k-1)} = \begin{cases} 1 & i = s \leqslant k - 2, \\ 0 & i \neq s \leqslant k - 2, \\ c_{s+1} & i = k - 1, \end{cases}$$
$$= \omega_{i+1}^{(s+1)}.$$

Assume now that $i \geqslant k$. By a routine calculation, we obtain

$$\omega_i^{(s)} + c_{s+1} \omega_i^{(k-1)} = \sum_{l=0}^{k-1} c_l \omega_{i-k+l}^{(s)} + c_{s+1} \sum_{l=0}^{k-1} c_l \omega_{i-k+l}^{(k-1)} \quad \text{(by Eq. (4))}$$

$$= \sum_{l=0}^{k-1} c_l \left[\omega_{i-k+l}^{(s)} + c_{s+1} \omega_{i-k+l}^{(k-1)} \right]$$

$$= \sum_{l=0}^{k-1} c_l \omega_{i-k+l+1}^{(s+1)} \quad \text{(using the induction hypothesis)}$$

$$= \omega_{i+1}^{(s+1)} \quad \text{(by Eq. (4))}$$

as desired. For the case when $i < 0$, we verify the result as follows

$$\omega_i^{(s)} + c_{s+1}\omega_i^{(k-1)} = c_0^{-1}\left(\omega_{i+k}^{(s)} - \sum_{l=1}^{k-1} c_l \omega_{i+l}^{(s)}\right)$$

$$+ c_{s+1}\left\{c_0^{-1}\left(\omega_{i+k}^{(k-1)} - \sum_{l=1}^{k-1} c_l \omega_{i+l}^{(k-1)}\right)\right\} \quad \text{(by Eq. (5))}$$

$$= c_0^{-1}\left\{\left(\omega_{i+k}^{(s)} + c_{s+1}\omega_{i+k}^{(k-1)}\right) - \sum_{l=1}^{k-1} c_l \left(\omega_{i+l}^{(s)} + c_{s+1}\omega_{i+l}^{(k-1)}\right)\right\}$$

$$= c_0^{-1}\left\{\omega_{i+k+1}^{(s+1)} - \sum_{l=1}^{k-1} c_l \omega_{i+l+1}^{(s+1)}\right\} \quad \text{(using the induction hypothesis)}$$

$$= \omega_{i+1}^{(s+1)} \quad \text{(by Eq. (5))}.$$

(iii) We obtain

$$c_0 \omega_{i-1}^{(k-1)} + c_1 \omega_i^{(k-1)} = \omega_i^{(0)} + c_1 \omega_i^{(k-1)} \quad \text{(by part (}i\text{))}$$

$$= \omega_{i+1}^{(1)} \quad \text{(by part (}ii\text{) for } s = 0)$$

as desired. This completes the proof of the lemma.   □

3 Main Results and Some Applications

Before stating our main results we introduce some notation. Let $\lambda = (\lambda_i)_{i\in\mathbb{Z}}$ be a doubly infinite sequence satisfying the linear recurrence relation (1). Given an integer m, we will consider the row vector

$$\mathbf{u}_m := (\lambda_m, \lambda_{m+1}, \ldots, \lambda_{m+k-1}),$$

which is called the *mth state vector* of λ. The state vector $\mathbf{u}_0 = (\lambda_0, \lambda_1, \ldots, \lambda_{k-1})$ is also referred to as the *initial state vector*.

We start with the following known result of Er [1], which is obtained by using the approach of Kalman [4]. Here, we provide a proof for the sake of completeness.

Theorem 1 ([1]) *For any integer m, we have $A^m = A_m$. Furthermore, we have $\mathbf{u}_m = \mathbf{u}_0 A^m$ and*

$$A^m [c_0 \; c_1 \; c_2 \; \ldots \; c_{k-1}]^T = \left[\omega_{k+m}^{(0)} \; \omega_{k+m}^{(1)} \; \omega_{k+m}^{(2)} \; \ldots \; \omega_{k+m}^{(k-1)}\right]^T,$$

where T denotes transpose.

Proof First we assume that $m \geqslant 0$. In this case, we conclude the result by induction on m. For $m = 0$ and $m = 1$, the result is straightforward. Assume that $m \geqslant 2$. Using the definition of A and induction hypothesis, we have

$$A^m = A^{m-1} \cdot A = \left[\omega_{j+m-1}^{(i)}\right] \cdot \left[\omega_{j+1}^{(i)}\right].$$

Therefore, for $0 \leqslant i, j \leqslant k - 1$, we have

$$(A^m)_{i,j} = \sum_{l=0}^{k-1} \omega_{l+m-1}^{(i)} \omega_{j+1}^{(l)}$$

$$= \omega_{j+m}^{(i)} \quad \text{(by Lemma 1)}$$

as desired. Now, suppose that $m < 0$. Since $-m > 0$, from the previous part we have

$$A^{-m} = \left[\omega_{j-m}^{(i)}\right]_{0 \leqslant i, j \leqslant k-1}.$$

Thus, using Lemma 1, for $0 \leqslant i, j \leqslant k - 1$, we get

$$\left(\left[\omega_{j+m}^{(i)}\right] \cdot A^{-m}\right)_{i,j} = \sum_{l=0}^{k-1} \omega_{l+m}^{(i)} \omega_{j-m}^{(l)} = \omega_j^{(i)} = \begin{cases} 1 & i = j, \\ 0 & i \neq j. \end{cases}$$

Therefore, we deduce

$$A^m = (A^{-m})^{-1} = \left[\omega_{j+m}^{(i)}\right]_{0 \leqslant i, j \leqslant k-1}$$

which completes the proof of the first part. The rest of conclusion follows immediately from previous part and Lemma 1. □

Theorem 2 *For integers x, y_i, and z_i, $1 \leqslant i \leqslant k - 1$, we have the following matrix decomposition:*

$$R_k(x, y_1, \ldots, y_{k-1}, z_1, \ldots, z_{k-1}) = S_k(y_1, \ldots, y_{k-1})^T R_k A^x S_k(z_1, \ldots, z_{k-1}),$$

where X^T is the transpose of X.

Proof Set $D = S_k(y_1, \ldots, y_{k-1})^T R_k A^x S_k(z_1, \ldots, z_{k-1})$. For the proof of the claim we compute the (i, j)-entry of D. Let $0 \leqslant i, j \leqslant k - 1$. Then, by direct computation, we obtain

$$D_{i,j} = \sum_{r=0}^{k-1} \left(S_k(y_1, \ldots, y_{k-1})^T\right)_{i,r} (R_k A^x S_k(z_1, \ldots, z_{k-1}))_{r,j}$$

$$= \sum_{r=0}^{k-1} \omega_{i y_i}^{(r)} \sum_{s=0}^{k-1} (R_k)_{r,s} (A^x S_k(z_1, \ldots, z_{k-1}))_{s,j}$$

$$= \sum_{r=0}^{k-1} \omega_{i\,y_i}^{(r)} \sum_{s=0}^{k-1} \lambda_{r+s} \sum_{t=0}^{k-1} (A^x)_{s,t} (S_k(z_1, \ldots, z_{k-1}))_{t,j}$$

$$= \sum_{r=0}^{k-1} \omega_{i\,y_i}^{(r)} \sum_{s=0}^{k-1} \lambda_{r+s} \sum_{t=0}^{k-1} \omega_{t+x}^{(s)} \omega_{j\,z_j}^{(t)} \quad \text{(by Theorem 1)}$$

$$= \sum_{r=0}^{k-1} \omega_{i\,y_i}^{(r)} \sum_{t=0}^{k-1} \left(\sum_{s=0}^{k-1} \lambda_{r+s} \omega_{t+x}^{(s)} \right) \omega_{j\,z_j}^{(t)}$$

$$= \sum_{r=0}^{k-1} \omega_{i\,y_i}^{(r)} \sum_{t=0}^{k-1} \lambda_{r+t+x} \omega_{j\,z_j}^{(t)} \quad \text{(by Lemma 1)}$$

$$= \sum_{r=0}^{k-1} \omega_{i\,y_i}^{(r)} \lambda_{r+x+jz_j} \quad \text{(by Lemma 1)}$$

$$= \lambda_{x+iy_i+jz_j} \quad \text{(by Lemma 1)}$$

$$= (R_k(x, y_1, \ldots, y_{k-1}, z_1, \ldots, z_{k-1}))_{i,j}.$$

This completes the proof of the theorem.   □

It is worthwhile to point out that Theorem 2 enables us to find many identities between the terms of a linear recurrence sequence of any order. In the next result, we will illustrate this situation for a linear recurrence sequence of order 2. A similar situation occurs with higher order linear recurrence sequences.

Theorem 3 *Let x, y, z, and t be arbitrary integers. Then the following identities hold between the terms of linear recurrence sequences* λ, $\omega^{(0)}$ *and* $\omega^{(1)}$ *of order* 2:

(i) $\lambda_x \lambda_{x+y+z} - \lambda_{x+y} \lambda_{x+z} = (-c_0)^x (\lambda_0 \lambda_2 - \lambda_1^2) \omega_y^{(1)} \omega_z^{(1)}$.

(ii) $\lambda_{x+t} \lambda_{x+y+z+t} - \lambda_{x+y+t} \lambda_{x+z+t} = (-c_0)^t (\lambda_x \lambda_{x+y+z} - \lambda_{x+y} \lambda_{x+z})$.

Proof (i) The identity is obtained by the following calculations:

$$\lambda_x \lambda_{x+y+z} - \lambda_{x+y} \lambda_{x+z}$$
$$= \det[R_2(x, y, z)]$$
$$= \det[S_2(y)^T R_2 A^x S_2(z)] \quad \text{(by Theorem 2)}$$
$$= \det[S_2(y)^T] \det[R_2] \det[A^x] \det[S_2(z)]$$
$$= \left(\omega_0^{(0)} \omega_y^{(1)} - \omega_0^{(1)} \omega_y^{(0)} \right) (\lambda_0 \lambda_2 - \lambda_1^2)(-c_0)^x \left(\omega_0^{(0)} \omega_z^{(1)} - \omega_0^{(1)} \omega_z^{(0)} \right)$$
$$= (-c_0)^x (\lambda_0 \lambda_2 - \lambda_1^2) \omega_y^{(1)} \omega_z^{(1)} \quad \text{(Since } \omega_0^{(0)} = 1 \text{ and } \omega_0^{(1)} = 0).$$

(ii) Again, by a routine calculation, we obtain

$$\lambda_{x+t} \lambda_{x+y+z+t} - \lambda_{x+y+t} \lambda_{x+z+t}$$
$$= (-c_0)^{x+t} (\lambda_0 \lambda_2 - \lambda_1^2) \omega_y^{(1)} \omega_z^{(1)} \quad \text{(by part } (i))$$

$$= (-c_0)^t \left\{ (-c_0)^x (\lambda_0 \lambda_2 - \lambda_1^2) \omega_y^{(1)} \omega_z^{(1)} \right\}$$

$$= (-c_0)^t (\lambda_x \lambda_{x+y+z} - \lambda_{x+y} \lambda_{x+z}) \quad \text{(by part } (i)).$$

This completes the proof of the theorem. □

In what follows, we have focused on some practical applications of the techniques provided in this paper. First of all, we investigate some relations between Fibonacci, Lucas, Pell, and Jacobsthal numbers.

Lemma 3 *Let m and n be two arbitrary integers. Then there hold the following identities:*

(i) $F_{m-1} F_{m+1} - F_m^2 = (-1)^m$.
(ii) $F_{m-1} F_n + F_m F_{m+1} = F_{m+n}$.
(iii) $F_{m-1} + F_{m+1} = F_{2m}/F_m$.

Proof (i) If $k = 2$ and $c_0 = c_1 = 1$, then we have

$$\omega_m^{(0)} = F_{m-1} \quad \text{and} \quad \omega_m^{(1)} = F_m.$$

On the other hand, it follows from Theorem 1 that

$$\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}^m = \begin{bmatrix} F_{m-1} & F_m \\ F_m & F_{m+1} \end{bmatrix}. \tag{6}$$

Taking the determinant of both sides yields

$$F_{m-1} F_{m+1} - F_m^2 = (-1)^m.$$

(ii) From the identity $A^{m+n} = A^m \cdot A^n$ and using Eq. (6), we get

$$\begin{bmatrix} F_{m+n-1} & F_{m+n} \\ F_{m+n} & F_{m+n+1} \end{bmatrix} = \begin{bmatrix} F_{m-1} & F_m \\ F_m & F_{m+1} \end{bmatrix} \begin{bmatrix} F_{n-1} & F_n \\ F_n & F_{n+1} \end{bmatrix}$$

$$= \begin{bmatrix} F_{m-1} F_{n-1} + F_m F_n & F_{m-1} F_n + F_m F_{n+1} \\ F_m F_{n-1} + F_{m+1} F_n & F_m F_n + F_{m+1} F_{n+1} \end{bmatrix}.$$

Comparing $(0, 1)$-entries on both sides of this equation, we obtain

$$F_{m+n} = F_{m-1} F_n + F_m F_{n+1}.$$

(iii) Setting $n = m$ in the previous part, we conclude that

$$F_{m-1} + F_{m+1} = F_{2m}/F_m,$$

which completes the proof. □

Lemma 4 *For any two integers m and n, there hold:*

(i) $P_{m-1}P_{m+1} - P_m^2 = (-1)^m$.
(ii) $P_{m-1}P_n + P_m P_{m+1} = P_{m+n}$.
(iii) $P_{m-1} + P_{m+1} = P_{2m}/P_m$.

Proof By taking $k = 2$, $c_0 = 1$ and $c_1 = 2$, we conclude that

$$\omega_m^{(0)} = P_{m-1} \quad \text{and} \quad \omega_m^{(1)} = P_m.$$

Moreover, by Theorem 1, we have

$$\begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix}^m = \begin{bmatrix} P_{m-1} & P_m \\ P_m & P_{m+1} \end{bmatrix}.$$

Now, the result follows immediately with arguments analogous to those of Lemma 3.
□

Lemma 5 *Let m and n be two arbitrary integers. Then the following identities hold true:*

(i) $J_m + J_{m+1} = 2^m$.
(ii) $[J_m + (-1)^m]J_n + J_m J_{n+1} = J_{m+n}$.
(iii) $2^n J_m + (-1)^m J_n = J_{m+n}$.
(iv) $J_{2m}/J_m = 2^m + (-1)^m$.
(v) $J_m = (2^m + (-1)^{m+1})/3$.

Proof (i) If $k = c_0 = 2$ and $c_1 = 1$, then we have

$$\omega_m^{(0)} = J_m + (-1)^m \quad \text{and} \quad \omega_m^{(1)} = J_m.$$

Again by Theorem 1, we obtain

$$\begin{bmatrix} 0 & 2 \\ 1 & 1 \end{bmatrix}^m = \begin{bmatrix} J_m + (-1)^m & J_{m+1} + (-1)^{m+1} \\ J_m & J_{m+1} \end{bmatrix}. \tag{7}$$

Taking the determinant of both sides yields

$$J_m + J_{m+1} = 2^m. \tag{8}$$

(ii) Comparing $(0, 1)$-entries on both sides of the identity $A^{m+n} = A^m \cdot A^n$, where

$$A = \begin{bmatrix} 0 & 2 \\ 1 & 1 \end{bmatrix},$$

and by Eq. (7), we obtain

$$[J_m + (-1)^m]J_n + J_m J_{n+1} = J_{m+n}. \tag{9}$$

(iii) Applying Eq. (8), we obtain

$$J_{m+n} = [J_m + (-1)^m]J_m J_{n+1}$$
$$= J_m[J_n + J_{n+1}] + (-1)^m J_n$$
$$= 2^n J_m + (-1)^m J_n.$$

(iv) Setting $n = m$ in part (iii), we get $J_{2m}/J_m = 2^m + (-1)^m$.

(v) We proceed as follows:

$$2^{m+n} = J_{m+n} + J_{m+n+1} \quad \text{(by Eq. (8))}$$
$$= [J_m + (-1)^m]J_n + J_m J_{n+1}$$
$$+ [J_m + (-1)^m]J_{n+1} + J_m J_{n+2} \quad \text{(by Eq. (9))}$$
$$= [J_m + (-1)^m][J_n + J_{n+1}] + J_m[J_{n+1} + J_{n+2}]$$
$$= [J_m + (-1)^m]2^n + J_m 2^{n+1} \quad \text{(by Eq. (8))},$$

and so $2^m = J_m + (-1)^m + 2J_m$, or equivalently

$$J_m = \left(2^m + (-1)^{m+1}\right)/3.$$

This completes the proof of lemma. □

A consequence of Theorem 3(i) is the following result stated again on Fibonacci, Lucas, Pell, and Jacobsthal numbers.

Corollary 1 *For any three integers x, y, and z, the following identities hold true:*

(i) $F_{x+y}F_{x+z} - F_x F_{x+y+z} = (-1)^x F_y F_z$ (See [2]).
(ii) $L_x L_{x+y+z} - L_{x+y}L_{x+z} = (-1)^x 5 F_y F_z$.
(iii) $P_{x+y}P_{x+z} - P_x P_{x+y+z} = (-1)^x P_y P_z$.
(iv) $J_{x+y}J_{x+z} - J_x J_{x+y+z} = (-2)^x J_y J_z$.
(v) $L_x L_{x+y+z} - L_{x+y}L_{x+z} = 5(F_{x+y}F_{x+z} - F_x F_{x+y+z})$.

Proof All parts $(i) - (iv)$ of this corollary can be derived from Theorem 3(i) by choosing suitable values for k, λ_0, λ_1, c_0, and c_1:

(i) $k = 2$, $\lambda_0 = 0$, $\lambda_1 = c_0 = c_1 = 1$: In this case, we have $\lambda_i = F_i$, $\omega_i^{(0)} = F_{i-1}$ and $\omega_i^{(1)} = F_i$.

(ii) $k = \lambda_0 = 2$, $\lambda_1 = c_0 = c_1 = 1$: In this case, we have $\lambda_i = L_i$, $\omega_i^{(0)} = F_{i-1}$ and $\omega_i^{(1)} = F_i$.

(iii) $k = c_1 = 2$, $\lambda_0 = 0$, $\lambda_1 = c_0 = 1$: In this case, we have $\lambda_i = P_i$, $\omega_i^{(0)} = P_{i-1}$ and $\omega_i^{(1)} = P_i$.

(iv) $k = c_0 = 2$, $\lambda_0 = 0$, $\lambda_1 = c_1 = 1$: In this case, we have $\lambda_i = J_i$, $\omega_i^{(0)} = J_i + (-1)^i$ and $\omega_i^{(1)} = J_i$.

Finally, the part (v) is obtained by using parts (i) and (ii). □

In the next corollary, we point out some well-known identities which are special cases of the previous results in Corollary 1, and consequently, Theorem 3(*i*).

Corollary 2 *For any integers m, n, and p, there hold*

(i) $F_{m+n} = F_{m+1}F_{n+1} - F_{m-1}F_{n-1}$ (See [6]).
(ii) $F_{2m+1}F_{2n+1} = F_{m+n+1}^2 + F_{m-n}^2$ (See [8,9]).
(iii) $F_{m+2n}^2 - F_m^2 = F_{2n}F_{2m+2n}$ (See [8]).
(iv) $F_{n-1}F_{n+1} - F_n^2 = (-1)^n$ (Cassini's identity).
(v) $F_{m+n}F_{m-n} - F_m^2 = (-1)^{m+n+1}F_n^2$ (Catalan's identity).
(vi) $F_mF_{n+p} - F_pF_{n+m} = (-1)^p F_{m-p}F_n$ (See [10]).
(vii) $2L_{m+n} = L_mL_n + 5F_mF_n$ (See [3]).
(viii) $L_{2m}L_{2n} = L_{m+n}^2 + 5F_{m-n}^2$.

In the following table we show how to obtain the identities in Corollaries 1 and 2 directly from Theorem 3(*i*).

λ_0	λ_1	c_0	c_1	x	y	z	Identity	Refs.
0	1	1	1	x	y	z	$F_{x+y}F_{x+z} - F_xF_{x+y+z} = (-1)^x F_yF_z$	[2]
0	1	1	1	2	$m-1$	$n-1$	$F_{m+n} = F_{m+1}F_{n+1} - F_{m-1}F_{n-1}$	[6]
0	1	1	1	$2n+1$	$m-n$	$m-n$	$F_{2m+1}F_{2n+1} = F_{m+n+1}^2 + F_{m-n}^2$	[8,9]
0	1	1	1	$2n$	m	m	$F_{m+2n}^2 - F_m^2 = F_{2n}F_{2m+2n}$	[8]
0	1	1	1	$n-1$	1	1	$F_{n-1}F_{n+1} - F_n^2 = (-1)^n$	Cassini's identity
0	1	1	1	$m-n$	n	n	$F_{m+n}F_{m-n} - F_m^2 = (-1)^{m+n+1}F_n^2$	Catalan's identity
0	1	1	1	p	n	$m-p$	$F_mF_{n+p} - F_pF_{n+m} = (-1)^p F_{m-p}F_n$	[10]
2	1	1	1	x	y	z	$L_xL_{x+y+z} - L_{x+y}L_{x+z} = (-1)^x 5F_yF_z$	
2	1	1	1	0	m	n	$2L_{m+n} = L_mL_n + 5F_mF_n$	[3]
2	1	1	1	$2n$	$m-n$	$m-n$	$L_{2m}L_{2n} = L_{m+n}^2 + 5F_{m-n}^2$	
0	1	1	2	x	y	z	$P_{x+y}P_{x+z} - P_xP_{x+y+z} = (-1)^x P_yP_z$	
0	1	2	1	x	y	z	$J_{x+y}J_{x+z} - J_xJ_{x+y+z} = (-2)^x J_yJ_z$	

References

1. Er, M.C.: Sums of Fibonacci numbers by matrix methods. Fibonacci Quart **22**(3), 204–207 (1984)
2. Everman, D., Danese, A.E., Venkannayah, K., Scheuer, E.M.: Elementary problems and solutions: solutions: E1396. Am. Math. Monthly **67**(7), 694 (1960)
3. Ferns, H.H.: Elementary problems and solutions: problem B-106. Fibonacci Quart **5**(1), 107 (1967)
4. Kalman, D.: Generalized Fibonacci numbers by matrix methods. Fibonacci Quart **20**(1), 73–76 (1982)
5. Kilic, E., Stănică, P.: A matrix approach for general higher order linear recurrences. Bull. Malays. Math. Sci. Soc. **34**(1), 51–67 (2011)
6. Mana, P.: Elementary problems and solutions: problem B-152. Fibonacci Quart **7**(3), 336 (1969)
7. Sloane, N.J.A.: The on-line encyclopedia of integer sequences. Published electronically at http://www.research.att.com/~njas/sequences/
8. Sharpe, B.: On sums $F_x^2 \pm F_y^2$. Fibonacci Quart **3**(1), 63 (1965)
9. Tadlock, S.B.: Products of odds. Fibonacci Quart **3**(1), 54–56 (1965)
10. Taylor, L.: Elementary problems and solutions: problem B-460. Fibonacci Quart **20**(4), 368–369 (1982)