

# Neighbor Sum Distinguishing Edge Colorings of Graphs with Small Maximum Average Degree

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**Abstract** A proper edge- $k$ -coloring of a graph  $G$  is an assignment of  $k$  colors  $1, 2, \dots, k$  to the edges of  $G$  such that no two adjacent edges receive the same color. A neighbor sum distinguishing edge- $k$ -coloring of  $G$  is a proper edge- $k$ -coloring of  $G$  such that for each edge  $uv \in E(G)$ , the sum of colors taken on the edges incident with  $u$  is different from the sum of colors taken on the edges incident with  $v$ . By  $ndi_{\Sigma}(G)$ , we denote the smallest value  $k$  in such a coloring of  $G$ . The maximum average degree of  $G$  is  $mad(G) = \max\{2|E(H)|/|V(H)|\}$ , where the maximum is taken over all the non-empty subgraphs  $H$  of  $G$ . In this paper, we obtain that if  $G$  is a graph without isolated edges and  $mad(G) < 8/3$ , then  $ndi_{\Sigma}(G) \leq k$  where  $k = \max\{\Delta(G) + 1, 6\}$ . It partially confirms the conjecture proposed by Flandrin et al. (Graphs Comb 29:1329–1336, 2013).

**Keywords** Proper edge coloring · Neighbor sum distinguishing edge coloring · Maximum average degree

## 1 Introduction

In this paper, all graphs considered are finite, simple, and undirected. The terminology and notation used but undefined in this paper can be found in [1]. Let  $G = (V, E)$

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be a graph. We use  $V(G)$ ,  $E(G)$ ,  $\Delta(G)$ , and  $\delta(G)$  to denote the vertex set, edge set, maximum degree, and minimum degree of  $G$ , respectively. Let  $d_G(v)$  or simply  $d(v)$  denote the degree of a vertex  $v$  in  $G$ . A vertex  $v$  is called a  $k$ -vertex (resp.  $k^-$ -vertex, or  $k^+$ -vertex) if  $d(v) = k$  [resp.  $d(v) \leq k$ , or  $d(v) \geq k$ ]. A vertex is called a leaf of  $G$  if  $d(v) = 1$ . A 2-vertex is called bad if it is adjacent to a 2-vertex, otherwise we call it good. A 5-vertex is called bad if it is adjacent to four bad 2-vertices, otherwise we call it good. The girth of a graph  $G$  is the length of a smallest cycle in  $G$ , and we denote it by  $g(G)$ . The maximum average degree of  $G$  is  $mad(G) = \max\{2|E(H)|/|V(H)|\}$ , where the maximum is taken over all the non-empty subgraphs  $H$  of  $G$ .

A proper edge- $k$ -coloring of a graph  $G$  is an assignment of  $k$  colors  $1, 2, \dots, k$  to the edges of  $G$  such that no two adjacent edges receive the same color. Let  $c$  be a proper edge- $k$ -coloring of  $G$ . By  $w(v)$  [resp.  $S(v)$ ], we denote the sum (resp. set) of colors taken on the edges incident with  $v$ , i.e.,  $w(v) = \sum_{uv \in E(G)} c(uv)$  [resp.  $S(v) = \{c(uv) | uv \in E(G)\}$ ]. We call the coloring  $c$  such that  $w(u) \neq w(v)$  [resp.  $S(u) \neq S(v)$ ] for each edge  $uv \in E(G)$  a neighbor sum distinguishing (resp. neighbor distinguishing) edge- $k$ -coloring of  $G$ . For simplicity, we use  $nsd$ - $k$ -coloring (resp.  $nd$ - $k$ -coloring) to denote the neighbor sum distinguishing (resp. neighbor distinguishing) edge- $k$ -coloring of  $G$ . By  $ndi_{\Sigma}(G)$  [resp.  $ndi(G)$ ], we denote the smallest value  $k$  such that  $G$  has an  $nsd$ - $k$ -coloring (resp.  $nd$ - $k$ -coloring) of  $G$ .

Obviously, a graph  $G$  has a neighbor sum distinguishing (neighbor distinguishing) coloring if and only if  $G$  has no isolated edges (we call it normal). Apparently, for any normal graph  $G$ ,  $ndi(G) \leq ndi_{\Sigma}(G)$ . In 2002, Zhang et al. [2] proposed the following conjecture.

**Conjecture 1.1** [2] *If  $G$  is a normal graph with at least six vertices, then  $ndi(G) \leq \Delta(G) + 2$ .*

Balister et al. [3] proved Conjecture 1.1 for bipartite graphs and for graphs  $G$  with  $\Delta(G) = 3$ . If  $G$  is bipartite planar with maximum degree  $\Delta(G) \geq 12$ , Conjecture 1.1 was confirmed by Edwards et al. [4]. Hatami [5] showed that if  $G$  is a normal graph and  $\Delta(G) > 10^{20}$ , then  $ndi(G) \leq \Delta(G) + 300$ . Akbari et al. [6] proved that  $ndi(G) \leq 3\Delta(G)$  for any normal graph. Wang et al. [7, 8] confirmed Conjecture 1.1 for sparse graphs and  $K_4$ -minor free graphs. More precisely, in [7] they showed that if  $G$  is a normal graph and  $mad(G) < 5/2$ , then  $ndi(G) \leq \Delta(G) + 1$ . Furthermore,  $ndi(G) = \Delta(G) + 1$  if and only if  $G$  has two adjacent maximum degree vertices. Recently, Hocquard et al. [9] proved that for every normal graph with  $\Delta(G) \geq 5$  and  $mad(G) < 13/5$ , we have  $ndi(G) \leq \Delta(G) + 1$ . Later, in [10] they proved that if  $G$  is a normal graph with  $\Delta(G) \geq 5$  and  $mad(G) < 3 - 2/\Delta(G)$ , then  $ndi(G) \leq \Delta(G) + 1$ .

Recently, Flandrin et al. [11] studied the neighbor sum distinguishing colorings of cycles, trees, complete graphs, and complete bipartite graphs. Based on these examples, they proposed the following conjecture.

**Conjecture 1.2** [11] *If  $G$  is a connected graph on at least three vertices and  $G \neq C_5$ , then  $ndi_{\Sigma}(G) \leq \Delta(G) + 2$ .*

Flandrin et al. [11] also proved that for each connected graph  $G$  with maximum degree  $\Delta \geq 2$ , we have  $ndi_{\Sigma}(G) \leq \lceil (7\Delta - 4)/2 \rceil$ . Dong et al. [12] considered the

neighbor sum distinguishing colorings of planar graphs and showed that if  $G$  is a normal planar graph, then  $ndi_{\Sigma}(G) \leq \max\{2\Delta(G) + 1, 25\}$ . In [13], Dong et al. proved that if  $G$  is a normal graph and  $mad(G) \leq 5/2$ , then  $ndi_{\Sigma}(G) \leq k$  where  $k = \max\{\Delta(G) + 1, 6\}$ . Other results on graph coloring problems are referred to [14–16].

In this paper, we will prove the following results.

**Theorem 1.3** *Let  $G$  be a normal graph. If  $mad(G) < \frac{8}{3}$ , then  $ndi_{\Sigma}(G) \leq k$  where  $k = \max\{\Delta(G) + 1, 6\}$ .*

**Corollary 1.4** *Let  $G$  be a normal graph. If  $mad(G) < \frac{8}{3}$ ,  $\Delta(G) \geq 5$ , then  $ndi_{\Sigma}(G) \leq \Delta(G) + 1$ .*

In [17], the authors obtained that  $mad(G) < 2g/(g - 2)$  if  $G$  is a planar graph with girth  $g$ . The following corollary is obvious.

**Corollary 1.5** *Let  $G$  be a normal planar graph. If  $g(G) \geq 8$  and  $\Delta(G) \geq 5$ , then  $ndi_{\Sigma}(G) \leq \Delta(G) + 1$ .*

We note that if  $G$  contains two adjacent vertices of maximum degree, then  $ndi_{\Sigma}(G) \geq \Delta(G) + 1$ . So the bound  $\Delta(G) + 1$  in Corollary 1.4 is sharp. Furthermore, Corollary 1.4 implies a result of Hocquard et al. [9] about the neighbor distinguishing coloring of sparse graphs.

## 2 Proof of Theorem 1.3

Firstly, we give two lemmas obtained by Dong et al. in [13], all the elements in each set are integers.

**Lemma 2.1** [13] *Let  $S_1, S_2$  be two sets and  $S_3 = \{\alpha + \beta \mid \alpha \in S_1, \beta \in S_2, \alpha \neq \beta\}$ .*

- (i) *If  $|S_1| = 2$  and  $|S_2| = 3$ , then  $|S_3| \geq 3$ .*
- (ii) *If  $|S_1| = 2$  and  $|S_2| = 4$ , then  $|S_3| \geq 4$ .*
- (iii) *If  $|S_1| = |S_2| = 2$  and  $S_1 \neq S_2$ , then  $|S_3| \geq 3$ .*

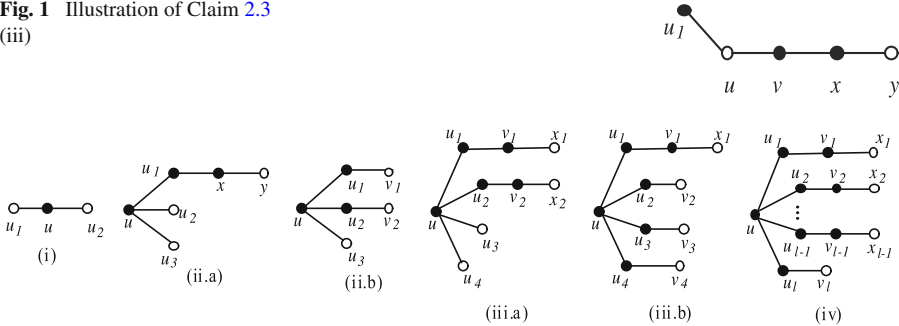
**Lemma 2.2** [13] *Let  $S$  be a set of size  $k + 1$ . If  $S_1 = \left\{ \sum_{i=1}^k x_i \mid x_i \in S, x_i \neq x_j \text{ if } 1 \leq i < j \leq k \right\}$ , then  $|S_1| \geq k + 1$ .*

Let  $k = \max\{\Delta(G) + 1, 6\}$  and  $[k] = \{1, 2, \dots, k\}$ . Suppose to the contrary that  $G$  is a counterexample to Theorem 1.3, such that  $|E(G)|$  is minimum. By the choice of  $G$ , it is clear that  $G$  is connected and any normal subgraph  $G'$  has an  $nsd$ - $k$ -coloring  $c$ . We use  $w(v)$  and  $S(v)$  to denote the sum and the set of colors taken on the edges incident with  $v$  in the coloring  $c$  of  $G'$ , i.e.,  $w(v) = \sum_{v \in e, e \in E(G')} c(e)$  and  $S(v) = \{c(e) \mid v \in e, e \in E(G')\}$ . In the following, we will extend  $c$  to the whole graph  $G$ .

Let  $H$  be the graph obtained by removing all the leaves of  $G$ . Obviously,  $H$  is a connected graph and  $mad(H) < 8/3$ . In the following, we give some properties of  $H$ .

**Fig. 1** Illustration of Claim 2.3

(iii)



**Fig. 2** Illustration of Claim 2.4

**Claim 2.3** *H* has the following properties:

- (i) [13]  $\delta(H) \geq 2$ , where  $\delta(H)$  is the minimum degree of *H*.
- (ii) [13] Let  $v \in V(H)$  such that  $d_H(v) = 2$ , then  $d_G(v) = 2$ .
- (iii) Let  $uvxy$  be a path in *H* such that  $d_H(v) = d_H(x) = 2$ , then  $d_G(u) = d_H(u)$  and  $d_G(y) = d_H(y)$  (Fig. 1).

*Proof* (iii) Let  $uvxy$  be a path in *H* such that  $d_H(v) = d_H(x) = 2$ . By Claim 2.3 (ii),  $d_G(v) = d_G(x) = 2$ . By contradiction suppose  $d_G(u) \neq d_H(u)$  [it follows from Claim 2.3 (i) and construction of *H* that  $d_G(u) \geq 3$ ]. Hence there exists at least one 1-vertex adjacent to *u* in *G*, say  $u_1$ . Consider  $G' = G \setminus \{vx\}$ . By the minimality of *G*,  $G'$  admits an *nsd-k*-coloring *c*. If  $c(uv) \neq c(xy)$ , then we color  $vx$  with a color distinct from  $c(uv)$ ,  $c(xy)$ ,  $w(u) - w(v)$ ,  $w(y) - w(x)$ , then we obtain an *nsd-k*-coloring of *G*. Otherwise, we permute the colors assigned to  $uu_1$  and  $uv$ . The obtained coloring is still an *nsd-k*-coloring of  $G'$ . We then extend this coloring to *G* as previously. This is a contradiction. □

**Claim 2.4** Let  $u \in V(H)$ ,  $d_H(u) = l$ ,  $uu_i \in E(H)$ ,  $i = 1, 2, \dots, l$ .

- (i) [13] If  $l = 2$ , then *u* is adjacent to at most one 2-vertex.
- (ii) (a) [13] If  $l = 3$  and  $d_H(u) < d_G(u)$ , then *u* is adjacent to at most one 2-vertex.  
 (b) If  $l = 3$ , then *u* is not adjacent to any bad 2-vertex. Furthermore, *u* is adjacent to at most one good 2-vertex.
- (iii) If  $l = 4$ , then *u* is adjacent to at most one bad 2-vertex. Furthermore, if *u* is adjacent to one bad 2-vertex, then *u* is adjacent to at most two good 2-vertices.
- (iv) If  $l \geq 5$  and *u* is adjacent to  $(l - 1)$  bad 2-vertices, then *u* is adjacent to at most  $(l - 1)$  2-vertices (Fig. 2).

*Proof* (ii) (b) Firstly, we prove that *u* is not adjacent to any bad 2-vertex. Suppose to the contrary that  $u_1$  is a bad 2-vertex. Let  $x$  be the other neighbor of  $u_1$  with  $d_H(x) = 2$ , and  $xy \in E(H)$ ,  $y \neq u_1$ . By Claim 2.3 (iii),  $d_G(u) = d_H(u) = 3$ . Consider the graph  $G' = G \setminus \{uu_1\}$ , then  $G'$  admits an *nsd-k*-coloring *c*. Color  $uu_1$  with a color  $\alpha$  in  $S = [k] \setminus (\{c(xy)\} \cup \{c(uu_2), c(uu_3)\} \cup \{w(u_2) - w(u)\} \cup \{w(u_3) - w(u)\})$ .

Recolor  $u_1x$  with a color distinct from  $\alpha$ ,  $c(xy)$ ,  $w(y) - c(xy)$ ,  $c(uu_2) + c(uu_3)$  and we obtain an  $nsd$ - $k$ -coloring of  $G$ , a contradiction.

In the following, we prove that  $u$  is adjacent to at most one good 2-vertex. Suppose to the contrary that  $d_H(u_1) = d_H(u_2) = 2$ ,  $v_i u_i \in E(H)$ ,  $v_i \neq u$ ,  $i = 1, 2$ .

**Case 1**  $d_G(u) > d_H(u) = 3$ . By Claim 2.4 (ii) (a), this Claim holds.

**Case 2**  $d_G(u) = d_H(u) = 3$ .

**Subcase 2.1**  $k \geq 7$ . Consider the graph  $G' = G \setminus \{uu_1, uu_2\}$ , then  $G'$  has an  $nsd$ - $k$ -coloring  $c$ . Let  $S_i = [k] \setminus (\{c(u_i v_i)\} \cup \{c(uu_3)\} \cup \{w(v_i) - w(u_i)\} \cup \{w(u_{3-i}) - c(uu_3)\})$ ,  $i = 1, 2$ , then  $|S_i| \geq 3$ ,  $i = 1, 2$ . By Lemma 2.1 (ii), we can choose  $\alpha_i \in S_i$ ,  $i = 1, 2$  such that  $\alpha_1 \neq \alpha_2$ ,  $\alpha_1 + \alpha_2 + c(uu_3) \neq w(u_3)$ . We obtain an  $nsd$ - $k$ -coloring of  $G$ , which is a contradiction.

**Subcase 2.2**  $k = 6$ . From the above discussion,  $d_G(v_i) \geq 3$ ,  $i = 1, 2$ . If  $d_G(v_1) = 5$ , then  $u_1$  can be distinguished from  $v_1$  under an arbitrary proper edge coloring of  $G$ . Consider  $G' = G \setminus \{uu_1, uu_2\}$ , then  $G'$  has an  $nsd$ -6-coloring  $c$ . The colors in  $\{c(u_1 v_1)\} \cup \{c(uu_3)\} \cup \{c(u_2 v_2) - c(uu_3)\}$  are forbidden for  $uu_1$ . Let  $S_1 = [6] \setminus (\{c(u_1 v_1)\} \cup \{c(uu_3)\} \cup \{c(u_2 v_2) - c(uu_3)\})$ ,  $S_2 = [6] \setminus (\{c(u_2 v_2)\} \cup \{c(uu_3)\} \cup \{w(v_2) - w(u_2)\} \cup \{c(u_1 v_1) - c(uu_3)\})$ , then  $|S_1| \geq 3$ ,  $|S_2| \geq 2$ . By Lemma 2.1 (ii), we can choose  $\alpha_i \in S_i$ ,  $i = 1, 2$  such that  $\alpha_1 \neq \alpha_2$ ,  $\alpha_1 + \alpha_2 + c(uu_3) \neq w(u_3)$ . We obtain an  $nsd$ -6-coloring of  $G$ , which is a contradiction. Therefore,  $d_G(v_1) \neq 5$ . Similarly,  $d_G(v_2) \neq 5$ .

If  $d_G(v_1) = 3$  and  $x_1, y_1$  are the other two neighbors of  $v_1$ . Consider  $G' = G \setminus \{uu_1, uu_2, u_1 v_1\}$ , then  $G'$  has an  $nsd$ -6-coloring  $c$ . Let  $S_1 = [6] \setminus (\{c(v_1 x_1) + c(v_1 y_1)\} \cup \{c(uu_3)\})$ ,  $S_2 = [6] \setminus (\{c(uu_3)\} \cup \{c(u_2 v_2)\} \cup \{w(v_2) - w(u_2)\})$ ,  $S_3 = [6] \setminus (\{c(v_1 x_1), c(v_1 y_1)\} \cup \{w(x_1) - w(v_1)\} \cup \{w(y_1) - w(v_1)\})$ , then  $|S_1| \geq 4$ ,  $|S_2| \geq 3$ ,  $|S_3| \geq 2$ . We can choose  $\alpha_i \in S_i$ ,  $i = 1, 2, 3$  such that  $\alpha_1 \neq \alpha_2$ ,  $\alpha_1 \neq \alpha_3$ ,  $u$  can be distinguished from  $u_1, u_2, u_3$ , and  $v_1$  can be distinguished from  $x_1, y_1$ . We obtain an  $nsd$ -6-coloring of  $G$ , a contradiction. Therefore,  $d_G(v_1) \neq 3$ . Similarly,  $d_G(v_2) \neq 3$ .

Now we assume that  $d_G(v_1) = d_G(v_2) = 4$ ,  $x_i, y_i, z_i$  are the other three neighbors of  $v_i$ ,  $i = 1, 2$ . Consider  $G' = G \setminus \{uu_1, uu_2\}$ , then  $G'$  has an  $nsd$ -6-coloring  $c$ . If  $c(v_1 x_1) + c(v_1 y_1) + c(v_1 z_1) > 6$ , then  $u_1$  and  $v_1$  can be distinguished. Let  $S_1 = [6] \setminus (\{c(uu_3)\} \cup \{c(u_1 v_1)\} \cup \{w(u_2) - c(uu_3)\})$ ,  $S_2 = [6] \setminus (\{c(uu_3)\} \cup \{c(u_2 v_2)\} \cup \{w(v_2) - w(u_2)\} \cup \{w(u_1) - c(uu_3)\})$ , then  $|S_1| \geq 3$ ,  $|S_2| \geq 2$ . We can choose  $\alpha_i \in S_i$ ,  $i = 1, 2$  such that  $\alpha_1 \neq \alpha_2$ ,  $\alpha_1 + \alpha_2 + c(uu_3) \neq w(u_3)$ . We obtain an  $nsd$ -6-coloring of  $G$ , which is a contradiction. Therefore,  $c(v_1 x_1) + c(v_1 y_1) + c(v_1 z_1) = 6$ . Similarly,  $c(v_2 x_2) + c(v_2 y_2) + c(v_2 z_2) = 6$ . Without loss of generality we assume that  $c(v_i x_i) = 1$ ,  $c(v_i y_i) = 2$ ,  $c(v_i z_i) = 3$ ,  $i = 1, 2$ . Suppose that  $c(u_1 v_1) \neq c(u_2 v_2)$  or  $c(uu_3) = 6$ , then we can obtain an  $nsd$ -6-coloring of  $G$  as previously. Hence,  $c(u_1 v_1) = c(u_2 v_2)$ ,  $c(uu_3) \neq 6$ . From the above discussion,  $d_G(u_3) \geq 3$ . If  $d_G(u_3) = 3$ , let  $x_3, y_3$  be the neighbors of  $u_3$  distinct from  $u$ . Consider the graph  $G' = G \setminus \{uu_1, uu_2, uu_3\}$ , then  $G'$  has an  $nsd$ -6-coloring  $c$ . Let  $S_1 = [5] \setminus \{c(u_1 v_1)\}$ ,  $S_2 = [5] \setminus \{c(u_2 v_2)\}$ ,  $S_3 = [6] \setminus (\{c(v_3 x_3), c(v_3 y_3)\} \cup \{w(x_3) - w(u_3)\} \cup \{w(y_3) - w(u_3)\})$ , then  $|S_1| \geq 4$ ,  $|S_2| \geq 4$ ,  $|S_3| \geq 2$ . From the above discussion, we know that  $S_1 = S_2 = \{1, 2, 3, 4\}$  or  $\{1, 2, 3, 5\}$ , so we can choose  $\alpha_i \in S_i$ ,  $i = 1, 2, 3$  such that  $\alpha_1, \alpha_2, \alpha_3$  are pairwise distinct and  $u$  can be distinguished from  $u_1, u_2, u_3$ . We obtain an  $nsd$ -6-coloring of  $G$ ,

which is a contradiction. Therefore,  $d_G(u_3) \geq 4$ . If  $c(uu_3) \in \{1, 2, 3\}$ , color  $uu_1, uu_2$  properly with  $\{1, 2, 3\} \setminus \{c(uu_3)\}$ . Otherwise, properly color  $uu_1, uu_2$  with colors in  $\{1, 2, 3\}$ . In both cases, we obtain an  $nsd$ -6-coloring of  $G$ , a contradiction.

(iii) Suppose to the contrary that  $d_H(u_1) = d_H(u_2) = 2, u_i v_i \in E(H), d_H(v_i) = 2, i = 1, 2, x_i$  is the other neighbor of  $v_i, i = 1, 2$ . By Claim 2.3 (iii),  $d_G(u) = d_H(u) = 4$ . Consider the graph  $G' = G \setminus \{uu_1, uu_2\}$ , then  $G'$  has an  $nsd$ - $k$ -coloring  $c$ . Let  $S_i = [k] \setminus (\{c(uu_3), c(uu_4)\} \cup \{c(v_i x_i)\}), i = 1, 2$ , then  $|S_i| \geq 3$ . By Lemma 2.1 (i), we can choose  $\alpha_i \in S_i, i = 1, 2$  such that  $\alpha_1 \neq \alpha_2$  and  $u$  can be distinguished from  $u_3, u_4$ . Recolor  $u_i v_i$  with a color distinct from  $\alpha_i, c(v_i x_i), w(x_i) - c(v_i x_i), \alpha_1 + \alpha_2 + c(uu_3) + c(uu_4) - \alpha_i, i = 1, 2$ , then  $u$  can be distinguished from  $u_1, u_2$ . We obtain an  $nsd$ - $k$ -coloring of  $G$ , a contradiction.

Now assume that  $u$  is adjacent to a bad 2-vertex  $u_1$  with  $u_1 v_1 \in E(H), d_H(v_1) = 2, v_1 x_1 \in E(H), x_1 \neq v_1$ . Suppose to the contrary that  $d_H(u_i) = 2, u_i v_i \in E(H), v_i \neq u, i = 2, 3, 4$ . By Claim 2.3 (iii),  $d_G(u) = d_H(u) = 4$ . Consider  $G' = G \setminus \{uu_1\}$ , then  $G'$  has an  $nsd$ - $k$ -coloring  $c$ . Let  $S = [k] \setminus (\{c(uu_2), c(uu_3), c(uu_4)\} \cup \{c(v_1 x_1)\})$ . When  $k \leq 7$ , if  $1 \in S$ , then color  $uu_1$  with  $\alpha \in S \setminus \{1\}$ , otherwise color  $uu_1$  with  $\alpha \in S \setminus \{2\}$ . In both cases  $w(u) + \alpha > w(u_i), i = 2, 3, 4$ . Then recolor  $u_1 v_1$  with a color distinct from  $\alpha, c(v_1 x_1), w(u), w(x_1) - c(v_1 x_1)$ . We obtain an  $nsd$ - $k$ -coloring of  $G$ , a contradiction. When  $k \geq 8, |S| \geq 4$ , we can choose  $\alpha \in S$  such that  $\alpha + w(u) \neq w(u_i), i = 2, 3, 4$ . Then recolor  $u_1 v_1$  with a color distinct from  $\alpha, c(v_1 x_1), w(u), w(x_1) - c(v_1 x_1)$ . We obtain an  $nsd$ - $k$ -coloring of  $G$ , a contradiction.

(iv) Suppose to the contrary that  $d_H(u_i) = 2, i = 1, 2, \dots, l, u_i v_i \in E(H), v_i \neq u, i = 1, 2, \dots, l$  and  $d_H(v_j) = 2, v_j x_j \in E(H), x_j \neq v_j, j = 1, 2, \dots, l - 1$ . By Claim 2.3 (iii),  $d_G(u) = d_H(u) = l$ . Let  $G' = G \setminus \{uu_1\}$ , then  $G'$  has an  $nsd$ - $k$ -coloring  $c$ . If  $l < \Delta = k - 1$ , color  $uu_1$  with  $\alpha \in [k] \setminus (\{c(uu_2), \dots, c(uu_l)\} \cup \{c(v_1 x_1)\} \cup \left\{ \alpha + \sum_{i=2}^{l-1} c(uu_i) \right\})$ . Otherwise color  $uu_1$  with  $\alpha \in [k] \setminus (\{c(uu_2), \dots, c(uu_l)\} \cup \{c(v_1 x_1)\})$ . In both cases,  $u$  can be distinguished from  $u_l$ . Properly recolor  $u_i v_i$  such that  $u$  can be distinguished from  $u_i$  and  $v_i$  can be distinguished from  $x_i, i = 1, 2, \dots, l - 1$ . We obtain an  $nsd$ - $k$ -coloring of  $G$ , a contradiction. □

**Claim 2.5** Let  $u \in V(H), d_H(u) = 5, uu_i \in E(H), i = 1, 2, 3, 4, 5$ .

(i) If  $\Delta(G) \geq 6$ , then  $u$  is adjacent to at most two bad 2-vertices. If  $\Delta(G) = 5$  and  $u$  is adjacent to three bad 2-vertices, then  $u$  is adjacent to at most one good 2-vertex.

Furthermore, if  $\Delta(G) = 5$  and  $u$  is a bad 5-vertex, then by Claim 2.3 (iii),  $d_G(u) = d_H(u) = 5$ . Let  $d_H(u_i) = 2, u_i v_i \in E(H), d_H(v_i) = 2, x_i$  be the other neighbor of  $v_i, i = 1, 2, 3, 4$ , we have

(ii)  $d_H(u_5) \geq 4$ .

(iii) If  $d_H(u_5) = 4$ , then  $u_5$  is adjacent to no bad 2-vertex.

(iv) If  $d_H(u_5) = 5$ , then  $u_5$  is adjacent to at most two bad 2-vertices (Fig. 3).

*Proof* (i) Assume  $\Delta(G) \geq 6$ . Suppose to the contrary that  $d_H(u_1) = d_H(u_2) = d_H(u_3) = 2, v_i$  is the other neighbor of  $u_i$  with  $d_H(v_i) = 2, x_i$  is the other

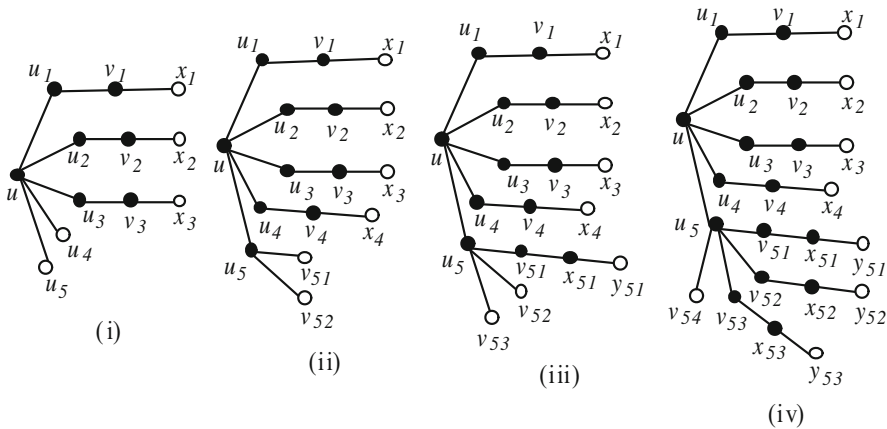


Fig. 3 Illustration of Claim 2.5

neighbor of  $v_i, i = 1, 2, 3$ . By Claim 2.3 (iii),  $d_G(u) = d_H(u) = 5$ . Consider the graph  $G' = G \setminus \{uu_1, uu_2, uu_3\}$ , then  $G'$  has an  $nsd-k$ -coloring  $c$ . Let  $S_i = [k] \setminus (\{c(uu_4), c(uu_5)\} \cup \{c(v_i x_i)\}), i = 1, 2, 3$ , then  $|S_i| \geq 4, i = 1, 2, 3$ . We can choose  $\alpha_i \in S_i, i = 1, 2, 3$  such that  $\alpha_1, \alpha_2, \alpha_3$  are pairwise distinct and  $u$  can be distinguished by  $u_4, u_5$ . Recolor  $u_i v_i$  with a color distinct from  $\alpha_i, c(v_i x_i), w(x_i) - c(v_i x_i), \sum_{i=1}^3 \alpha_i + w(u) - \alpha_i, i = 1, 2, 3$ , then  $u$  can be distinguished from  $u_1, u_2, u_3$ . We obtain an  $nsd-k$ -coloring of  $G$ , a contradiction.

Assume that  $\Delta(G) = 5$ . Suppose to the contrary that  $d_H(u_i) = 2, u_i v_i \in E(H), v_i \neq u, i = 1, 2, \dots, 5$  and  $d_H(v_j) = 2, v_j x_j \in E(H), j = 1, 2, 3$ . By Claim 2.3 (iii),  $d_G(u) = d_H(u) = 5$ . Consider the graph  $G' = G \setminus \{uu_1\}$ , then  $G'$  has an  $nsd-6$ -coloring  $c$ . Color  $uu_1$  with  $\alpha \in [6] \setminus (\{c(uu_2), \dots, c(uu_5)\} \cup \{c(v_1 x_1)\})$ . Then recolor  $u_1 v_1$  with a color distinct from  $\alpha, c(v_1 x_1), w(x_1) - c(v_1 x_1), \sum_{i=2}^5 c(uu_i)$ . It can be seen that  $w(u) + \alpha > w(u_i), i = 2, \dots, 5$ , so we obtain an  $nsd-6$ -coloring of  $G$ , a contradiction.

(ii) By Claim 2.4 (iv),  $d_H(u_5) \geq 3$ . Suppose to the contrary that  $d_H(u_5) = 3, v_{51}, v_{52}$  are the other two neighbors of  $u_5$ . Consider the graph  $G' = G \setminus \{uu_1, uu_2, uu_3, uu_4, uu_5\}$ , then  $G'$  has an  $nsd-6$ -coloring  $c$ . Let  $S_i = [6] \setminus \{c(v_i x_i)\}, i = 1, 2, 3, 4, S_5 = [6] \setminus (\{c(u_5 v_{51}), c(u_5 v_{52})\} \cup \{w(v_{51}) - w(u_5)\} \cup \{w(v_{52}) - w(u_5)\})$ , then  $|S_i| \geq 5, i = 1, 2, 3, 4, |S_5| \geq 2$ . We can choose  $\alpha_i \in S_i, i = 1, 2, 3, 4, 5$  such that  $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5$  are pairwise distinct and  $u$  can be distinguished by  $u_5$ . Obviously,  $u$  can be distinguished by  $u_1, u_2, u_3, u_4$ . Recolor  $u_i v_i$  with a color distinct from  $\alpha_i, c(v_i x_i), w(x_i) - c(v_i x_i), i = 1, 2, 3, 4$  and we obtain an  $nsd-6$ -coloring of  $G$ , a contradiction.

(iii) Let  $v_{51}, v_{52}, v_{53}$  be the other three neighbors of  $u_5$ . Suppose to the contrary that  $d_H(v_{51}) = d_H(x_{51}) = 2, v_{51} x_{51} \in E(H), x_{51} y_{51} \in E(H), y_{51} \neq v_{51}$ . By Claim 2.3 (iii),  $d_G(u_5) = d_H(u_5) = 4$ . Consider the graph  $G' = G \setminus \{uu_1, uu_2, uu_3, uu_4\}$ , then  $G'$  has an  $nsd-6$ -coloring  $c$ . Assume that  $c(uu_5) = \alpha_5, c(u_5 v_{5j}) = \beta_j, j = 1, 2, 3$ ,

$c(v_i x_i) = \gamma_i, i = 1, 2, 3, 4, c(x_{51} y_{51}) = \eta$ . Let  $S_i = [6] \setminus (\{\gamma_i\} \cup \{\alpha_5\}), i = 1, 2, 3, 4$ , then  $|S_i| \geq 4, i = 1, 2, 3, 4$ .

If there exists some  $\gamma_i = \alpha_5 (i \in \{1, 2, 3, 4\})$  or  $\gamma_i \neq \gamma_j (i \neq j, i, j \in \{1, 2, 3, 4\})$ , then we can choose  $\alpha_i \in S_i, i = 1, 2, 3, 4$  such that  $u$  can be distinguished by  $u_5$ . Recolor  $u_i v_i$  with a color distinct from  $\gamma_i, \alpha_i, w(x_i) - c(v_i x_i), i = 1, 2, 3, 4$ , we can obtain an  $nsd$ -6-coloring of  $G$ , a contradiction. Therefore, we assume that  $\gamma_1 = \gamma_2 = \gamma_3 = \gamma_4 = \gamma, \alpha_5 \neq \gamma$ . Then  $S_1 = S_2 = S_3 = S_4 = [6] \setminus \{\alpha_5, \gamma\}$ . Assume that  $S_i = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}, i = 1, 2, 3, 4$ . Color  $uu_i$  with  $\alpha_i, i = 1, 2, 3, 4$ .

If  $|\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\} \cap \{\beta_1, \beta_2, \beta_3\}| = 3$ , then  $u$  can be distinguished by  $u_5$ , recolor  $u_i v_i, i = 1, 2, 3, 4$  as previously and we obtain an  $nsd$ -6-coloring of  $G$ , a contradiction. Furthermore,  $|\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\} \cap \{\beta_1, \beta_2, \beta_3\}| \geq 2$  because there are six colors in total. Therefore,  $|\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\} \cap \{\beta_1, \beta_2, \beta_3\}| = 2$ . Without loss of generality we assume that  $\{\alpha_1, \alpha_2\} \subseteq \{\beta_1, \beta_2, \beta_3\}$ .

If  $\beta_1 = \gamma$ , then  $\{\alpha_1, \alpha_2\} = \{\beta_2, \beta_3\}$ . If  $\alpha_5 \neq \eta$ , suppose  $u$  can not be distinguished by  $u_5$ , then  $\gamma = \alpha_3 + \alpha_4$ . Recolor  $uu_5$  with  $\gamma$  and recolor  $u_5 v_{51}$  with  $\alpha_5$ , we obtain an  $nsd$ -6-coloring of  $G$ , a contradiction. Therefore,  $\alpha_5 = \eta$ . Recolor  $u_5 v_{51}$  with one of  $\alpha_3, \alpha_4$ , or exchange the colors of  $uu_3$  and  $uu_5$ , or exchange the colors of  $uu_4$  and  $uu_5$  such that  $u_5$  can be distinguished by  $u, v_{52}, v_{53}$ . It is easy to see that  $u$  can be distinguished by  $u_1, u_2, u_3, u_4$ . Recolor  $u_i v_i, i = 1, 2, 3, 4$ , and  $v_{51} x_{51}$  as previously and we can obtain an  $nsd$ -6-coloring of  $G$ , a contradiction.

If  $\beta_1 \neq \gamma$ . Without loss of generality we assume that  $\beta_1 = \alpha_1, \beta_2 = \alpha_2$ . Recolor  $u_5 v_{51}$  with one of  $\alpha_3, \alpha_4$ , or exchange the colors of  $uu_3$  and  $uu_5$ , or exchange the colors of  $uu_4$  and  $uu_5$  such that  $u_5$  can be distinguished by  $u, v_{52}, v_{53}$ . It is easy to see that  $u$  can be distinguished by  $u_1, u_2, u_3, u_4$ . Recolor  $u_i v_i, i = 1, 2, 3, 4$ , and  $v_{51} x_{51}$  as previously and we can obtain an  $nsd$ -6-coloring of  $G$ , a contradiction.

(iv) Let  $v_{51}, v_{52}, v_{53}, v_{54}$  be the other four neighbors of  $u_5$ . Suppose to the contrary that  $d_H(v_{5i}) = d_H(x_{5i}) = 2, v_{5i} x_{5i} \in E(H), x_{5i} y_{5i} \in E(H), y_{5i} \neq v_{5i}, i = 1, 2, 3$ . By Claim 2.3 (iii),  $d_G(u_5) = d_H(u_5) = 5$ . Consider the graph  $G' = G \setminus \{uu_1, uu_2, uu_3, uu_4\}$ , then  $G'$  has an  $nsd$ -6-coloring  $c$ . Assume that  $c(uu_5) = \alpha_5, c(u_5 v_{5j}) = \beta_j, j = 1, 2, 3, 4, c(v_i x_i) = \gamma_i, i = 1, 2, 3, 4$ . Let  $S_i = [6] \setminus (\{\gamma_i\} \cup \{\alpha_5\}), i = 1, 2, 3, 4$ , then  $|S_i| \geq 4, i = 1, 2, 3, 4$ .

If there exists some  $\gamma_i = \alpha_5 (i \in \{1, 2, 3, 4\})$  or  $\gamma_i \neq \gamma_j (i \neq j, i, j \in \{1, 2, 3, 4\})$ , then we can choose  $\alpha_i \in S_i, i = 1, 2, 3, 4$  such that  $u$  can be distinguished by  $u_5$ . Recolor  $u_i v_i$  with a color distinct from  $\alpha_i, \gamma_i, w(x_i) - c(v_i x_i), i = 1, 2, 3, 4$ . We obtain an  $nsd$ -6-coloring of  $G$ , a contradiction. Therefore, we assume that  $\gamma_1 = \gamma_2 = \gamma_3 = \gamma_4 = \gamma, \alpha_5 \neq \gamma$ . Then  $S_1 = S_2 = S_3 = S_4 = [6] \setminus \{\alpha_5, \gamma\}$ . Assume that  $S_i = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$ . Color  $uu_i$  with  $\alpha_i$  and recolor  $u_i v_i$  with a color distinct from  $\gamma, \alpha_i, w(x_i) - c(v_i x_i), i = 1, 2, 3, 4$ .

If  $\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\} \neq \{\beta_1, \beta_2, \beta_3, \beta_4\}$ , then  $u$  can be distinguished by  $u_5$ , it is easy to see that  $u$  can be distinguished from  $u_1, u_2, u_3, u_4$ . We obtain an  $nsd$ -6-coloring of  $G$ , a contradiction. Therefore,  $\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\} = \{\beta_1, \beta_2, \beta_3, \beta_4\}$ . Without loss of generality we assume that  $\alpha_i = \beta_i, i = 1, 2, 3, 4$ . If  $c(x_{51} y_{51}) \neq \gamma$ , then recolor  $u_5 v_{51}$  with  $\gamma$ . We can see that  $u$  can be distinguished from  $u_5$ . If  $u_5$  can not be distinguished from  $v_{54}$ , then exchange the colors of  $uu_1$  and  $uu_5$ . Recolor  $v_{51} x_{51}$  with a color distinct from  $\gamma_1, c(x_{51} y_{51}), w(y_{51}) - c(x_{51} y_{51})$ . We obtain an  $nsd$ -6-coloring of  $G$ , a contradiction. Similarly,  $c(x_{52} y_{52}) = c(x_{53} y_{53}) = \gamma$ . Recolor  $uu_5$  with  $\gamma$ .



Then recolor  $u_5v_{51}$  or recolor  $u_5u_{52}$  with  $\alpha_5$  such that  $u_5$  can be distinguished by  $v_{54}$ . Recolor  $v_{5i}x_{5i}$  with a color distinct from  $\gamma, \alpha_5, \alpha_i, w(y_{5i}) - c(x_{5i}y_{5i}), i = 1, 2$ . We obtain an  $nsd$ -6-coloring of  $G$ , a contradiction.  $\square$

In order to complete the proof, we use a discharging procedure. For every  $v \in V(H)$ , we define the original charge of  $v$  to be  $ch(v) = d_H(v) = l$ . We then redistribute the charges according to the rules R1, R2, and R3 (below). To complete the proof, our aim is to prove that, for every vertex  $v$ , the new charge  $ch^*(v)$  is at least  $8/3$ .

The discharging rules are defined as follows:

- (R1) Every  $4^+$ -vertex gives  $\frac{2}{3}$  to each adjacent bad 2-vertex.  
 (R2) Every  $3^+$ -vertex gives  $\frac{1}{3}$  to each adjacent good 2-vertex.  
 (R3) If  $u$  is a bad 5-vertex,  $u_i, i = 1, 2, 3, 4, 5$ , are the neighbors of  $u$ , and if  $u_1, u_2, u_3, u_4$  are bad 2-vertices, then  $u_5$  gives  $\frac{1}{3}$  to  $u$ .

**Case  $l = 2$ .** Observe that  $ch(v) = 2$ . Suppose  $v$  is a good 2-vertex. Hence, by (R2),  $ch^*(v) \geq 2 + 2 \times \frac{1}{3} = \frac{8}{3}$ . Suppose  $v$  is bad. By Claim 2.4 (i) and Claim 2.4 (ii),  $v$  is adjacent to at most one 2-vertex and is adjacent to a  $4^+$ -vertex. Hence, by (R1),  $ch^*(v) = 2 + 1 \times \frac{2}{3} = \frac{8}{3}$ .

**Case  $l = 3$ .** Observe that  $ch(v) = 3$ . By Claim 2.4 (ii),  $v$  is adjacent to no bad 2-vertex and is adjacent to at most one good 2-vertex. By Claim 2.5 (ii),  $v$  is adjacent to no bad 5-vertex. By (R2) and (R3),  $ch^*(v) \geq 3 - 1 \times \frac{1}{3} = \frac{8}{3}$ .

**Case  $l = 4$ .** Observe that  $ch(v) = 4$ . Suppose  $v$  is not adjacent to a bad 2-vertex. Then, by (R2) and (R3),  $ch^*(v) \geq 4 - 4 \times \frac{1}{3} = \frac{8}{3}$ . Assume now,  $v$  is adjacent to a bad 2-vertex. By Claim 2.4 (iii),  $v$  is adjacent to at most two good 2-vertices. By Claim 2.5 (iii),  $v$  is adjacent to no bad 5-vertex. Hence by (R1), (R2), and (R3),  $ch^*(v) \geq 4 - 1 \times \frac{2}{3} - 2 \times \frac{1}{3} = \frac{8}{3}$ .

**Case  $l = 5$ .** Observe that  $ch(v) = 5$ . Suppose  $v$  is adjacent to at most two bad 2-vertices, then by (R1), (R2), and (R3),  $ch^*(v) \geq 5 - 2 \times \frac{2}{3} - 3 \times \frac{1}{3} = \frac{8}{3}$ . If  $v$  is adjacent to three bad 2-vertices, then by Claim 2.5 (i) and (iv),  $v$  is adjacent to at most one good 2-vertex and  $v$  is adjacent to no bad 5-vertex. Hence by (R1), (R2), and (R3),  $ch^*(v) \geq 5 - 3 \times \frac{2}{3} - 1 \times \frac{1}{3} = \frac{8}{3}$ . Assume now,  $v$  is a bad 5-vertex. By (R1), (R2), and (R3),  $ch^*(v) \geq 5 - 4 \times \frac{2}{3} + 1 \times \frac{1}{3} = \frac{8}{3}$ .

**Case  $l \geq 6$ .** Observe that  $ch(v) = l$ . By Claim 2.5 (i), any 5-vertex in  $H$  is good. By Claim 2.4 (iv),  $v$  is adjacent to at most  $(l - 1)$  bad 2-vertices. Moreover if  $v$  is adjacent to  $(l - 1)$  bad 2-vertices, then its last neighbor has degree at least 3. It follows by (R1),  $ch^*(v) \geq l - (l - 1) \times \frac{2}{3} \geq \frac{8}{3}$ .

This completes the proof of Theorem 1.3.

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