Bull. Malays. Math. Sci. Soc. (2016) 39:1005–1041
DOI 10.1007/s40840-015-0206-1

BULLETIN *of the*
MALAYSIAN MATHEMATICAL
SCIENCES SOCIETY
http://www.springer.com/mathematics/journal/40840

Stepanov-Like Weighted Pseudo Almost Automorphic Functions Via Measure Theory

Yong-Kui Chang[1] · G. M. N'Guérékata[2] · Rui Zhang[3]

Received: 28 September 2013 / Revised: 2 January 2014 / Published online: 7 August 2015

Abstract In this article, we introduce and study the concept of μ-Stepanov-like pseudo almost automorphic function using the measure theory. We present new results on completeness and composition theorems for the space of such functions. To illustrate our main results, we provide some applications to a nonautonomous semilinear evolution equation.

Keywords Measure theory · μ-Pseudo almost automorphic function · μ-Stepanov-like pseudo almost automorphic function · Fixed point theorem

Mathematics Subject Classification 34K14 · 60H10 · 35B15 · 34F05

Communicated by Shangjiang Guo.

✉ Yong-Kui Chang
lzchangyk@163.com

G. M. N'Guérékata
gaston.n'guerekata@morgan.edu

Rui Zhang
zhangrui_2008aoyun@163.com

1 School of Mathematics and Statistics, Xidian University, Xi'an 710071, People's Republic of China

2 Department of Mathematics, Morgan State University, 1700 E. Cold Spring Lane, Baltimore, MD 21251, USA

3 Department of Mathematics, Lanzhou Jiaotong University, Lanzhou 730070, People's Republic of China

1 Introduction

The concept of almost automorphy was first introduced in the literature by Bochner in 1960s, and it is a natural generalization of almost periodicity [1,2]; for more details about this topic, we refer to [3–6]. N'Guérékata and Pankov introduced the concept of Stepanov-like almost automorphy and applied this concept to study the existence and uniqueness of an almost automorphic solution to the autonomous semilinear equation in [6]. Moreover, Blot introduced the notion of weighted pseudo almost automorphic functions with values in a Banach space in [7], and Mophou studied the existence and uniqueness of a weighted pseudo almost automorphic mild solution to a semilinear fractional equation in [8]. Xia and Fan presented the notation of Stepanov-like weighted pseudo almost automorphic function in [9]. Zhang, Chang, and N'Guérékata investigated some properties and new composition theorems of Stepanov-like weighted pseudo almost automorphic functions in [10,11] and then used these results to study the existence of weighted pseudo almost automorphic solutions for some differential equations in [12,13] and integral equations in [14].

Recently, Blot et al. in [15] applied the measure theory to define an ergodic function and investigated many interesting properties of μ-pseudo almost automorphic functions. To the best of our knowledge, there is no work reported in the literature on S^p-weighted pseudo almost automorphic functions in the light of the measure theory. To close this gap, motivated by the above-mentioned works, the purpose of this work is to present the concept of μ-S^p-pseudo almost automorphic functions and establish completeness and composition theorems for the space of such functions. And then, we apply our main results to investigate the existence of μ-pseudo almost automorphic mild solutions with μ-S^p-pseudo almost automorphic coefficients to the following nonautonomous semilinear evolution equation:

$$u'(t) = A(t)u(t) + f(t, u(t)), \quad t \in \mathbb{R}, \tag{1.1}$$

where $\{A(t)\}_{t\in\mathbb{R}}$ satisfies the *Acquistapace–Terreni* condition in [16], $U(t, s)$ generated by $A(t)$ is exponentially stable, and $f \in PAA^p(\mathbb{R}, \mathbb{X}, \mu) \cap C(\mathbb{R}, \mathbb{X})$ for $p > 1$ will be specified later.

The rest of this paper is organized as follows. In Sect. 2, we present some basic definitions, lemmas, and preliminary results which will be used throughout this paper. In Sect. 3, we establish some composition theorems of μ-S^p-pseudo almost automorphic functions. In Sect. 4, we prove the existence of μ-pseudo almost automorphic mild solutions to the nonautonomous semilinear evolution Eq. (1.1).

2 Preliminaries and μ-S^p-Pseudo Almost Automorphic Functions

In this section, we define new notion of the μ-ergodic functions and the μ-Stepanov-like pseudo almost automorphic functions and then give some fundamental properties of these functions that we use in differential equations. Recall that the notion of μ-Stepanov-like pseudo almost automorphy will be a generalization of the Stepanov-like weighted pseudo almost automorphy.

Let $(\mathbb{X}, \|\cdot\|)$, $(\mathbb{Y}, \|\cdot\|_{\mathbb{Y}})$, be two Banach spaces and $BC(\mathbb{R}, \mathbb{X})$ denote the Banach space of bounded continuous functions from $\mathbb{R}$ to $\mathbb{X}$, equipped with the supremum norm $\|f\|_\infty = \sup_{t\in\mathbb{R}} \|f(t)\|$. Throughout this work, we denote by $\mathfrak{B}$ the Lebesgue σ-field of $\mathbb{R}$ and by $\mathfrak{M}$ the set of all positive measures μ on $\mathfrak{B}$ satisfying $\mu(\mathbb{R}) = +\infty$ and $\mu([a, b]) < +\infty$, for all $a, b \in \mathbb{R}(a < b)$.

Definition 2.1 [4] A continuous function $f : \mathbb{R} \to \mathbb{X}$ is said to be almost automorphic if for every sequence of real numbers $\{s'_n\}_{n\in\mathbb{N}}$, there exists a subsequence $\{s_n\}_{n\in\mathbb{N}}$ such that

$$g(t) := \lim_{n\to\infty} f(t + s_n)$$

is well defined for each $t \in \mathbb{R}$, and

$$\lim_{n\to\infty} g(t - s_n) = f(t)$$

for each $t \in \mathbb{R}$. The collection of all such functions will be denoted by $AA(\mathbb{X})$.

Definition 2.2 [5] A continuous function $f(t, s) : \mathbb{R} \times \mathbb{R} \to \mathbb{X}$ is called bi-almost automorphic if for every sequence of real numbers $\{s'_n\}_{n\in\mathbb{N}}$, there exists a subsequence $\{s_n\}_{n\in\mathbb{N}}$ such that

$$g(t, s) := \lim_{n\to\infty} f(t + s_n, s + s_n)$$

is well defined for each $t, s \in \mathbb{R}$, and

$$\lim_{n\to\infty} g(t - s_n, s - s_n) = f(t, s)$$

for each $t, s \in \mathbb{R}$. The collection of all such functions will be denoted by $bAA(\mathbb{R} \times \mathbb{R}, \mathbb{X})$.

Definition 2.3 [4,17] A continuous function $f : \mathbb{R} \times \mathbb{X} \to \mathbb{X}$ is said to be almost automorphic if $f(t, x)$ is almost automorphic for each $t \in \mathbb{R}$ uniformly for all $x \in B$, where B is any bounded subset of $\mathbb{X}$. The collection of all such functions will be denoted by $AA(\mathbb{R} \times \mathbb{X}, \mathbb{X})$.

Let $\mathbb{U}$ denote the set of all functions $\rho : \mathbb{R} \to (0, \infty)$, which are locally integrable over $\mathbb{R}$ such that $\rho > 0$ almost everywhere. For a given $r > 0$ and for each $\rho \in \mathbb{U}$, we set $m(r, \rho) := \int_{-r}^{r} \rho(t)\mathrm{d}t$.

Thus the space of weights $\mathbb{U}_\infty$ is defined by

$$\mathbb{U}_\infty := \left\{\rho \in \mathbb{U} : \lim_{r\to\infty} m(r, \rho) = \infty\right\}.$$

Now for $\rho \in \mathbb{U}_\infty$, we define

$$PAA_0(\mathbb{X}, \rho) := \left\{ f \in BC(\mathbb{R}, \mathbb{X}) : \lim_{r\to\infty} \frac{1}{m(r,\rho)} \int_{-r}^{r} \|f(t)\|\rho(t)\mathrm{d}t = 0 \right\};$$

$$PAA_0(\mathbb{Y}, \mathbb{X}, \rho) := \Big\{ f \in C(\mathbb{R}\times\mathbb{Y}, \mathbb{X}) : f(\cdot, y) \text{ is bounded for each } y \in \mathbb{Y} \text{ and } \lim_{r\to\infty} \frac{1}{m(r,\rho)} \int_{-r}^{r} \|f(t,y)\|\rho(t)\mathrm{d}t = 0 \text{ uniformly in } y \in \mathbb{Y} \Big\}.$$

Remark 2.1 When $\rho(t) = 1$ for each $t \in \mathbb{R}$, one retrieves the so-called ergodic space that is, $AA_0(\mathbb{X})$ and $AA_0(\mathbb{X}) = \left\{ f \in BC(\mathbb{R}, \mathbb{X}) : \lim_{r\to\infty} \frac{1}{2r} \int_{-r}^{r} \|f(t)\|\mathrm{d}t = 0 \right\}$. Note that the spaces $PAA_0(\mathbb{X}, \rho)$ are richer than $AA_0(\mathbb{X})$.

Definition 2.4 [7] Let $\rho \in \mathbb{U}_\infty$. A function $f \in BC(\mathbb{R}, \mathbb{X})$ [respectively, $f \in BC(\mathbb{R}\times\mathbb{Y}, \mathbb{X})$] is called weighted pseudo almost automorphic if it can be expressed as $f = g + \phi$, where $g \in AA(\mathbb{X})$ [respectively, $AA(\mathbb{R}\times\mathbb{Y}, \mathbb{X})$] and $\phi \in PAA_0(\mathbb{X}, \rho)$ [respectively, $PAA_0(\mathbb{Y}, \mathbb{X}, \rho)$]. We denote by $WPAA(\mathbb{X})$ [respectively, $WPAA(\mathbb{R}\times\mathbb{Y}, \mathbb{X})$] the set of all such functions.

Definition 2.5 [15] Let $\mu \in \mathfrak{M}$. A bounded continuous function $f : \mathbb{R} \to \mathbb{X}$ is said to be μ-ergodic if

$$\lim_{r\to+\infty} \frac{1}{\mu([-r,r])} \int_{[-r,r]} \|f(t)\| \,\mathrm{d}\mu(t) = 0.$$

We denote the space of all such functions by $\varepsilon(\mathbb{R}, \mathbb{X}, \mu)$.

Definition 2.6 [15] Let $\mu \in \mathfrak{M}$. A continuous function $f : \mathbb{R} \to \mathbb{X}$ is said to be μ-pseudo almost automorphic if f is written in the form:

$$f = g + \phi,$$

where $g \in AA(\mathbb{R}, \mathbb{X})$ and $\phi \in \varepsilon(\mathbb{R}, \mathbb{X}, \mu)$. We denote the space of all such functions by $PAA(\mathbb{R}, \mathbb{X}, \mu)$.

Thus, we have

$$AA(\mathbb{R}, \mathbb{X}) \subset PAA(\mathbb{R}, \mathbb{X}, \mu) \subset BC(\mathbb{R}, \mathbb{X}).$$

Lemma 2.1 [15] *Let $\mu \in \mathfrak{M}$. Then $(\varepsilon(\mathbb{R}, \mathbb{X}, \mu), \|\cdot\|_\infty)$ is a Banach space.*

For $\mu \in \mathfrak{M}$ and $\tau \in \mathbb{R}$, we denote μ_τ the positive measure on $(\mathbb{R}, \mathfrak{B})$ defined by

$$\mu_\tau(A) = \mu(\{a + \tau : a \in A\}) \quad for \;\; A \in \mathfrak{B}. \tag{2.1}$$

From $\mu \in \mathfrak{M}$, we list the following hypothesis.

(**H0**) For all $\tau \in \mathbb{R}$, there exist $\beta > 0$ and a bounded interval I such that

$$\mu_\tau(A) \leq \beta\mu(A),$$

when $A \in \mathfrak{B}$ satisfies $A \cap I = \emptyset$.

Lemma 2.2 [15] *Let $\mu \in \mathfrak{M}$ satisfy (H0). Then $\varepsilon(\mathbb{R}, \mathbb{X}, \mu)$ is translation invariant, and therefore $PAA(\mathbb{R}, \mathbb{X}, \mu)$ is also translation invariant.*

Lemma 2.3 [15] *Let $\mu \in \mathfrak{M}$. Assume that $PAA(\mathbb{R}, \mathbb{X}, \mu)$ is translation invariant. Then the decomposition of a μ-pseudo almost automorphic function in the form $f = g + \phi$, where $g \in AA(\mathbb{R}, \mathbb{X})$ and $\phi \in \varepsilon(\mathbb{R}, \mathbb{X}, \mu)$, is unique.*

Lemma 2.4 [15] *Let $\mu \in \mathfrak{M}$. Assume that $PAA(\mathbb{R}, \mathbb{X}, \mu)$ is translation invariant. Then $(PAA(\mathbb{R}, \mathbb{X}, \mu), \|\cdot\|_\infty)$ is a Banach space.*

Definition 2.7 [6,18] The Bochner transform $f^b(t, s)$, $t \in \mathbb{R}$, $s \in [0, 1]$, of a function $f : \mathbb{R} \to \mathbb{X}$ is defined by

$$f^b(t, s) := f(t + s).$$

Remark 2.2 [18] (i) A function $\varphi(t, s)$, $t \in \mathbb{R}$, $s \in [0, 1]$, is the Bochner transform of a certain function f, $\varphi(t, s) = f^b(t, s)$, if and only if $\varphi(t + \tau, s - \tau) = \varphi(s, t)$ for all $t \in \mathbb{R}$, $s \in [0, 1]$ and $\tau \in [s - 1, s]$.

(ii) Note that if $f = h + \varphi$, then $f^b = h^b + \varphi^b$. Moreover, $(\lambda f)^b = \lambda f^b$ for each scalar λ.

Definition 2.8 [18] The Bochner transform $f^b(t, s, u)$, $t \in \mathbb{R}$, $s \in [0, 1]$, $u \in \mathbb{X}$ of a function $f : \mathbb{R} \times \mathbb{X} \to \mathbb{X}$ is defined by

$$f^b(t, s, u) := f(t + s, u) \quad \text{for each} \quad u \in \mathbb{X}.$$

Definition 2.9 [6,18] Let $p \in [1, \infty)$. The space $BS^p(\mathbb{X})$ of all Stepanov bounded functions, with the exponent p, consists of all measurable functions $f : \mathbb{R} \to \mathbb{X}$ such that $f^b \in L^\infty(\mathbb{R}, L^p(0, 1; \mathbb{X}))$. This is a Banach space with the norm

$$\|f\|_{S^p} = \|f^b\|_{L^\infty(\mathbb{R}, L^p)} = \sup_{t\in\mathbb{R}} \left(\int_t^{t+1} \|f(\tau)\|^p \, d\tau \right)^{\frac{1}{p}}.$$

Definition 2.10 [6,19] The space $AS^p(\mathbb{X})$ of Stepanov-like almost automorphic (or S^p-almost automorphic) functions consists of all $f \in BS^p(\mathbb{X})$ such that $f^b \in AA(L^p(0, 1; \mathbb{X}))$. In other words, a function $f \in L^p_{loc}(\mathbb{R}, \mathbb{X})$ is said to be S^p-almost automorphic if its Bochner transform $f^b : \mathbb{R} \to L^p(0, 1; \mathbb{X})$ is almost automorphic in the sense that for every sequence of real numbers $\{s'_n\}_{n\in\mathbb{N}}$, there exist a subsequence $\{s_n\}_{n\in\mathbb{N}}$ and a function $g \in L^p_{loc}(\mathbb{R}, \mathbb{X})$ such that

$$\lim_{n\to\infty}\left(\int_t^{t+1}\|f(s+s_n)-g(s)\|^p\,\mathrm{d}s\right)^{\frac{1}{p}}=0\quad\text{and}$$
$$\lim_{n\to\infty}\left(\int_t^{t+1}\|g(s-s_n)-f(s)\|^p\,\mathrm{d}s\right)^{\frac{1}{p}}=0$$

pointwise on $\mathbb{R}$.

Definition 2.11 [6,19] A function $f:\mathbb{R}\times\mathbb{Y}\to\mathbb{X}$, $(t,u)\to f(t,u)$ with $f(\cdot,u)\in L^p_{loc}(\mathbb{R},\mathbb{X})$ for each $u\in\mathbb{Y}$ is said to be S^p-almost automorphic in $t\in\mathbb{R}$ uniformly in $u\in\mathbb{Y}$ if $t\to f(t,u)$ is S^p-almost automorphic for each $u\in\mathbb{Y}$. That means, for every sequence of real numbers $\{s'_n\}_{n\in\mathbb{N}}$, there exist a subsequence $\{s_n\}_{n\in\mathbb{N}}$ and a function $g(\cdot,u)\in L^p_{loc}(\mathbb{R},\mathbb{X})$ such that

$$\lim_{n\to\infty}\left(\int_t^{t+1}\|f(s+s_n,u)-g(s,u)\|^p\mathrm{d}s\right)^{\frac{1}{p}}=0$$

and

$$\lim_{n\to\infty}\left(\int_t^{t+1}\|g(s-s_n,u)-f(s,u)\|^p\mathrm{d}s\right)^{\frac{1}{p}}=0$$

pointwise on $\mathbb{R}$ and for each $u\in\mathbb{Y}$. We denote by $AS^p(\mathbb{R}\times\mathbb{Y},\mathbb{X})$ the set of all such functions.

Definition 2.12 [20] A function $f\in BS^p(\mathbb{X})$ is said to be Stepanov-like pseudo almost automorphic if it can be decomposed as $f=g+\varphi$ where $g\in AS^p(\mathbb{X})$ and $\varphi^b\in AA_0(L^p(0,1;\mathbb{X}))$. Denote by $PAA^p(\mathbb{X})$ the set of all such functions.

Definition 2.13 [20] A function $F:\mathbb{R}\times\mathbb{Y}\to\mathbb{X}$, $(t,u)\to F(t,u)$ with $F(\cdot,u)\in L^p_{loc}(\mathbb{R},\mathbb{X})$ for each $u\in\mathbb{Y}$ is said to be Stepanov-like pseudo almost automorphic in $t\in\mathbb{R}$, if it can be decomposed as $F(t,u)=G(t,u)+H(t,u)$ with $G\in AS^p(\mathbb{R}\times\mathbb{Y},\mathbb{X})$ and $H^b\in AA_0(\mathbb{Y},L^p(0,1;\mathbb{X}))$. Denote by $PAA^p(\mathbb{R}\times\mathbb{Y},\mathbb{X})$ the set of all such functions.

Definition 2.14 [11] Let $\rho\in\mathbb{U}_\infty$. A function $f\in BS^p(\mathbb{X})$ is said to be Stepanov-like weighted pseudo almost automorphic (or S^p-weighted pseudo almost automorphic) if it can be expressed as $f=g+h$, where $g\in AS^p(\mathbb{X})$ and $h^b\in PAA_0(L^p(0,1;\mathbb{X}),\rho)$. In other words, a function $f\in L^p_{loc}(\mathbb{R},\mathbb{X})$ is said to be Stepanov-like weighted pseudo almost automorphic relatively to the weight $\rho\in\mathbb{U}_\infty$, if its Bochner transform $f^b:\mathbb{R}\to L^p(0,1;\mathbb{X})$ is weighted pseudo almost automorphic in the sense that there exist two functions $g,h:\mathbb{R}\to\mathbb{X}$ such that $f=g+h$, where $g\in AS^p(\mathbb{X})$ and $h^b\in PAA_0(L^p(0,1;\mathbb{X}),\rho)$. We denote by $WPAAS^p(\mathbb{X})$ the set of all such functions.

Definition 2.15 [11] Let $\rho\in\mathbb{U}_\infty$. A function $f:\mathbb{R}\times\mathbb{Y}\to\mathbb{X}$, $(t,u)\to f(t,u)$ with $f(\cdot,u)\in L^p_{loc}(\mathbb{R},\mathbb{X})$ for each $u\in\mathbb{Y}$ is said to be Stepanov-like weighted pseudo almost automorphic (or S^p-weighted pseudo almost automorphic) if it can be expressed as $f = g + h$, where $g \in AS^p(\mathbb{R} \times \mathbb{Y}, \mathbb{X})$ and $h^b \in PAA_0(\mathbb{Y}, L^p(0, 1; \mathbb{X}), \rho)$. We denote by $WPAAS^p(\mathbb{R} \times \mathbb{Y}, \mathbb{X})$ the set of all such functions.

Definition 2.16 Let $\mu \in \mathfrak{M}$. A function $f \in BS^p(\mathbb{X})$ is said to be μ-Stepanov-like pseudo almost automorphic (or μ-S^p-pseudo almost automorphic) if it can be expressed as $f = g + \phi$, where $g \in AS^p(\mathbb{X})$ and $\phi^b \in \varepsilon(L^p(0, 1; \mathbb{X}), \mu)$. In other words, a function $f \in L^p_{loc}(\mathbb{R}, \mathbb{X})$ is said to be μ-Stepanov-like pseudo almost automorphic relatively to the measure μ, if its Bochner transform $f^b : \mathbb{R} \to L^p(0, 1; \mathbb{X})$ is μ-pseudo almost automorphic in the sense that there exist two functions $g, \phi : \mathbb{R} \to \mathbb{X}$ such that $f = g + \phi$, where $g \in AS^p(\mathbb{X})$ and $\phi^b \in \varepsilon(L^p(0, 1; \mathbb{X}), \mu)$, that is $\phi^b \in BC(L^p(0, 1; \mathbb{X}))$ and

$$\lim_{r\to\infty} \frac{1}{\mu([-r, r])} \int_{[-r,r]} \left(\int_t^{t+1} \|\phi(s)\|^p \, \mathrm{d}s \right)^{\frac{1}{p}} \mathrm{d}\mu(t) = 0.$$

We denote by $PAA^p(\mathbb{R}, \mathbb{X}, \mu)$ the set of all such functions.

Definition 2.17 Let $\mu \in \mathfrak{M}$. A function $f : \mathbb{R} \times \mathbb{Y} \to \mathbb{X}, (t, u) \to f(t, u)$ with $f(\cdot, u) \in L^p_{loc}(\mathbb{R}, \mathbb{X})$ for each $u \in \mathbb{Y}$ is said to be μ-Stepanov-like pseudo almost automorphic (or μ-S^p-pseudo almost automorphic) if it can be expressed as $f = g + \phi$, where $g \in AS^p(\mathbb{R} \times \mathbb{Y}, \mathbb{X})$ and $\phi^b \in \varepsilon(\mathbb{Y}, L^p(0, 1; \mathbb{X}), \mu)$. We denote by $PAA^p(\mathbb{R} \times \mathbb{Y}, \mathbb{X}, \mu)$ the set of all such functions.

[15] One can observe that an S^p-weighted pseudo almost automorphic function is μ-S^p-pseudo almost automorphic, where the measure μ is absolutely continuous with respect to the Lebesgue measure and its Radon–Nikodym derivative is $\rho : \frac{\mathrm{d}\mu(t)}{\mathrm{d}t} = \rho(t)$. Moreover, a S^p-pseudo almost automorphic function is an μ-S^p-pseudo almost automorphic function in the particular case where the measure μ is the Lebesgue measure.

Remark 2.3 [15] From $\mu \in \mathfrak{M}$ and the fact that $\mu([-r, r]) = \mu([-r, r] \setminus I) + \mu(I)$ for r sufficiently large, we deduce that $\lim_{r\to+\infty} \mu([-r, r] \setminus I) = +\infty$.

Theorem 2.1 *Let $\mu \in \mathfrak{M}$ and I be a bounded interval (eventually $I = \emptyset$). Assume that $f(\cdot) \in BS^p(\mathbb{R}, \mathbb{X})$. Then the following assertions are equivalent:*

(i) $f^b(\cdot) \in \varepsilon(L^p(0, 1; \mathbb{X}), \mu)$.

(ii) $\lim_{r\to+\infty} \dfrac{1}{\mu([-r, r] \setminus I)} \int_{[-r,r]\setminus I} \left(\int_t^{t+1} \|f(s)\|^p \, \mathrm{d}s\right)^{\frac{1}{p}} \mathrm{d}\mu(t) = 0.$

(iii) *For any* $\epsilon > 0$, $\lim_{r\to+\infty} \dfrac{\mu\left(\left\{t \in [-r, r]\setminus I : \left(\int_t^{t+1} \|f(s)\|^p \, \mathrm{d}s\right)^{\frac{1}{p}} > \epsilon\right\}\right)}{\mu([-r, r] \setminus I)} = 0.$

Proof To prove the theorem, we refer to [15, Theorem 2.14], and first we prove $(i) \Longleftrightarrow (ii)$. Denote by $A = \mu(I)$ and $B = \displaystyle\int_I \left(\int_t^{t+1} \|f(s)\|^p \, \mathrm{d}s\right)^{\frac{1}{p}} \mathrm{d}\mu(t)$. Since the interval I is bounded and the function $f \in BS^p(\mathbb{X})$, then A and B are finite. Let $r > 0$ be such that $I \subset [-r, r]$ and $\mu([-r, r] \setminus I) > 0$. Then we have

$$\frac{1}{\mu([-r,r]\setminus I)}\int_{[-r,r]\setminus I}\left(\int_t^{t+1}\|f(s)\|^p\,\mathrm{d}s\right)^{\frac{1}{p}}\,\mathrm{d}\mu(t)$$
$$=\frac{1}{\mu([-r,r])-A}\left(\int_{[-r,r]}\left(\int_t^{t+1}\|f(s)\|^p\,\mathrm{d}s\right)^{\frac{1}{p}}\,\mathrm{d}\mu(t)-B\right)$$
$$=\frac{\mu([-r,r])}{\mu([-r,r])-A}\left(\frac{1}{\mu([-r,r])}\int_{[-r,r]}\left(\int_t^{t+1}\|f(s)\|^p\,\mathrm{d}s\right)^{\frac{1}{p}}\,\mathrm{d}\mu(t)\right.$$
$$\left.-\frac{B}{\mu([-r,r])}\right). \tag{2.2}$$

From the equality (2.2) and the fact that $\mu(\mathbb{R})=+\infty$, we deduce that (ii) is equivalent to

$$\lim_{r\to+\infty}\frac{1}{\mu([-r,r])}\int_{[-r,r]}\left(\int_t^{t+1}\|f(s)\|^p\,\mathrm{d}s\right)^{\frac{1}{p}}\,\mathrm{d}\mu(t)=0,$$

that is (i).

(iii)$\Longrightarrow$(ii) Denote by $A_r^\epsilon(f)$ and $B_r^\epsilon(f)$ the following sets:

$$A_r^\epsilon(f)=\left\{t\in[-r,r]\setminus I:\left(\int_t^{t+1}\|f(s)\|^p\,\mathrm{d}s\right)^{\frac{1}{p}}>\epsilon\right\}$$

and

$$B_r^\epsilon(f)=\left\{t\in[-r,r]\setminus I:\left(\int_t^{t+1}\|f(s)\|^p\,\mathrm{d}s\right)^{\frac{1}{p}}\le\epsilon\right\}.$$

Assume that (iii) holds, that is

$$\lim_{r\to+\infty}\frac{\mu(A_r^\epsilon(f))}{\mu([-r,r]\setminus I)}=0. \tag{2.3}$$

From the following equality

$$\int_{[-r,r]\setminus I}\left(\int_t^{t+1}\|f(s)\|^p\,\mathrm{d}s\right)^{\frac{1}{p}}\,\mathrm{d}\mu(t)=\int_{A_r^\epsilon(f)}\left(\int_t^{t+1}\|f(s)\|^p\,\mathrm{d}s\right)^{\frac{1}{p}}\,\mathrm{d}\mu(t)$$
$$+\int_{B_r^\epsilon(f)}\left(\int_t^{t+1}\|f(s)\|^p\,\mathrm{d}s\right)^{\frac{1}{p}}\,\mathrm{d}\mu(t),$$

we deduce for r large enough that

$$\frac{1}{\mu([-r,r])\setminus I}\int_{[-r,r]\setminus I}\left(\int_t^{t+1}\|f(s)\|^p\,\mathrm{d}s\right)^{\frac{1}{p}}\mathrm{d}\mu(t)$$
$$\leq \|f\|_{S^p}\frac{\mu(A_r^{\epsilon}(f))}{\mu([-r,r]\setminus I)}+\epsilon,$$

then for all $\epsilon > 0$,

$$\limsup_{r\to+\infty}\frac{1}{\mu([-r,r]\setminus I)}\int_{[-r,r]\setminus I}\left(\int_t^{t+1}\|f(s)\|^p\,\mathrm{d}s\right)^{\frac{1}{p}}\mathrm{d}\mu(t)\leq\epsilon,$$

so (ii) holds.

(ii)$\Longrightarrow$ (iii) Assume that (ii) holds. From the following inequality

$$\frac{1}{\mu([-r,r]\setminus I)}\int_{[-r,r]\setminus I}\left(\int_t^{t+1}\|f(s)\|^p\,\mathrm{d}s\right)^{\frac{1}{p}}\mathrm{d}\mu(t)$$
$$\geq\frac{1}{\mu([-r,r]\setminus I)}\int_{A_r^{\epsilon}(f)}\left(\int_t^{t+1}\|f(s)\|^p\,\mathrm{d}s\right)^{\frac{1}{p}}\mathrm{d}\mu(t)$$
$$\geq\epsilon\frac{\mu(A_r^{\epsilon}(f))}{\mu([-r,r]\setminus I)},$$

for r sufficiently large, we obtain (2.3), that is (iii). This completes the proof. □

Definition 2.18 [15] Let μ_1 and $\mu_2\in\mathfrak{M}$. μ_1 is said to be equivalent to $\mu_2(\mu_1\sim\mu_2)$ if there exist constants α and $\beta>0$ and a bounded interval I (eventually $I=\emptyset$) such that

$$\alpha\mu_1(A)\leq\mu_2(A)\leq\beta\mu_1(A),$$

for $A\in\mathfrak{B}$ satisfying $A\cap I=\emptyset$.

Theorem 2.2 *Let $\mu_1,\mu_2\in\mathfrak{M}$. If μ_1 and μ_2 are equivalent, then*

$$\varepsilon(L^p(0,1;\mathbb{X}),\mu_1)=\varepsilon(L^p(0,1;\mathbb{X}),\mu_2)$$

and

$$PAA^p(\mathbb{R},\mathbb{X},\mu_1)=PAA^p(\mathbb{R},\mathbb{X},\mu_2).$$

Proof Let us show that $\varepsilon(L^p(0,1;\mathbb{X}),\mu_1)=\varepsilon(L^p(0,1;\mathbb{X}),\mu_2)$. Since $\mu_1\sim\mu_2$ and $\mathfrak{B}$ is the Lebesgue σ-field, we obtain for r sufficiently large

$$\frac{\alpha}{\beta}\frac{\mu_1\left(\left\{t\in[-r,r]\setminus I:\left(\int_t^{t+1}\|f(s)\|^p\,\mathrm{d}s\right)^{\frac{1}{p}}>\epsilon\right\}\right)}{\mu_1([-r,r]\setminus I)}$$
$$\le\frac{\mu_2\left(\left\{t\in[-r,r]\setminus I:\left(\int_t^{t+1}\|f(s)\|^p\,\mathrm{d}s\right)^{\frac{1}{p}}>\epsilon\right\}\right)}{\mu_2([-r,r]\setminus I)}$$
$$\le\frac{\beta}{\alpha}\frac{\mu_1\left(\left\{t\in[-r,r]\setminus I:\left(\int_t^{t+1}\|f(s)\|^p\,\mathrm{d}s\right)^{\frac{1}{p}}>\epsilon\right\}\right)}{\mu_1([-r,r]\setminus I)}.$$

Using Theorem 2.1, we deduce that $\varepsilon(L^p(0,1;\mathbb{X}),\mu_1) = \varepsilon(L^p(0,1;\mathbb{X}),\mu_2)$. From the definition of a μ-S^p-pseudo almost automorphic function, we deduce that $PAA^p(\mathbb{R},\mathbb{X},\mu_1) = PAA^p(\mathbb{R},\mathbb{X},\mu_2)$. □

We give sufficient conditions for the translation invariance of the spaces of μ-S^p-pseudo almost automorphic functions.

[15] Hypothesis (H0) holds if and only if, for all $\tau\in\mathbb{R}$, there exist a constant $\beta>0$ and a bounded interval I such that

$$\rho(t+\tau)\le\beta\rho(t)\quad a.e.\ \text{on}\ \mathbb{R}\setminus I.$$

Lemma 2.5 [15] *Let $\mu\in\mathfrak{M}$. Then μ satisfies (H0) if and only if the measures μ and μ_τ are equivalent for all $\tau\in\mathbb{R}$.*

Lemma 2.6 [15] *Hypothesis (H0) implies for all $\sigma>0$,*

$$\limsup_{r\to+\infty}\frac{\mu([-r-\sigma,r+\sigma])}{\mu([-r,r])}<+\infty.$$

Theorem 2.3 *Let $\mu\in\mathfrak{M}$ satisfy (H0). Then $\varepsilon(L^p(0,1;\mathbb{X}),\mu)$ is translation invariant, and therefore $PAA^p(\mathbb{R},\mathbb{X},\mu)$ is also translation invariant.*

Proof The proof of this theorem is similar to that of [15, Theorem 3.5]. First, it is clear that $AS^p(\mathbb{X})$ is translation invariant, and it remains to prove that if $f\in\varepsilon(L^p(0,1;\mathbb{X}),\mu)$ then $f_\tau\in\varepsilon(L^p(0,1;\mathbb{X}),\mu)$ for all $\tau\in\mathbb{R}$. Let $f\in\varepsilon(L^p(0,1;\mathbb{X}),\mu)$ and $\tau\in\mathbb{R}$. Since $\mu(\mathbb{R})=+\infty$, there exists $r_0>0$ such that $\mu([-r-|\tau|,r+|\tau|])>0$ for all $r\ge r_0$. In this proof, we assume that $r\ge r_0$. Let us denote by

$$K_\tau(r)=\frac{1}{\mu_\tau([-r,r])}\int_{[-r,r]}\left(\int_t^{t+1}\|f(s)\|^p\,\mathrm{d}s\right)^{\frac{1}{p}}\mathrm{d}\mu_\tau(t)\quad\text{for}\quad r>0\ \text{and}\ \tau\in\mathbb{R},\tag{2.4}$$

where μ_τ is the positive measure defined by (2.1). Using Lemma 2.5, it follows that μ_τ and μ are equivalent, then by using Theorem 2.2 we have $\varepsilon(L^p(0,1;\mathbb{X}),\mu_\tau) = \varepsilon(L^p(0,1;\mathbb{X}),\mu)$, therefore $f \in \varepsilon(L^p(0,1;\mathbb{X}),\mu_\tau)$, that is

$$\lim_{r\to+\infty} K_\tau(r) = 0, \quad \text{for all } \tau \in \mathbb{R}. \tag{2.5}$$

For all $A \in \mathfrak{B}$, we denote by χ_A the characteristic function of A. Using definition of the measure μ_τ, we obtain that $\int_{[-r,r]} \chi_A(t)d\mu_\tau(t) = \int_{[-r+\tau,r+\tau]} \chi_A(t-\tau)d\mu(t)$ for all $A \in \mathfrak{B}$, and since $t \mapsto \left(\int_t^{t+1} \|f(s)\|^p ds\right)^{\frac{1}{p}}$ is the pointwise limit of an increasing sequence of linear combinations of characteristic functions [Theorem 1.17], we deduce that

$$\int_{[-r,r]} \left(\int_t^{t+1} \|f(s)\|^p \,\mathrm{d}s\right)^{\frac{1}{p}} \mathrm{d}\mu_\tau(t) = \int_{[-r+\tau,r+\tau]} \left(\int_t^{t+1} \|f(s-\tau)\|^p \,\mathrm{d}s\right)^{\frac{1}{p}} \times \mathrm{d}\mu(t). \tag{2.6}$$

From (2.1), (2.4), and (2.6), we obtain

$$K_\tau(r) = \frac{1}{\mu([-r+\tau, r+\tau])} \int_{[-r+\tau,r+\tau]} \left(\int_t^{t+1} \|f(s-\tau)\|^p \,\mathrm{d}s\right)^{\frac{1}{p}} \mathrm{d}\mu(t).$$

If we denote by $\tau^+ := \max(\tau, 0)$ and $\tau^- := \max(-\tau, 0)$, we have $|\tau| + \tau = 2\tau^+$ and $|\tau| - \tau = 2\tau^-$; and then $[-r+\tau-|\tau|, r+\tau+|\tau|] = [-r-2\tau^-, r+2\tau^+]$. Therefore, we obtain

$$K_\tau(r+|\tau|) = \frac{1}{\mu([-r-2\tau^-, r+2\tau^+])} \int_{[-r-2\tau^-,r+2\tau^+]} \left(\int_t^{t+1} \|f(s-\tau)\|^p \,\mathrm{d}s\right)^{\frac{1}{p}} \times \mathrm{d}\mu(t). \tag{2.7}$$

From (2.7) and the following inequality

$$\frac{1}{\mu([-r,r])} \int_{[-r,r]} \left(\int_t^{t+1} \|f(s-\tau)\|^p \,\mathrm{d}s\right)^{\frac{1}{p}} \mathrm{d}\mu(t) \le \frac{1}{\mu([-r,r])} \int_{[-r-2\tau^-,r+2\tau^+]} \left(\int_t^{t+1} \|f(s-\tau)\|^p \,\mathrm{d}s\right)^{\frac{1}{p}} \mathrm{d}\mu(t),$$

we get

$$\frac{1}{\mu([-r,r])} \int_{[-r,r]} \left(\int_t^{t+1} \|f(s-\tau)\|^p \,\mathrm{d}s\right)^{\frac{1}{p}} \mathrm{d}\mu(t) \le \frac{\mu([-r-2\tau^-, r+2\tau^+])}{\mu([-r,r])} \times K_\tau(r+|\tau|),$$

which implies

$$\frac{1}{\mu([-r,r])}\int_{[-r,r]}\left(\int_t^{t+1}\|f(s-\tau)\|^p\,\mathrm{d}s\right)^{\frac{1}{p}}\mathrm{d}\mu(t) \leq \frac{\mu([-r-2|\tau|,r+2|\tau|])}{\mu([-r,r])} \times K_\tau(r+|\tau|). \tag{2.8}$$

From (2.5) and (2.8) and by using Lemma 2.6, we deduce that

$$\lim_{r\to+\infty}\frac{1}{\mu([-r,r])}\int_{[-r,r]}\left(\int_t^{t+1}\|f(s-\tau)\|^p\,\mathrm{d}s\right)^{\frac{1}{p}}\mathrm{d}\mu(t)=0,$$

that is $f_{-\tau}\in\varepsilon(L^p(0,1;\mathbb{X}),\mu)$ for all $\tau\in\mathbb{R}$. Then $\varepsilon(L^p(0,1;\mathbb{X}),\mu)$ is translation invariant. This ends the proof. □

Theorem 2.4 *Let* $\mu\in\mathfrak{M}$ *satisfy* (*H0*). *If* $f\in PAA(\mathbb{R},\mathbb{X},\mu)$, *then* $f\in PAA^p(\mathbb{R},\mathbb{X},\mu)$ *for each* $1\leq p<\infty$. *In other words,* $PAA(\mathbb{R},\mathbb{X},\mu)\subset PAA^p(\mathbb{R},\mathbb{X},\mu)$.

Proof In the proof of this theorem, we follow the same reasoning as in the proof of [11, Lemma 2.4]. Let $f=g+h$ where $g\in AA(\mathbb{X})$ and $h\in\varepsilon(\mathbb{R},\mathbb{X},\mu)$. From [6, Remark 2.3], we know that the function $g\in AA(\mathbb{X})\subset AS^p(\mathbb{X})$.

Next, let us show that $h^b\in\varepsilon\,(L^p(0,1;\mathbb{X}),\mu)$. For $r>0$, we see that

$$\begin{aligned}&\frac{1}{\mu([-r,r])}\int_{[-r,r]}\left(\int_0^1\|h(t+s)\|^p\,\mathrm{d}s\right)^{\frac{1}{p}}\mathrm{d}\mu(t)\\ &\leq\frac{1}{\mu([-r,r])}\int_{[-r,r]}\left(\int_0^1\sup_{s\in[0,1]}\|h(t+s)\|^p\,\mathrm{d}s\right)^{\frac{1}{p}}\mathrm{d}\mu(t)\\ &\leq\frac{1}{\mu([-r,r])}\int_{[-r,r]}\left(\sup_{s\in[0,1]}\|h(t+s)\|^p\right)^{\frac{1}{p}}\mathrm{d}\mu(t).\end{aligned}$$

Let $s_0\in[0,1]$ such that $\sup_{s\in[0,1]}\|h(t+s)\|=\|h(t+s_0)\|$. Then, we deduce

$$\begin{aligned}&\frac{1}{\mu([-r,r])}\int_{[-r,r]}\left(\int_0^1\|h(t+s)\|^p\,\mathrm{d}s\right)^{\frac{1}{p}}\mathrm{d}\mu(t)\\ &\leq\frac{1}{\mu([-r,r])}\int_{[-r,r]}\left(\sup_{s\in[0,1]}\|h(t+s)\|^p\right)^{\frac{1}{p}}\mathrm{d}\mu(t)\\ &\leq\frac{1}{\mu([-r,r])}\int_{[-r,r]}\left(\|h(t+s_0)\|^p\right)^{\frac{1}{p}}\mathrm{d}\mu(t)\\ &=\frac{1}{\mu([-r,r])}\int_{[-r,r]}\|h(t+s_0)\|\,\mathrm{d}\mu(t).\end{aligned}$$

Using the fact that $\varepsilon(\mathbb{R}, \mathbb{X}, \mu)$ is translation invariant, it follows that $\lim_{r\to\infty} \frac{1}{\mu([-r,r])} \int_{[-r,r]} \|h(t + s_0)\| \,\mathrm{d}\mu(t) = 0$. Hence, $h^b \in \varepsilon(L^p(0, 1; \mathbb{X}), \mu)$. The proof is then completed. □

Theorem 2.5 *Let $\mu \in \mathfrak{M}$ and $f \in PAA^p(\mathbb{R}, \mathbb{X}, \mu)$ be such that $f = g + h$, where $g \in AS^p(\mathbb{X})$ and $h^b \in \varepsilon(L^p(0, 1; \mathbb{X}), \mu)$. If $PAA^p(\mathbb{R}, \mathbb{X}, \mu)$ is translation invariant, then*

$$\{g(t) : t \in \mathbb{R}\} \subset \overline{\{f(t) : t \in \mathbb{R}\}}, \quad \text{(the closure of range } f).$$

Proof The proof is an adaptation of [15, Theorem 4.1]. Suppose that the above claim is not true, then there exist constants $t_0 \in \mathbb{R}$ such that $g(t_0) \notin \overline{\{f(t) : t \in \mathbb{R}\}}$. Since the space $AS^p(\mathbb{X})$ and $\varepsilon(L^p(0, 1; \mathbb{X}), \mu)$ are translation invariant, we can assume that $t_0 = 0$, then there exists a constant $\epsilon > 0$ such that

$$\|g(0) - f(t)\|_p > 2\epsilon, \quad \text{for all } t \in \mathbb{R},$$

where $\|\cdot\|_p$ denotes the norm in $L^p(0, 1; \mathbb{X})$. Since $g^b \in AA\,(L^p(0, 1; \mathbb{X}))$, for $\epsilon > 0$, let

$$C_\epsilon = \left\{t \in \mathbb{R} : \|g(t) - g(0)\|_p < \epsilon\right\}.$$

By [8, Lemma 2.12], there exist constants $s_1, \ldots, s_m \in \mathbb{R}$ such that $\bigcup_{i=1}^m (s_i + C_\epsilon) = \mathbb{R}$. From the fact that $f = g + h$ and the Minkowski inequality, for all $t \in C_\epsilon$, we have

$$\|h(t)\|_p = \|f(t) - g(t)\|_p \geq \|g(0) - f(t)\|_p - \|g(t) - g(0)\|_p > \epsilon.$$

Then it follows that

$$\|h(t - s_i)\|_p > \epsilon \quad \text{for all } i = 1, \ldots, m \text{ and } t \in s_i + C_\epsilon.$$

Let $\mathbb{H}(t) := \sum_{i=1}^m \|h(t - s_i)\|_p$. From the previous inequalities, we have the fact that

$$\mathbb{H}(t) > \epsilon, \text{ for all } t \in \mathbb{R}. \tag{2.9}$$

In view of $\varepsilon(L^p(0, 1; \mathbb{X}), \mu)$ is translation invariant, then $[t \longmapsto h(t - s_i)] \in \varepsilon(L^p(0, 1; \mathbb{X}), \mu)$ for all $i \in \{1, \ldots, m\}$. Hence $\mathbb{H} \in \varepsilon(L^p(0, 1; \mathbb{X}), \mu)$, which contradicts the relation (2.9). This finishes the proof. □

Theorem 2.6 *Let $\mu \in \mathfrak{M}$. Assume that $PAA^p(\mathbb{R}, \mathbb{X}, \mu)$ is translation invariant. Then $(PAA^p(\mathbb{R}, \mathbb{X}, \mu), \|\cdot\|_{S^p})$ is a Banach space.*

Proof Let $(f_n)_{n\in\mathbb{N}} \subset PAA^p(\mathbb{R}, \mathbb{X}, \mu)$ be a Cauchy sequence for the norm $\|\cdot\|_{S^p}$. By definition, we can write $f_n = g_n + h_n$, where $(g_n)_{n\in\mathbb{N}} \subset AS^p(\mathbb{X})$ and $(h_n^b)_{n\in\mathbb{N}} \subset \varepsilon(L^p(0, 1; \mathbb{X}), \mu)$. From Theorem 2.5, we obtain that

$$\{g_n(t) : t \in \mathbb{R}\} \subset \overline{\{f_n(t) : t \in \mathbb{R}\}}.$$

Hence, we easily deduce that $(g_n)_{n\in\mathbb{N}}$ is also a Cauchy sequence for the norm $\|\cdot\|_{S^p}$. Thus there exists a function $g \in AS^p(\mathbb{X})$ such that $\|g_n - g\|_{S^p} \to 0$ as $n \to \infty$. Using the previous fact, it follows that $h_n = f_n - g_n$ is a Cauchy sequence with respect to the norm $\|\cdot\|_{S^p}$. So there exists a function $h \in BS^p(\mathbb{X})$ such that $\|h_n - h\|_{S^p} \to 0$ as $n \to \infty$.

Now for $r > 0$,

$$\begin{aligned} &\frac{1}{\mu([-r,r])}\int_{[-r,r]}\left(\int_t^{t+1}\|h(s)\|^p\,\mathrm{d}s\right)^{\frac{1}{p}}\mathrm{d}\mu(t)\\ &\quad\le \frac{1}{\mu([-r,r])}\int_{[-r,r]}\left(\int_t^{t+1}\|h_n(s)-h(s)\|^p\,\mathrm{d}s\right)^{\frac{1}{p}}\mathrm{d}\mu(t)\\ &\qquad+\frac{1}{\mu([-r,r])}\int_{[-r,r]}\left(\int_t^{t+1}\|h_n(s)\|^p\,\mathrm{d}s\right)^{\frac{1}{p}}\mathrm{d}\mu(t)\\ &\quad\le \|h_n-h\|_{S^p}+\frac{1}{\mu([-r,r])}\int_{[-r,r]}\left(\int_t^{t+1}\|h_n(s)\|^p\,\mathrm{d}s\right)^{\frac{1}{p}}\mathrm{d}\mu(t). \end{aligned}$$

It follows that

$$\limsup_{r\to\infty}\frac{1}{\mu([-r,r])}\int_{[-r,r]}\left(\int_t^{t+1}\|h(s)\|^p\,\mathrm{d}s\right)^{\frac{1}{p}}\mathrm{d}\mu(t)\le\|h_n-h\|_{S^p}\quad\text{for all } n\in\mathbb{N}.$$

Since $\lim_{n\to\infty}\|h_n-h\|_{S^p}=0$, we deduce that

$$\lim_{r\to\infty}\frac{1}{\mu([-r,r])}\int_{[-r,r]}\left(\int_t^{t+1}\|h(s)\|^p\,\mathrm{d}s\right)^{\frac{1}{p}}\mathrm{d}\mu(t)=0,$$

that is, $f = g + h \in PAA^p(\mathbb{R}, \mathbb{X}, \mu)$. So $PAA^p(\mathbb{R}, \mathbb{X}, \mu, \|\cdot\|_{S^p})$ is a Banach space. □

From Theorem 2.5 and the proofs of [15, Theorem 4.7], we have the following result.

Theorem 2.7 *Let $\mu \in \mathfrak{M}$. Assume that $PAA^p(\mathbb{R}, \mathbb{X}, \mu)$ is translation invariant. Then the decomposition of a μ-S^p-pseudo almost automorphic function in the form $f = g + h$, where $g \in AS^p(\mathbb{X})$ and $h^b \in \varepsilon(L^p(0, 1; \mathbb{X}), \mu)$, is unique.*

Lemma 2.7 [11] *Assume that $f \in AS^p(\mathbb{R}\times\mathbb{X}, \mathbb{X})$ and $f(t, x)$ is uniformly continuous on each bounded subset $K' \subset \mathbb{X}$ uniformly for $t \in \mathbb{R}$. If $u \in AS^p(\mathbb{X})$ and $K = \overline{\{u(t) : t \in \mathbb{R}\}}$ is compact. Then $f(\cdot, u(\cdot)) \in AS^p(\mathbb{X})$.*

(H1) There exists a constant $L > 0$ such that for all $u, v \in \mathbb{X}$ and $t \in \mathbb{R}$,

$$\|f(t,u)-f(t,v)\|\le L\|u-v\|.$$

Lemma 2.8 [21] *Suppose that* $f \in AS^p(\mathbb{R} \times \mathbb{X}, \mathbb{X})$ *and the following condition holds.*
(**H2**) *There exists a constant* $L > 0$ *such that for all* $u, v \in \mathbb{X}$ *and* $t \in \mathbb{R}$,

$$\left(\int_t^{t+1} \| f(t, u) - f(t, v) \|^p \, ds \right)^{\frac{1}{p}} \le L \| u - v \|.$$

If $u \in AS^p(\mathbb{X})$ *and* $K_1 = \overline{\{u(t) : t \in \mathbb{R}\}}$ *is compact. Then* $f(\cdot, u(\cdot)) \in AS^p(\mathbb{X})$.

Lemma 2.9 [21] *Suppose that* $f = g + h \in PAA^p(\mathbb{R} \times \mathbb{X}, \mathbb{X})$ *with* $g \in AS^p(\mathbb{X})$, $h^b \in AA_0(L^p(0, 1; \mathbb{X})$ *and* f *satisfies condition* (*H1*)*, then the function* g *satisfies condition* (*H2*).

Now, we recall a useful compactness criterion.
Let $h : \mathbb{R} \to \mathbb{R}$ be a continuous function such that $h(t) \ge 1$ for all $t \in \mathbb{R}$ and $h(t) \to \infty$ as $|t| \to \infty$. We consider the space

$$C_h(\mathbb{X}) = \left\{ u \in C(\mathbb{R}, \mathbb{X}) : \lim_{|t| \to \infty} \frac{u(t)}{h(t)} = 0 \right\}.$$

Endowed with the norm $\|u\|_h = \sup_{t \in \mathbb{R}} \frac{\|u(t)\|}{h(t)}$, it is a Banach space (see [22]).

Lemma 2.10 [22] *A subset* $R \subseteq C_h(\mathbb{X})$ *is a relatively compact set if it verifies the following conditions:*

(c-1) *The set* $R(t) = \{u(t) : u \in R\}$ *is relatively compact in* $\mathbb{X}$ *for each* $t \in \mathbb{R}$.
(c-2) *The set* R *is equicontinuous.*
(c-3) *For each* $\epsilon > 0$ *there exists* $L > 0$ *such that* $\|u(t)\| \le \epsilon h(t)$ *for all* $u \in R$ *and all* $|t| > L$.

Lemma 2.11 [23] (Leray–Schauder alternative theorem) *Let* D *be a closed convex subset of a Banach space* $\mathbb{X}$ *such that* $0 \in D$. *Let* $F : D \to D$ *be a completely continuous map. Then the set* $\{x \in D : x = \lambda F(x), 0 < \lambda < 1\}$ *is unbounded or the map* F *has a fixed point in* D.

3 Composition Theorems of μ-S^p-Pseudo Almost Automorphic Functions

In this section, we prove some composition theorems for μ-Stepanov-like pseudo almost automorphic functions under suitable conditions.

Theorem 3.1 *Let* $\mu \in \mathfrak{M}$. *Suppose that* $f = g + h \in PAA^p(\mathbb{R} \times \mathbb{X}, \mathbb{X}, \mu)$ *with* $g \in AS^p(\mathbb{R} \times \mathbb{X}, \mathbb{X})$, $h^b \in \varepsilon(\mathbb{X}, L^p(0, 1; \mathbb{X}), \mu)$ *and* (*H1*) *holds. If* $\varphi = \alpha + \beta \in PAA^p(\mathbb{R}, \mathbb{X}, \mu)$ *with* $\alpha \in AS^p(\mathbb{X})$, $\beta^b \in \varepsilon(L^p(0, 1; \mathbb{X}), \mu)$ *and* $K_1 = \overline{\{\alpha(t); t \in \mathbb{R}\}}$ *is compact. Then* $f(\cdot, \varphi(\cdot)) \in PAA^p(\mathbb{R}, \mathbb{X}, \mu)$.

Proof Let $f(t,u) = g(t,u) + h(t,u)$, where $g \in AS^p(\mathbb{R}\times\mathbb{X},\mathbb{X})$, and $h^b \in \varepsilon\left(\mathbb{X}, L^p(0,1;\mathbb{X}),\mu\right)$. Moreover, let $\varphi(t) = \alpha(t) + \beta(t)$, where $\alpha \in AS^p(\mathbb{X})$, and $\beta^b \in \varepsilon\left(L^p(0,1;\mathbb{X}),\mu\right)$. It is easily verified that

$$\begin{aligned} f\left(t,\varphi(t)\right) &= g\left(t,\alpha(t)\right) + f\left(t,\varphi(t)\right) - g\left(t,\alpha(t)\right) \\ &= g\left(t,\alpha(t)\right) + f\left(t,\varphi(t)\right) - f\left(t,\alpha(t)\right) + h\left(t,\alpha(t)\right). \end{aligned}$$

Define

$$G(t) = g\left(t,\alpha(t)\right), \quad F(t) = f\left(t,\varphi(t)\right) - f\left(t,\alpha(t)\right), \quad H(t) = h\left(t,\alpha(t)\right).$$

Firstly, we show that $G(t) \in AS^p(\mathbb{X})$. In fact, by the same reason of Lemma 2.9, we have that the function g satisfies condition (H2). Note that $g \in AS^p(\mathbb{R}\times\mathbb{X},\mathbb{X})$, $\alpha \in AS^p(\mathbb{X})$ and $K_1 = \overline{\{\alpha(t) : t \in \mathbb{R}\}}$ is compact. Thus, by Lemma 2.8, we obtain $G(t) \in AS^p(\mathbb{X})$.

Secondly, we claim that $F^b(t) \in \varepsilon\left(L^p(0,1;\mathbb{X}),\mu\right)$. Actually, by (H1), we have

$$\begin{aligned} \left(\int_t^{t+1} \|F(s)\|^p \,\mathrm{d}s\right)^{\frac{1}{p}} &= \left(\int_t^{t+1} \|f\left(s,\varphi(s)\right) - f\left(s,\alpha(s)\right)\|^p \,\mathrm{d}s\right)^{\frac{1}{p}} \\ &\le L\left(\int_t^{t+1} \|\varphi(s) - \alpha(s)\|^p \,\mathrm{d}s\right)^{\frac{1}{p}} \\ &\le L\left(\int_t^{t+1} \|\beta(s)\|^p \,\mathrm{d}s\right)^{\frac{1}{p}}, \end{aligned}$$

thus, for $r > 0$,

$$\begin{aligned} \frac{1}{\mu([-r,r])}\int_{[-r,r]}\left(\int_t^{t+1} \|F(s)\|^p \,\mathrm{d}s\right)^{\frac{1}{p}} \mathrm{d}\mu(t) \le{}& \frac{L}{\mu([-r,r])}\int_{[-r,r]} \\ &\times\left(\int_t^{t+1} \|\beta(s)\|^p \mathrm{d}s\right)^{\frac{1}{p}} \mathrm{d}\mu(t). \end{aligned}$$

Note that $\beta^b \in \varepsilon\left(L^p(0,1;\mathbb{X}),\mu\right)$, we have

$$\lim_{r\to+\infty} \frac{1}{\mu([-r,r])}\int_{[-r,r]}\left(\int_t^{t+1} \|F(s)\|^p \,\mathrm{d}s\right)^{\frac{1}{p}} \mathrm{d}\mu(t) = 0,$$

which implies $F^b(\cdot) \in \varepsilon\left(L^p(0,1;\mathbb{X}),\mu\right)$.

Finally, we also claim that $H^b(\cdot) \in \varepsilon\left(L^p(0,1;\mathbb{X}),\mu\right)$. In fact, let $\epsilon > 0$. Since g satisfies condition (H2), there is a $\delta > 0$ such that

$$\left(\int_t^{t+1} \|g(s,u) - g(s,v)\|^p \,\mathrm{d}s\right)^{\frac{1}{p}} \le \epsilon$$

for all $t \in \mathbb{R}$, $u, v \in \mathbb{X}$ with $\|u - v\| \le \delta$. Put $\delta_0 = \min\{\epsilon, \delta\}$. Then

$$\begin{aligned}&\left(\int_t^{t+1} \|h(s,u) - h(s,v)\|^p \, \mathrm{d}s\right)^{\frac{1}{p}} \\ &\le \left(\int_t^{t+1} \|f(s,u) - f(s,v)\|^p \, \mathrm{d}s\right)^{\frac{1}{p}} + \left(\int_t^{t+1} \|g(s,u) - g(s,v)\|^p \, \mathrm{d}s\right)^{\frac{1}{p}} \\ &\le (L+1)\epsilon \end{aligned} \tag{3.1}$$

for all $t \in \mathbb{R}$, $u, v \in \mathbb{X}$ with $\|u - v\| \le \delta_0$.

Since $K_1 = \overline{\{\alpha(t) : t \in \mathbb{R}\}}$ is compact, there are finite open balls $U_k (k = 1, 2, \ldots, m)$ with center $x_k \in K_1$ and radius δ_0 (small enough) such that

$$\{\alpha(t) : t \in \mathbb{R}\} \subset \bigcup_{k=1}^{m} U_k.$$

Define and choose D_k such that

$$D_k = \{s \in \mathbb{R} : \alpha(s) \in U_k\}, \quad \mathbb{R} = \bigcup_{k=1}^{m} D_k,$$

and let

$$J_1 = D_1, \quad J_k = D_k \setminus \bigcup_{j=1}^{k-1} D_j \ (2 \le k \le m).$$

Then

$$J_i \cap J_j = \emptyset, \quad \text{when} \quad i \ne j, \ 1 \le i, j \le m.$$

Define the step function $\overline{x} : \mathbb{R} \to \mathbb{X}$ by $\overline{x}(s) = x_k, \ s \in J_k, \ k = 1, 2 \ldots, m$. It is easy to see that $\|\alpha(s) - \overline{x}(s)\| \le \delta_0$ for all $s \in \mathbb{R}$. It follows from (3.1) that

$$\begin{aligned}&\frac{1}{\mu([-r,r])} \int_{[-r,r]} \left(\int_t^{t+1} \|H(s)\|^p \, \mathrm{d}s\right)^{\frac{1}{p}} \mathrm{d}\mu(t) \\ &= \frac{1}{\mu([-r,r])} \int_{[-r,r]} \left(\int_t^{t+1} \|h(s,\alpha(s))\|^p \, \mathrm{d}s\right)^{\frac{1}{p}} \mathrm{d}\mu(t) \\ &\le \frac{1}{\mu([-r,r])} \int_{[-r,r]} \Bigg[\left(\int_t^{t+1} \|h(s,\alpha(s)) - h(s,\overline{x}(s))\|^p \, \mathrm{d}s\right)^{\frac{1}{p}} \\ &\quad + \left(\int_t^{t+1} \|h(s,\overline{x}(s))\|^p \, \mathrm{d}s\right)^{\frac{1}{p}}\Bigg] \mathrm{d}\mu(t)\end{aligned}$$

$$\leq (L+1)\epsilon + \frac{1}{\mu([-r,r])}\int_{[-r,r]}\left(\sum_{k=1}^{m}\int_{[t,t+1]\cap J_k}\|h(s,x_k)\|^p\,\mathrm{d}s\right)^{\frac{1}{p}}\mathrm{d}\mu(t)$$

$$\leq (L+1)\epsilon + \sum_{k=1}^{m}\frac{1}{\mu([-r,r])}\int_{[-r,r]}\left(\int_t^{t+1}\|h(s,x_k)\|^p\,\mathrm{d}s\right)^{\frac{1}{p}}\mathrm{d}\mu(t).$$

Using the arbitrariness of ϵ and $h^b \in \varepsilon\,(\mathbb{X}, L^p(0,1;\mathbb{X}),\mu)$, we obtain that

$$\lim_{r\to\infty}\frac{1}{\mu([-r,r])}\int_{[-r,r]}\left(\int_t^{t+1}\|H(s)\|^p\,\mathrm{d}s\right)^{\frac{1}{p}}\mathrm{d}\mu(t)=0.$$

That is, $H^b(\cdot)\in\varepsilon\,(L^p(0,1;\mathbb{X}),\mu)$. This completes the proof. □

Lemma 3.1 *Let $\mu \in \mathfrak{M}$. Assume that $x(t) \in AS^p(\mathbb{X})$, $K_2 = \overline{\{x(t) : t \in \mathbb{R}\}}$ is a compact subset of $\mathbb{X}$, and $f^b \in \varepsilon\,(\mathbb{X}, L^p(0,1;\mathbb{X}),\mu)$ satisfying that* $\forall\epsilon > 0$, $\exists\,\delta > 0$ *and $L(\cdot) \in BS^p(\mathbb{R})$ with $p > 1$ such that*

$$\left(\int_t^{t+1}\|f(s,x)-f(s,y)\|^p\,\mathrm{d}s\right)^{\frac{1}{p}} < L(t)\epsilon, \tag{3.2}$$

for all $x, y \in K_2$ with $\|x-y\| < \delta$. Then

$$\lim_{r\to\infty}\frac{1}{\mu([-r,r])}\int_{[-r,r]}\left(\int_t^{t+1}\|f(s,x(s))\|^p\,\mathrm{d}s\right)^{\frac{1}{p}}\mathrm{d}\mu(t)=0$$

whenever

$$\lim_{r\to\infty}\frac{1}{\mu([-r,r])}\int_{[-r,r]}L(t)\,\mathrm{d}\mu(t)<\infty. \tag{3.3}$$

Proof For $\forall\epsilon > 0$, let δ and $L(t)$ be as in the assumptions and $\delta_0 = \min\{\epsilon,\delta\}$ since K_2 is compact, there are finite open balls $O_k (k = 1, 2, \ldots, m)$ with center x_k and radius δ_0 such that

$$\{x(t) : t \in \mathbb{R}\} \subset \bigcup_{k=1}^{m} O_k.$$

Define and choose B_k, such that

$$B_k = \{t \in \mathbb{R} : \|x(t) - x_k\| < \delta_0\}, \quad k = 1, 2, \ldots, m.$$

Then $\mathbb{R} = \bigcup_{k=1}^{m} B_k$, and let $E_1 = B_1,\ E_k = B_k \setminus (\cup_{i=1}^{k-1} B_i)\ (2 \leq k \leq m)$. Then $\mathbb{R} = \cup_{k=1}^{m} E_k$ *and* $E_i \bigcap E_j = \emptyset, i \neq j,\ 1 \leq i, j \leq m$. Define the step function $\overline{x} : \mathbb{R} \to \mathbb{X}$, by $\overline{x}(t) = x_k$ for $t \in E_k, k = 1, 2, \ldots, m$. It is easy to see that $\|x(t) - \overline{x}(t)\| < \delta_0$, for all $t \in \mathbb{R}$. By the definition of $\varepsilon\,(L^p(0,1;\mathbb{X}),\mu)$, for the above $\epsilon > 0$, there is constant $r_0 > 0$ such that for all $r > r_0$ and $1 \leq k \leq m$,

$$\frac{1}{\mu([-r,r])}\int_{[-r,r]}\left(\int_t^{t+1}\|f(s,x_k)\|^p\,\mathrm{d}s\right)^{\frac{1}{p}}\,\mathrm{d}\mu(t)<\frac{\epsilon}{m}, \tag{3.4}$$

Then, by (3.2) we have

$$\begin{aligned}\left(\int_t^{t+1}\|f(s,x(s))\|^p\,\mathrm{d}s\right)^{\frac{1}{p}} &\leq \left(\int_t^{t+1}\|f(s,x(s))-f(s,\overline{x}(s))\|^p\,\mathrm{d}s\right)^{\frac{1}{p}}\\ &\quad+\left(\int_t^{t+1}\|f(s,\overline{x}(s))\|^p\,\mathrm{d}s\right)^{\frac{1}{p}}\\ &\leq L(t)\epsilon+\left(\sum_{k=1}^{m}\int_{E_k\bigcap[t,t+1]}\|f(s,x_k)\|^p\,\mathrm{d}s\right)^{\frac{1}{p}}.\end{aligned}$$

Now combining (3.3), (3.4) and the above inequality, we get

$$\begin{aligned}&\frac{1}{\mu([-r,r])}\int_{[-r,r]}\left(\int_t^{t+1}\|f(s,x(s))\|^p ds\right)^{\frac{1}{p}}d\mu(t)\\ &\leq \frac{\epsilon}{\mu([-r,r])}\int_{[-r,r]}L(t)d\mu(t)\\ &\quad+\frac{1}{\mu([-r,r])}\int_{[-r,r]}\sum_{k=1}^{m}\left(\int_{E_k\bigcap[t,t+1]}\|f(s,x_k)\|^p ds\right)^{\frac{1}{p}}d\mu(t)\\ &\leq \frac{\epsilon}{\mu([-r,r])}\int_{[-r,r]}L(t)d\mu(t)\\ &\quad+\sum_{k=1}^{m}\frac{1}{\mu([-r,r])}\int_{[-r,r]}\left(\int_t^{t+1}\|f(s,x_k)\|^p ds\right)^{\frac{1}{p}}d\mu(t)\\ &\leq \frac{\epsilon}{\mu([-r,r])}\int_{[-r,r]}L(t)d\mu(t)+\sum_{k=1}^{m}\frac{\epsilon}{m}\\ &\leq \frac{\epsilon}{\mu([-r,r])}\int_{[-r,r]}L(t)d\mu(t)+\epsilon\\ &\leq \left(\frac{1}{\mu([-r,r])}\int_{[-r,r]}L(t)d\mu(t)+1\right)\epsilon.\end{aligned}$$

For all $r>r_0$, we have

$$\lim_{r\to\infty}\frac{1}{\mu([-r,r])}\int_{[-r,r]}\left(\int_t^{t+1}\|f(s,x(s))\|^p\,\mathrm{d}s\right)^{\frac{1}{p}}\,\mathrm{d}\mu(t)=0.$$

□

Theorem 3.2 *Let $\mu \in \mathfrak{M}$ and let $f = g + h \in PAA^p(\mathbb{R} \times \mathbb{X}, \mathbb{X}, \mu)$ with $g \in AS^p(\mathbb{R} \times \mathbb{X}, \mathbb{X})$, $h^b \in \varepsilon(\mathbb{X}, L^p(0, 1; \mathbb{X}), \mu)$. Assume that the following conditions are satisfied*:

(i) *There exists a nonnegative function $L(\cdot) \in BS^p(\mathbb{R})$ satisfying (3.3) with $p > 1$ such that for all $u, v \in \mathbb{X}$ and $t \in \mathbb{R}$,*

$$\left(\int_t^{t+1} \|f(s,u) - f(s,v)\|^p \, \mathrm{d}s\right)^{\frac{1}{p}} < L(t)\|u - v\|.$$

(ii) *$g(t, x)$ is uniformly continuous in any bounded subset $K' \subset \mathbb{X}$ uniformly for $t \in \mathbb{R}$. If $u = u_1 + u_2 \in PAA^p(\mathbb{R}, \mathbb{X}, \mu)$, with $u_1 \in AS^p(\mathbb{X})$, $u_2^b \in \varepsilon(L^p(0, 1; \mathbb{X}), \mu)$ and $K_2 = \overline{\{u_1(t) : t \in \mathbb{R}\}}$ is compact, then $f(\cdot, u(\cdot))$ belongs to $PAA^p(\mathbb{R}, \mathbb{X}, \mu)$.*

Proof Since $f \in PAA^p(\mathbb{R} \times \mathbb{X}, \mathbb{X}, \mu)$ and $u(t) \in PAA^p(\mathbb{R}, \mathbb{X}, \mu)$, we have by definition that $f = g + h$ and $u = u_1 + u_2$ where $g \in AS^p(\mathbb{R} \times \mathbb{X}, \mathbb{X})$, $h^b \in \varepsilon(\mathbb{X}, L^p(0, 1; \mathbb{X}), \mu)$, $u_1 \in AS^p(\mathbb{X})$, and $u_2^b \in \varepsilon(L^p(0, 1; \mathbb{X}), \mu)$. Now, the function f can be decomposed as

$$\begin{aligned} f(t, u(t)) &= g(t, u_1(t)) + f(t, u(t)) - g(t, u_1(t)) \\ &= g(t, u_1(t)) + f(t, u(t)) - f(t, u_1(t)) + h(t, u_1(t)). \end{aligned}$$

Define

$$G(t) = g(t, u_1(t)), \quad F(t) = f(t, u(t)) - f(t, u_1(t)), \quad H(t) = h(t, u_1(t)).$$

Then $f(t, u(t)) = G(t) + F(t) + H(t)$. Since the function g satisfies condition (ii) and $K_2 = \overline{\{u_1(t) : t \in \mathbb{R}\}}$ is compact, it follows from Lemma 2.7 that the function $g(\cdot, u_1(\cdot)) \in AS^p(\mathbb{X})$. To show that $f(\cdot, u(\cdot)) \in PAA^p(\mathbb{R}, \mathbb{X}, \mu)$, it is sufficient to show that $F^b + H^b \in \varepsilon(L^p(0, 1; \mathbb{X}), \mu)$. First, we prove that $F^b \in \varepsilon(L^p(0, 1; \mathbb{X}), \mu)$. It is easy to see that $F(\cdot) \in BS^p(\mathbb{X})$. Assume that $\|F(t)\|_{S^p} \le M$ for $t \in \mathbb{R}$. For any $\epsilon > 0$, by (i) and $I = \emptyset$, we have

$$\begin{aligned} &\frac{1}{\mu([-r,r])} \int_{[-r,r]} \left(\int_t^{t+1} \|F(s)\|^p ds\right)^{\frac{1}{p}} d\mu(t) \\ &= \frac{1}{\mu([-r,r])} \int_{A_r^\epsilon(u_2)} \left(\int_t^{t+1} \|F(s)\|^p ds\right)^{\frac{1}{p}} d\mu(t) \\ &\quad + \frac{1}{\mu([-r,r])} \int_{B_r^\epsilon(u_2)} \left(\int_t^{t+1} \|f(s, u(s)) - f(s, u_1(s))\|^p ds\right)^{\frac{1}{p}} d\mu(t) \\ &\le M \frac{\mu(A_r^\epsilon(u_2))}{\mu([-r,r])} + \frac{1}{\mu([-r,r])} \int_{B_r^\epsilon(u_2)} L(t) \left(\int_t^{t+1} \|u_2(s)\|^p ds\right)^{\frac{1}{p}} d\mu(t) \\ &\le M \frac{\mu(A_r^\epsilon(u_2))}{\mu([-r,r])} + \epsilon \cdot \frac{1}{\mu([-r,r])} \int_{[-r,r]} L(t) d\mu(t), \end{aligned}$$

where I, $A_r^\epsilon(u_2)$, $B_r^\epsilon(u_2)$ are given in Theorem 2.1.

On the other hand, it follows from Theorem 2.1 that

$$\lim_{r\to\infty}\frac{\mu(A_r^{\epsilon}(u_2))}{\mu([-r,r])}=0.$$

So we get

$$\lim_{r\to\infty}\frac{1}{\mu([-r,r])}\int_{[-r,r]}\left(\int_t^{t+1}\|F(s)\|^p\,\mathrm{d}s\right)^{\frac{1}{p}}\,\mathrm{d}\mu(t)=0.$$

Therefore, $F^b \in \varepsilon\,(L^p(0,1;\mathbb{X}),\mu)$. Next we prove that $H^b \in \varepsilon\,(L^p(0,1;\mathbb{X}),\mu)$. $K_2 = \overline{\{u_1(t) : t\in\mathbb{R}\}}$ is compact in $\mathbb{X}$, and $g(t,x)$ is uniformly continuous in any bounded subset $K^{'}\subset\mathbb{X}$ uniformly for $t\in\mathbb{R}$. Thus for any $\epsilon>0$, there is a constant $\delta\in(0,\epsilon)$ such that

$$\left(\int_t^{t+1}\|g(s,u)-g(s,v)\|^p ds\right)^{\frac{1}{p}}<\epsilon,$$

$t\in\mathbb{R}$, $u,v\in K_2$ with $\|u-v\|\le\delta$. By (i), we have

$$\begin{aligned}\left(\int_t^{t+1}\|h(s,u)-h(s,v)\|^p\,\mathrm{d}s\right)^{\frac{1}{p}} &\le\left(\int_t^{t+1}\|f(s,u)-f(s,v)\|^p\,\mathrm{d}s\right)^{\frac{1}{p}}\\ &\quad+\left(\int_t^{t+1}\|g(s,u)-g(s,v)\|^p\,\mathrm{d}s\right)^{\frac{1}{p}}\\ &\le (L(t)+1)\epsilon.\end{aligned}$$

For all $t\in\mathbb{R}$ and $u,v\in K_2$ with $\|u-v\|\le\delta$. Noting that $(L(t)+1)\in BS^p(\mathbb{R})$, we know from Lemma 3.1 that

$$\lim_{r\to\infty}\frac{1}{\mu([-r,r])}\int_{[-r,r]}\left(\int_t^{t+1}\|h\left(s,u_1(s)\right)\|^p\,\mathrm{d}s\right)^{\frac{1}{p}}\,\mathrm{d}\mu(t)=0,$$

which means that $H^b\in\varepsilon\,(L^p(0,1;\mathbb{X}),\mu)$. This completes the proof. □

Theorem 3.3 *Let* $\mu\in\mathfrak{M}$ *and let* $f := g+\phi \in PAA^p\,(\mathbb{R}\times\mathbb{X},\mathbb{X},\mu)$ *with* $g\in AS^p\,(\mathbb{R}\times\mathbb{X},\mathbb{X})$*, and* $\phi^b\in\varepsilon\,(\mathbb{X},L^p(0,1;\mathbb{X}),\mu)$. *Assume that the following conditions are satisfied:*

(1) $f(t,x)$ *is uniformly continuous in any bounded subset* $K^{'}\subset\mathbb{X}$ *uniformly for* $t\in\mathbb{R}$.
(2) $g(t,x)$ *is uniformly continuous in any bounded subset* $K^{'}\subset\mathbb{X}$ *uniformly for* $t\in\mathbb{R}$.
(3) *For every bounded subset* $K^{'}\subset\mathbb{X}$,$\{f(\cdot,x):x\in K^{'}\}$ *is bounded in* $PAA^p(\mathbb{R}\times\mathbb{X},\mathbb{X},\mu)$.

If $x = \alpha + \beta \in PAA^p(\mathbb{R}, \mathbb{X}, \mu) \cap B(\mathbb{R}, \mathbb{X})$*, with* $\alpha \in AS^p(\mathbb{X})$*,* $\beta^b \in \varepsilon(L^p(0, 1; \mathbb{X}), \mu)$ *and* $Q = \overline{\{x(t) : t \in \mathbb{R}\}}$*,* $Q_1 = \overline{\{\alpha(t) : t \in \mathbb{R}\}}$ *are compact, then* $f(\cdot, x(\cdot))$ *belongs to* $PAA^p(\mathbb{R}, \mathbb{X}, \mu)$*.*

Proof Since $f \in PAA^p(\mathbb{R} \times \mathbb{X}, \mathbb{X}, \mu)$ and $x \in PAA^p(\mathbb{R}, \mathbb{X}, \mu)$, we have by definition that $f = g + \phi$ where $g \in AS^p(\mathbb{R} \times \mathbb{X}, \mathbb{X})$ and $\phi^b \in \varepsilon(\mathbb{X}, L^p(0, 1; \mathbb{X}), \mu)$. So, the function f can be written in the form:

$$\begin{aligned} f(t, x(t)) &= g(t, \alpha(t)) + f(t, x(t)) - g(t, \alpha(t)) \\ &= g(t, \alpha(t)) + f(t, x(t)) - f(t, \alpha(t)) + \phi(t, \alpha(t)). \end{aligned}$$

Define

$$G(t) = g(t, \alpha(t)), \quad H(t) = f(t, x(t)) - f(t, \alpha(t)), \quad \Lambda(t) = \phi(t, \alpha(t)).$$

Then $f(t, x(t)) = G(t) + H(t) + \Lambda(t)$. Since the function g satisfies condition (2) and $Q_1 = \overline{\{\alpha(t) : t \in \mathbb{R}\}}$ is compact, it follows from Lemma 2.7 that the function $g(\cdot, \alpha(\cdot)) \in AS^p(\mathbb{X})$. To show that $f(\cdot, x(\cdot)) \in PAA^p(\mathbb{R}, \mathbb{X}, \mu)$, it is enough to show that $H^b + \Lambda^b \in \varepsilon(L^p(0, 1; \mathbb{X}), \mu)$.

First, we prove that $H^b \in \varepsilon(L^p(0, 1; \mathbb{X}), \mu)$. Since $x(\cdot)$ and $\alpha(\cdot)$ are bounded, we can choose a bounded subset $K' \subseteq \mathbb{X}$, such that $x(\mathbb{R}), \alpha(\mathbb{R}) \subseteq K'$. Under assumption (3) that $H(\cdot) \in BS^p(\mathbb{X})$, from (1) we can see that f is uniformly continuous on the bounded subset $K' \subseteq \mathbb{X}$ uniformly for $t \in \mathbb{R}$. So given $\epsilon > 0$, there exists $\delta > 0$, such that $u, v \in K'$ and $\|u - v\| \leq \delta$ imply that $\|f(t, u) - f(t, v)\| \leq \epsilon$ for all $t \in \mathbb{R}$. Then we have

$$\left(\int_t^{t+1} \|f(s, u) - f(s, v)\|^p \, ds\right)^{\frac{1}{p}} \leq \epsilon.$$

Hence, for each $t \in \mathbb{R}$, $\|\beta(s)\|_{S^p} < \delta$, $s \in [t, t+1]$ implies that for all $t \in \mathbb{R}$,

$$\left(\int_t^{t+1} \|H(s)\|^p \, ds\right)^{\frac{1}{p}} = \left(\int_t^{t+1} \|f(s, x(s)) - f(s, \alpha(s))\|^p \, ds\right)^{\frac{1}{p}} \leq \epsilon.$$

Therefore, the following inequality holds:

$$\begin{aligned} &\frac{\mu\left\{t \in [-r, r] : \left(\int_t^{t+1} \|f(s, x(s)) - f(s, \alpha(s))\|^p \, ds\right)^{\frac{1}{p}} > \epsilon\right\}}{\mu([-r, r])} \\ &\leq \frac{\mu\left\{t \in [-r, r] : \left(\int_t^{t+1} \|\beta(s)\|^p \, ds\right)^{\frac{1}{p}} > \delta\right\}}{\mu([-r, r])}. \end{aligned}$$

Since β^b is μ-ergodic, Theorem 2.1 yields that for the above-mentioned δ we have

$$\lim_{r \to +\infty} \frac{\mu\left\{t \in [-r, r] : \left(\int_t^{t+1} \|\beta(s)\|^p \, ds\right)^{\frac{1}{p}} > \delta\right\}}{\mu([-r, r])} = 0,$$

and then we obtain

$$\lim_{r\to+\infty}\frac{\mu\left\{t\in[-r,r]:\left(\int_t^{t+1}\|f(s,x(s))-f(s,\alpha(s))\|^p\,\mathrm{d}s\right)^{\frac{1}{p}}>\epsilon\right\}}{\mu([-r,r])}=0. \quad (3.5)$$

With the help of Theorem 2.1, (3.5) shows that $t\to H^b$ is μ-ergodic.

Now to complete the proof, it is enough to prove that Λ^b is μ-ergodic. Since f, g satisfy conditions (1) and (2), then for any $\epsilon>0$, there exists $\delta>0$, such that $u, v\in Q_1$ imply that

$$\left(\int_t^{t+1}\|f(s,u)-f(s,v)\|^p\,\mathrm{d}s\right)^{\frac{1}{p}}<\frac{\epsilon}{16}\quad t\in\mathbb{R},$$

and

$$\left(\int_t^{t+1}\|g(s,u)-g(s,v)\|^p\,\mathrm{d}s\right)^{\frac{1}{p}}<\frac{\epsilon}{16}\quad t\in\mathbb{R}.$$

Now, we put $\delta_0=\min(\epsilon,\delta)$, then

$$\begin{aligned}\left(\int_t^{t+1}\|\phi(s,u)-\phi(s,v)\|^p\,\mathrm{d}s\right)^{\frac{1}{p}} &\le \left(\int_t^{t+1}\|f(s,u)-f(s,v)\|^p\,\mathrm{d}s\right)^{\frac{1}{p}}\\ &\quad+\left(\int_t^{t+1}\|g(s,u)-g(s,v)\|^p\,\mathrm{d}s\right)^{\frac{1}{p}}\\ &\le\frac{\epsilon}{8}\end{aligned}$$

for all $t\in\mathbb{R}$, and $u, v\in Q_1$ with $\|u-v\|\le\delta_0$.

Since $Q_1=\overline{\{\alpha(t):t\in\mathbb{R}\}}$ is compact, we find finite open balls $O_k(k=1,2,\ldots,m)$ with center $u_k\in Q_1$ and radius δ_0 given above, such that $\{\alpha(t):t\in\mathbb{R}\}\subset\cup_{k=1}^m O_k$. Define and choose $\mathfrak{B}_k$ such that $\mathfrak{B}_k=\{t\in\mathbb{R}:\|\alpha(t)-u_k\|<\delta_0\}$, $k=1,2,\ldots,m$, $\mathbb{R}=\cup_{k=1}^m\mathfrak{B}_k$, and set $\mathfrak{E}_1=\mathfrak{B}_1$, $\mathfrak{E}_k=\mathfrak{B}_k\setminus(\cup_{j=1}^{k-1}\mathfrak{B}_j)$ $(2\le k\le m)$. Then $\mathbb{R}=\cup_{k=1}^m\mathfrak{E}_k$ *and* $\mathfrak{E}_i\bigcap\mathfrak{E}_j=\emptyset, i\neq j,\ 1\le i,j\le m$. Define a function $\overline{u}:\mathbb{R}\to\mathbb{X}$ by $\overline{u}(t)=u_k$ for $t\in\mathfrak{E}_k,\ k=1,2,\ldots m$. Then $\|\alpha(t)-\overline{u}(t)\|<\delta_0$ for all $t\in\mathbb{R}$, it is easy to get from

$$\begin{aligned}&\left(\sum_{k=1}^m\int_{\mathfrak{E}_k\bigcap[t,t+1]}\|\phi(s,\alpha(s))-\phi(s,u_k)\|^p\,\mathrm{d}s\right)^{\frac{1}{p}}\\ &=\left(\int_t^{t+1}\|\phi(s,\alpha(s))-\phi(s,\overline{u}(s))\|^p\,\mathrm{d}s\right)^{\frac{1}{p}}\\ &<\frac{\epsilon}{8}.\end{aligned}$$

Since $\phi^b \in \varepsilon(\mathbb{X}, L^p(0,1;\mathbb{X}), \mu)$, there exists a constant $r_0 > 0$, such that

$$\frac{1}{\mu([-r,r])}\int_{[-r,r]}\left(\int_t^{t+1}\|\phi(s,u_k)\|^p\,\mathrm{d}s\right)^{\frac{1}{p}}\mathrm{d}\mu(t) < \frac{\epsilon}{8m^2}$$

for all $r > r_0$ and $1 \le k \le m$.

Now combing these estimates, we deduce that for all $r > r_0$

$$\begin{aligned}
&\frac{1}{\mu([-r,r])}\int_{[-r,r]}\left(\int_t^{t+1}\|\Lambda(s)\|^p\,\mathrm{d}s\right)^{\frac{1}{p}}\mathrm{d}\mu(t)\\
&=\frac{1}{\mu([-r,r])}\int_{[-r,r]}\left(\sum_{k=1}^{m}\left(\int_{\mathfrak{E}_k\bigcap[t,t+1]}\|\phi(s,\alpha(s))-\phi(s,u_k)+\phi(s,u_k)\|^p\,\mathrm{d}s\right)\right)^{\frac{1}{p}}\mathrm{d}\mu(t)\\
&\le\frac{1}{\mu([-r,r])}\int_{[-r,r]}\left[2^p\sum_{k=1}^{m}\left(\int_{\mathfrak{E}_k\bigcap[t,t+1]}\|\phi(s,\alpha(s))-\phi(s,u_k)\|^p\,\mathrm{d}s\right.\right.\\
&\quad\left.\left.+\int_{\mathfrak{E}_k\bigcap[t,t+1]}\|\phi(s,u_k)\|^p\,\mathrm{d}s\right)\right]^{\frac{1}{p}}\mathrm{d}\mu(t)\\
&\le\frac{2^{1+\frac{1}{p}}}{\mu([-r,r])}\int_{[-r,r]}\left(\int_t^{t+1}\|\phi(s,\alpha(s))-\phi(s,\overline{u}(s))\|^p\,\mathrm{d}s\right)^{\frac{1}{p}}\mathrm{d}\mu(t)\\
&\quad+\frac{2^{1+\frac{1}{p}}}{\mu([-r,r])}\int_{[-r,r]}\left(\sum_{k=1}^{m}\int_{\mathfrak{E}_k\bigcap[t,t+1]}\|\phi(s,u_k)\|^p\,\mathrm{d}s\right)^{\frac{1}{p}}\mathrm{d}\mu(t)\\
&<\frac{4}{\mu([-r,r])}\int_{[-r,r]}\frac{\epsilon}{8}\,\mathrm{d}\mu(t)+\sum_{k=1}^{m}\frac{4m^{\frac{1}{p}}}{\mu([-r,r])}\int_{[-r,r]}\left(\int_t^{t+1}\|\phi(s,u_k)\|^p\,\mathrm{d}s\right)^{\frac{1}{p}}\mathrm{d}\mu(t)\\
&<\frac{\epsilon}{2}+m^{\frac{1}{p}}\frac{\epsilon}{2m}<\epsilon,
\end{aligned}$$

which implies that $\Lambda^b \in \varepsilon(L^p(0,1;\mathbb{X}), \mu)$. This completes the proof. □

4 Existence of μ-Pseudo Almost Automorphic Solutions

In this section, we consider the existence of μ-pseudo almost automorphic mild solutions for the problem (1.1) under some suitable conditions.

Definition 4.1 A continuous function u is called a μ-pseudo almost automorphic mild solution of Eq. (1.1) on $\mathbb{R}$ if $u \in PAA(\mathbb{R}, \mathbb{X}, \mu)$ and $u(t)$ satisfies

$$u(t) = U(t,a)u(a) + \int_a^t U(t,s)f(s,u(s))\,\mathrm{d}s$$

for $t \ge a$.

First, we list the following basic assumptions:

In this paper, we assume that $\{A(t)\}_{t\in\mathbb{R}}$ satisfies the *Acquistapace–Terreni* conditions introduced in [16,24], that is,

(A1) there exist constants $\lambda_0 \geq 0$, $\theta \in \left(\frac{\pi}{2}, \pi\right)$, $\mathcal{L}$, $\mathcal{K} \geq 0$, and $\alpha, \beta \in (0, 1]$ with $\alpha + \beta > 1$ such that

$$\Sigma_\theta \cup \{0\} \subset \rho(A(t) - \lambda_0), \quad \|R(\lambda, A(t) - \lambda_0)\| \leq \frac{\mathcal{K}}{1 + |\lambda|}$$

and

$$\|(A(t) - \lambda_0)R(\lambda, A(t) - \lambda_0)[R(\lambda_0, A(t)) - R(\lambda_0, A(s))]\| \leq \mathcal{L}|t - s|^\alpha |\lambda|^{-\beta}$$

for $t, s \in \mathbb{R}$, $\lambda \in \Sigma_\theta := \{\lambda \in \mathbb{C} \setminus \{0\} : |\arg\lambda| \leq \theta\}$.

Remark 4.1 [16,25] If the condition (A1) holds, then there exists a unique evolution family $\{U(t, s)\}_{-\infty<s\leq t<\infty}$ on $\mathbb{X}$, which satisfies the homogeneous equation $u'(t) = A(t)u(t)$, $t \in \mathbb{R}$.

We further suppose that

(A2) the evolution family $U(t, s)$ generated by $A(t)$ is exponentially stable, that is, there are constants $K, \omega > 0$ such that $\|U(t, s)\| \leq K\,\mathrm{e}^{-\omega(t-s)}$ for all $t \geq s$. And the function $\mathbb{R} \times \mathbb{R} \mapsto \mathbb{X}$, $(t, s) \mapsto U(t, s)x \in bAA(\mathbb{R} \times \mathbb{R}, \mathbb{X})$ uniformly for all x in any bounded subset of $\mathbb{X}$.

(A3) There exists a constant $\mathcal{L}_f > 0$, such that

$$\|f(t, x) - f(t, y)\| \leq \mathcal{L}_f \|x - y\|$$

for all $t \in \mathbb{R}$ and each $x, y \in \mathbb{X}$.

(A4) There exists a nonnegative function $L_f(\cdot) \in BS^p(\mathbb{R})$ with $p > 1$ such that

$$\|f(t, x) - f(t, y)\| \leq L_f(t)\|x - y\|, \quad \lim_{r\to\infty} \frac{1}{\mu([-r, r])} \int_{[-r,r]} L_f(t)\,\mathrm{d}\mu(t) < \infty$$

for all $t \in \mathbb{R}$ and each $x, y \in \mathbb{X}$.

(A5) The function $f : \mathbb{R} \times \mathbb{X} \to \mathbb{X}$ satisfies the following conditions:

(I) There exists $\widetilde{L} > 0$ such that

$$M_f = \sup_{t\in\mathbb{R}, \|u\|\leq\widetilde{L}} \left(\int_t^{t+1} \|f(s, u(s))\|^p\,\mathrm{d}s\right)^{\frac{1}{p}} \leq \frac{\widetilde{L}}{\Delta(K, q, \omega)},$$

where $\Delta(K, q, \omega) = K \sqrt[q]{\frac{\mathrm{e}^{q\omega}-1}{q\omega}}\,\Sigma_{n=1}^\infty \mathrm{e}^{-\omega n}$.

(II) Let $\{x_n\} \subset PAA(\mathbb{R}, \mathbb{X}, \mu)$ be uniformly bounded in $\mathbb{R}$ and uniformly convergent in each compact subset of $\mathbb{R}$. Then $\{f(\cdot, x_n(\cdot))\}$ is relatively compact in $BS^p(\mathbb{X})$.

(A6) The function $f = g + h \in PAA^p(\mathbb{R} \times \mathbb{X}, \mathbb{X}, \mu)$ where $g \in AS^p(\mathbb{R} \times \mathbb{X}, \mathbb{X})$ is uniformly continuous in any bounded subset $M \subset \mathbb{X}$ uniformly in $t \in \mathbb{R}$ and $h^b \in \varepsilon(\mathbb{X}, L^p(0, 1; \mathbb{X}), \mu)$.

(A7) $f \in PAA^p(\mathbb{R} \times \mathbb{X}, \mathbb{X}, \mu)$ and $f(t, x)$ is uniformly continuous in any bounded subset $M \subset \mathbb{X}$ uniformly for $t \in \mathbb{R}$ and for every bounded subset $M \subset \mathbb{X}$, $\{f(\cdot, x) : x \in M\}$ is bounded in $PAA^p(\mathbb{R} \times \mathbb{X}, \mathbb{X}, \mu)$.

Consider the following abstract differential equation in the Banach space $(\mathbb{X}, \|\cdot\|)$

$$u'(t) = A(t)u(t) + f(t), \quad t \in \mathbb{R}, \tag{4.1}$$

where $\{A(t)\}_{t\in\mathbb{R}}$ satisfies the condition (A1) and $f \in PAA^p(\mathbb{R}, \mathbb{X}, \mu) \cap C(\mathbb{R}, \mathbb{X})$ for $p > 1$. Throughout this paper, we set $\frac{1}{q} = 1 - \frac{1}{p}$. Note that $q \neq 0$, as $p \neq 1$.

Lemma 4.1 *Let $\mu \in \mathfrak{M}$. Assume that (A1)–(A2) hold. Then the* Eq. (4.1) *admits a unique μ-pseudo almost automorphic mild solution given by*

$$u(t) = \int_{-\infty}^{t} U(t, \sigma) f(\sigma)\, \mathrm{d}\sigma. \tag{4.2}$$

Proof The proof of uniqueness has been given in [13]. Now let us investigate the existence. Since $f \in PAA^p(\mathbb{R}, \mathbb{X}, \mu)$, there exist $g \in AS^p(\mathbb{X})$ and $h^b \in \varepsilon(L^p(0, 1; \mathbb{X}), \mu)$ such that $f = g + h$. So

$$\begin{aligned} u(t) &= \int_{-\infty}^{t} U(t, \sigma) f(\sigma)\, \mathrm{d}\sigma = \int_{-\infty}^{t} U(t, \sigma) g(\sigma)\, \mathrm{d}\sigma + \int_{-\infty}^{t} U(t, \sigma) h(\sigma)\, \mathrm{d}\sigma \\ &= \Phi(t) + \Psi(t), \end{aligned}$$

where $\Phi(t) = \int_{-\infty}^{t} U(t, \sigma) g(\sigma)\, \mathrm{d}\sigma$, and $\Psi(t) = \int_{-\infty}^{t} U(t, \sigma) h(\sigma)\, \mathrm{d}\sigma$. We just need to verify $\Phi(t) \in AA(\mathbb{X})$ and $\Psi(t) \in \varepsilon(\mathbb{R}, \mathbb{X}, \mu)$. First we prove that $\Phi(t) \in AA(\mathbb{X})$. It follows from [5, Lemma 11.2] that $\Phi(t)$ is almost automorphic. Next, we prove that $\Psi(t) \in \varepsilon(\mathbb{R}, \mathbb{X}, \mu)$.

For this, we consider

$$\Psi_n(t) = \int_{t-n}^{t-n+1} U(t, \sigma) h(\sigma)\, \mathrm{d}\sigma,$$

for each $t \in \mathbb{R}$ and $n = 1, 2, 3 \ldots$. From assumption (A2) and Holder's inequality, it follows that

$$\begin{aligned} \|\Psi_n(t)\| &\leq K \int_{t-n}^{t-n+1} \mathrm{e}^{-\omega(t-\sigma)} \|h(\sigma)\|\, \mathrm{d}\sigma \\ &\leq K \left(\int_{t-n}^{t-n+1} \mathrm{e}^{-q\omega(t-\sigma)}\, \mathrm{d}\sigma \right)^{\frac{1}{q}} \left(\int_{t-n}^{t-n+1} \|h(\sigma)\|^p\, \mathrm{d}\sigma \right)^{\frac{1}{p}} \end{aligned}$$

$$\begin{aligned}
&\leq K\left(\int_{n-1}^{n} \mathrm{e}^{-q\omega\sigma}\,\mathrm{d}\sigma\right)^{\frac{1}{q}}\left(\int_{t-n}^{t-n+1}\|h(\sigma)\|^{p}\,\mathrm{d}\sigma\right)^{\frac{1}{p}}\\
&\leq \frac{K}{\sqrt[q]{q\omega}}\left(\mathrm{e}^{-q\omega(n-1)}-\mathrm{e}^{-q\omega n}\right)^{\frac{1}{q}}\left(\int_{t-n}^{t-n+1}\|h(\sigma)\|^{p}\,\mathrm{d}\sigma\right)^{\frac{1}{p}}\\
&\leq \frac{K\mathrm{e}^{-\omega n}}{\sqrt[q]{q\omega}}(\mathrm{e}^{q\omega}-1)^{\frac{1}{q}}\left(\int_{t-n}^{t-n+1}\|h(\sigma)\|^{p}\,\mathrm{d}\sigma\right)^{\frac{1}{p}}\\
&\leq \frac{K\mathrm{e}^{-\omega n}}{\sqrt[q]{q\omega}}(\mathrm{e}^{q\omega}+1)^{\frac{1}{q}}\left(\int_{t-n}^{t-n+1}\|h(\sigma)\|^{p}\,\mathrm{d}\sigma\right)^{\frac{1}{p}}.
\end{aligned}$$

Then for $r > 0$, we see that

$$\begin{aligned}
&\frac{1}{\mu([-r,r])}\int_{[-r,r]}\|\Psi_n(t)\|\,\mathrm{d}\mu(t)\\
&\leq \frac{K\mathrm{e}^{-\omega n}}{\sqrt[q]{q\omega}}(\mathrm{e}^{q\omega}+1)^{\frac{1}{q}}\frac{1}{\mu([-r,r])}\int_{[-r,r]}\left(\int_{t-n}^{t-n+1}\|h(\sigma)\|^{p}\,\mathrm{d}\sigma\right)^{\frac{1}{p}}\,\mathrm{d}\mu(t).
\end{aligned}$$

Since $h^b \in \varepsilon\,(L^p(0,1;\mathbb{X}),\mu)$, the above inequality leads to $\Psi_n \in \varepsilon\,(\mathbb{R},\mathbb{X},\mu)$. The above inequality leads also to

$$\|\Psi_n(t)\| \leq \frac{K\mathrm{e}^{-\omega n}}{\sqrt[q]{q\omega}}(\mathrm{e}^{q\omega}+1)^{\frac{1}{q}}\|h\|_{S^p}.$$

Since the series

$$\frac{K}{\sqrt[q]{q\omega}}(\mathrm{e}^{q\omega}+1)^{\frac{1}{q}}\times\sum_{n=1}^{\infty}\mathrm{e}^{-\omega n}$$

is convergent, then we deduce from the Weierstrass test that the series $\sum_{n=1}^{\infty}\Psi_n(t)$ is uniformly convergent on $\mathbb{R}$ and $\Psi(t)=\displaystyle\int_{-\infty}^{t}U(t,\sigma)h(\sigma)\,\mathrm{d}\sigma=\sum_{n=1}^{\infty}\Psi_n(t)$. Applying $\Psi_n \in \varepsilon\,(\mathbb{R},\mathbb{X},\mu)$ and the inequality

$$\begin{aligned}
\frac{1}{\mu([-r,r])}\int_{[-r,r]}\|\Psi(t)\|\,\mathrm{d}\mu(t) &\leq \frac{1}{\mu([-r,r])}\int_{[-r,r]}\|\Psi(t)-\sum_{k=1}^{n}\Psi_k(t)\|\,\mathrm{d}\mu(t)\\
&\quad+\sum_{k=1}^{n}\frac{1}{\mu([-r,r])}\int_{[-r,r]}\|\Psi_k(t)\|\,\mathrm{d}\mu(t),
\end{aligned}$$

we deduce that the uniform limit $\Psi(t)=\sum_{n=1}^{\infty}\Psi_n(t)\in\varepsilon\,(\mathbb{R},\mathbb{X},\mu)$. Therefore, $u(t)=\Phi(t)+\Psi(t)$ is μ-pseudo almost automorphic.

Finally, let us prove that $u(t)$ is a mild solution of the Eq. (4.1). Indeed, if we let

$$u(s) = \int_{-\infty}^{s} U(s,\sigma) f(\sigma)\, \mathrm{d}\sigma \tag{4.3}$$

and multiply both sides of (4.3) by $U(t,s)$, then

$$U(t,s)u(s) = \int_{-\infty}^{s} U(t,\sigma) f(\sigma)\, \mathrm{d}\sigma.$$

If $t \geq s$, then

$$\begin{aligned}\int_{s}^{t} U(t,\sigma) f(\sigma)\, \mathrm{d}\sigma &= \int_{-\infty}^{t} U(t,\sigma) f(\sigma)\, \mathrm{d}\sigma - \int_{-\infty}^{s} U(t,\sigma) f(\sigma)\, \mathrm{d}\sigma \\ &= u(t) - U(t,s)u(s).\end{aligned}$$

It follows that

$$u(t) = U(t,s)u(s) + \int_{s}^{t} U(t,\sigma) f(\sigma)\, \mathrm{d}\sigma.$$

□

This completes the proof of the theorem.

Theorem 4.1 *Let* $\mu \in \mathfrak{M}$. *Assume the condition (H0), (A1)–(A3) are satisfied and the function* $f = g + h \in PAA^p(\mathbb{R} \times \mathbb{X}, \mathbb{X}, \mu)$ *with* $g \in AS^p(\mathbb{R} \times \mathbb{X}, \mathbb{X})$, *and* $h^b \in \varepsilon(\mathbb{X}, L^p(0,1;\mathbb{X}), \mu)$. *Then* Eq. (1.1) *has a unique* μ*-pseudo almost automorphic mild solution on* $\mathbb{R}$ *provided that* $\frac{K\mathcal{L}_f}{\omega} < 1$.

Proof Let $\Gamma : PAA(\mathbb{R}, \mathbb{X}, \mu) \to PAA(\mathbb{R}, \mathbb{X}, \mu)$ be the nonlinear operator defined by

$$(\Gamma u)(t) = \int_{-\infty}^{t} U(t,s) f(s, u(s))\, \mathrm{d}s, \quad t \in \mathbb{R}.$$

First, let us prove that $\Gamma(PAA(\mathbb{R}, \mathbb{X}, \mu)) \subset PAA(\mathbb{R}, \mathbb{X}, \mu)$. For each $u \in PAA(\mathbb{R}, \mathbb{X}, \mu)$, by using the fact that the range of an almost automorphic function is relatively compact combined with the above Theorem 2.4 and Theorem 3.1, one can easily see that $f(\cdot, u(\cdot)) \in PAA^p(\mathbb{R}, \mathbb{X}, \mu)$. Hence, from the proof of Lemma 4.1, we know that $(\Gamma u)(\cdot) \in PAA(\mathbb{R}, \mathbb{X}, \mu)$. That is, Γ maps $PAA(\mathbb{R}, \mathbb{X}, \mu)$ into $PAA(\mathbb{R}, \mathbb{X}, \mu)$.

Now, let us prove that Γ has a unique fixed point. To this end, for each $t \in \mathbb{R}$, $u, v \in PAA(\mathbb{R}, \mathbb{X}, \mu)$, we have

$$\begin{aligned}\|(\Gamma u)(t) - (\Gamma v)(t)\| &\leq \int_{-\infty}^{t} \|U(t,s)[f(s,u(s)) - f(s,v(s))]\|\, \mathrm{d}s \\ &\leq K \int_{-\infty}^{t} \mathrm{e}^{-\omega(t-s)} \|f(s,u(s)) - f(s,v(s))\|\, \mathrm{d}s\end{aligned}$$

$$\leq K\mathcal{L}_f \int_{-\infty}^{t} \mathrm{e}^{-\omega(t-s)} \|u(s) - v(s)\| \,\mathrm{d}s$$
$$\leq K\mathcal{L}_f \int_{-\infty}^{t} \mathrm{e}^{-\omega(t-s)} \,\mathrm{d}s \|u - v\|_\infty$$
$$\leq \frac{K\mathcal{L}_f}{\omega} \|u - v\|_\infty.$$

So $\|\Gamma u - \Gamma v\|_\infty \leq \frac{K\mathcal{L}_f}{\omega}\|u - v\|_\infty$. Hence by the Banach contraction principle with $\frac{K\mathcal{L}_f}{\omega} < 1$, Γ has a unique fixed point u in $PAA(\mathbb{R}, \mathbb{X}, \mu)$ which is the μ-pseudo almost automorphic solution to Eq. (1.1). □

A different Lipschitz condition is considered in the following result.

Theorem 4.2 *Let $\mu \in \mathfrak{M}$. Assume that (H0), (A1), (A2), (A4), and (A6) hold, then* Eq. (1.1) *admits a unique μ-pseudo almost automorphic mild solution whenever* $\|L_f\|_{S^p} < \frac{1-\mathrm{e}^{-\omega}}{K}\left(\frac{\omega q}{1-\mathrm{e}^{-\omega q}}\right)^{\frac{1}{q}}$.

Proof Consider the nonlinear operator Γ given by

$$(\Gamma u)(t) = \int_{-\infty}^{t} U(t, s) f(s, u(s)) \,\mathrm{d}s, \quad t \in \mathbb{R}.$$

Let $u \in PAA(\mathbb{R}, \mathbb{X}, \mu)$, with Theorem 2.4 and Theorem 3.2, it follows that the function $s \to f(s, u(s))$ is in $PAA^p(\mathbb{R}, \mathbb{X}, \mu)$. Moreover, from Lemma 4.1, we infer that $\Gamma u \in PAA(\mathbb{R}, \mathbb{X}, \mu)$, that is, Γ maps $PAA(\mathbb{R}, \mathbb{X}, \mu)$ into itself. Next, we prove that the operator Γ has a unique fixed point in $PAA(\mathbb{R}, \mathbb{X}, \mu)$. Indeed, for each $t \in \mathbb{R}$, $u, v \in PAA(\mathbb{R}, \mathbb{X}, \mu)$, we have

$$\begin{aligned}
\|\Gamma u(t) - \Gamma v(t)\| &\leq \left\| \int_{-\infty}^{t} U(t, s)[f(s, u(s)) - f(s, v(s))] \,\mathrm{d}s \right\| \\
&\leq K \int_{-\infty}^{t} \mathrm{e}^{-\omega(t-s)} \|f(s, u(s)) - f(s, v(s))\| \,\mathrm{d}s \\
&\leq K \int_{-\infty}^{t} \mathrm{e}^{-\omega(t-s)} L_f(s) \,\mathrm{d}s \|u - v\|_\infty \\
&= \sum_{n=1}^{\infty} \int_{t-n}^{t-n+1} K\mathrm{e}^{-\omega(t-s)} L_f(s) \,\mathrm{d}s \|u - v\|_\infty \\
&\leq \sum_{n=1}^{\infty} \left(\int_{t-n}^{t-n+1} K^q \mathrm{e}^{-\omega q(t-s)} \,\mathrm{d}s \right)^{\frac{1}{q}} \|L_f\|_{S^p} \|u - v\|_\infty \\
&\leq \frac{K}{1 - \mathrm{e}^{-\omega}} \left(\frac{1 - \mathrm{e}^{-q\omega}}{\omega q} \right)^{\frac{1}{q}} \|L_f\|_{S^p} \|u - v\|_\infty,
\end{aligned}$$

which gives

$$\|(\Gamma u)(t) - (\Gamma v)(t)\|_\infty \leq \frac{K}{1-\mathrm{e}^{-\omega}} \left(\frac{1-\mathrm{e}^{-q\omega}}{\omega q}\right)^{\frac{1}{q}} \|L_f\|_{S^p} \|u - v\|_\infty.$$

Since $\|L_f\|_{S^p} < \frac{1-\mathrm{e}^{-\omega}}{K} \left(\frac{\omega q}{1-\mathrm{e}^{-\omega q}}\right)^{\frac{1}{q}}$, Γ has a unique fixed point $u \in PAA(\mathbb{R}, \mathbb{X}, \mu)$. □

We next study the existence of μ-pseudo almost automorphic mild solutions of Eq. (1.1) when the perturbation f is not Lipschitz continuous.

Theorem 4.3 *Let $\mu \in \mathfrak{M}$. Assume that the conditions (H0), (A1)–(A2), and (A5)–(A7) are satisfied, and moreover $U(t, s)$ is compact for $t > s$. Then the problem (1.1) has at least one μ-pseudo almost automorphic mild solution on $\mathbb{R}$.*

Proof Consider the nonlinear operator Γ given by

$$(\Gamma x)(t) = \int_{-\infty}^{t} U(t, s) f(s, x(s))\, \mathrm{d}s, \quad t \in \mathbb{R}.$$

First, we show that the nonlinear operator Γ is well defined and continuous. From Theorem 2.4 and Theorem 3.3, we can see that $f(s, x(s)) \in PAA^p(\mathbb{R}, \mathbb{X}, \mu)$. Hence from Lemma 4.1, we can see that $(\Gamma x)(\cdot) \in PAA(\mathbb{R}, \mathbb{X}, \mu)$, that is, Γ maps $PAA(\mathbb{R}, \mathbb{X}, \mu)$ into $PAA(\mathbb{R}, \mathbb{X}, \mu)$.

Now, let us show that Γ is continuous on $PAA(\mathbb{R}, \mathbb{X}, \mu)$. Let $\{x_n\} \subset PAA(\mathbb{R}, \mathbb{X}, \mu)$ be a sequence which converges to some $x \in PAA(\mathbb{R}, \mathbb{X}, \mu)$, that is $\|x_n - x\| \to 0$ as $n \to \infty$. We may find a bounded subset $M \subset \mathbb{X}$ such that $x_n(t), x(t) \in M$ for $t \in \mathbb{R}, n = 1, 2, \ldots$. By (A7), for any $\epsilon > 0$, there exists $\omega > 0$ such that $u, v \in M$ and $\|u - v\| < \omega$ imply that

$$\|f(t, u) - f(t, v)\| < \frac{\omega\epsilon}{K} \quad \text{for each } t \in \mathbb{R},$$

where ω, K are given in (A2). For the above $\omega > 0$, there exists $N > 0$ such that $\|x_n(t) - x(t)\| < \omega$ for all $n > N$ and all $t \in \mathbb{R}$. Therefore,

$$\|f(t, x_n(t)) - f(t, x(t))\| < \frac{\omega\epsilon}{K} \quad \text{for each } t \in \mathbb{R},$$

for all $n > N$ and all $t \in \mathbb{R}$. Then by the dominated convergence theorem, we have

$$\begin{aligned}\|(\Gamma x_n)(t) - (\Gamma x)(t)\| &= \left\| \int_{-\infty}^{t} U(t, s)[f(s, x_n(s)) - f(s, x(s))]\, \mathrm{d}s \right\| \\ &\leq K \int_{-\infty}^{t} \mathrm{e}^{-\omega(t-s)} \|f(s, x_n(s)) - f(s, x(s))\|\, \mathrm{d}s \\ &< K \int_{-\infty}^{t} \mathrm{e}^{-\omega(t-s)} \frac{\omega\epsilon}{K}\, \mathrm{d}s \leq \epsilon\end{aligned}$$

for all $n > N$ and all $t \in \mathbb{R}$. This implies that Γ is continuous.

For the sake of convenience, we divide the remaining proof into several steps.

Step 1 Let $\mathbb{B} = \{x \in PAA(\mathbb{R}, \mathbb{X}, \mu) : \|x\|_\infty \leq \widetilde{L}\}$. Then $\mathbb{B}$ is a closed convex subset of $PAA(\mathbb{R}, \mathbb{X}, \mu)$. We claim that $\Gamma\mathbb{B} \subset \mathbb{B}$. In fact, for $x \in \mathbb{B}$ and $t \in \mathbb{R}$, we get

$$\begin{aligned}
\|(\Gamma x)(t)\| &= \left\| \int_{-\infty}^{t} U(t,s) f(s, x(s))\, \mathrm{d}s \right\| \\
&\leq \sum_{n=1}^{\infty} \left\| \int_{t-n}^{t-n+1} U(t,s) f(s, x(s))\, \mathrm{d}s \right\| \\
&\leq \sum_{n=1}^{\infty} K \int_{t-n}^{t-n+1} \mathrm{e}^{-\omega(t-s)} \|f(s, x(s))\|\, \mathrm{d}s \\
&\leq \sum_{n=1}^{\infty} K \left(\int_{t-n}^{t-n+1} \mathrm{e}^{-\omega q(t-s)}\, \mathrm{d}s \right)^{\frac{1}{q}} \left(\int_{t-n}^{t-n+1} \|f(s, x(s))\|^p\, \mathrm{d}s \right)^{\frac{1}{p}} \\
&\leq \sum_{n=1}^{\infty} K \sqrt[q]{\frac{\mathrm{e}^{q\omega} - 1}{q\omega}} \mathrm{e}^{-\omega n} M_f \leq \widetilde{L},
\end{aligned}$$

which implies that $\|\Gamma x\|_\infty \leq \widetilde{L}$. Thus $\Gamma\mathbb{B} \subset \mathbb{B}$.

Step 2 We prove that the operator Γ is completely continuous on $\mathbb{B}$. It is sufficient to prove that the following statements are true.

(B1) $V(t) = \{(\Gamma x)(t) : x \in \mathbb{B}\}$ is relatively compact in $\mathbb{X}$ for each $t \in \mathbb{R}$.
(B2) $\{\Gamma x : x \in \mathbb{B} \subset PAA(\mathbb{R}, \mathbb{X}, \mu)\}$ is a family of equicontinuous functions.

First, we show that (B1) holds. Let $0 < \epsilon < 1$ be given. For each $t \in \mathbb{R}$ and $x \in \mathbb{B}$, we define

$$\begin{aligned}
(\Gamma_\epsilon x)(t) &= \int_{-\infty}^{t-\epsilon} U(t,s) f(s, x(s))\, \mathrm{d}s \\
&= U(t, t-\epsilon) \int_{-\infty}^{t-\epsilon} U(t-\epsilon, s) f(s, x(s))\, \mathrm{d}s \\
&= U(t, t-\epsilon)[(\Gamma x)(t-\epsilon)].
\end{aligned}$$

Since $U(t,s)(t > s)$ is compact, then the set $V_\epsilon(t) : \{(\Gamma_\epsilon x)(t) : x \in \mathbb{B}\}$ is relatively compact in $\mathbb{X}$ for each $t \in \mathbb{R}$. Moreover, for each $x \in \mathbb{B}$, we get

$$\begin{aligned}
\|(\Gamma x)(t) - (\Gamma_\epsilon x)(t)\| &= \left\| \int_{t-\epsilon}^{t} U(t,s) f(s, x(s))\, \mathrm{d}s \right\| \\
&\leq K \int_{t-\epsilon}^{t} \mathrm{e}^{-\omega(t-s)} \|f(s, x(s))\|\, \mathrm{d}s \\
&\leq K \left(\int_{t-\epsilon}^{t} \mathrm{e}^{-q\omega(t-s)}\, \mathrm{d}s \right)^{\frac{1}{q}} \left(\int_{t-\epsilon}^{t} \|f(s, x(s))\|^p\, \mathrm{d}s \right)^{\frac{1}{p}} \\
&\leq K M_f \left(\int_{t-\epsilon}^{t} \mathrm{e}^{-q\omega(t-s)}\, \mathrm{d}s \right)^{\frac{1}{q}}.
\end{aligned}$$

Letting $\epsilon \to 0$, it follows that there is relatively compact set $V_\epsilon(t)$ arbitrarily close to $V(t)$ and hence $V(t)$ is also relatively compact in $\mathbb{X}$ for each $t \in \mathbb{R}$.

Next we prove that (B2) holds. Let $\epsilon > 0$ be small enough and $-\infty < t_1 < t_2 < \infty$. Since $\{U(t,s)\}$ is exponentially stable and compact for $t > s$, there exists $\delta = \delta(\epsilon) < \widetilde{\epsilon}$ such that $t_2 - t_1 < \delta$ implies that

$$\left\| U\left(t_1, t_1 - \frac{t}{2}\right) - U\left(t_2, t_1 - \frac{t}{2}\right) \right\| < \frac{\epsilon}{\gamma} \quad \text{for all } t > 0,$$

where $\widetilde{\epsilon} = (\frac{\epsilon}{6KM_f})^q \le 1$ and $\gamma = 3KM_f \sqrt[q]{\frac{2(\mathrm{e}^{\frac{q\omega}{2}} - 1)}{q\omega}} \sum_{n=1}^{\infty} \mathrm{e}^{-\frac{\omega(\widetilde{\epsilon}+n)}{2}}$.

Indeed, for $x \in \mathbb{B}$ and $t_2 - t_1 < \delta$, we have

$$\begin{aligned}
&\|(\Gamma x)(t_2) - (\Gamma x)(t_1)\| \\
&\le \left\| \int_{-\infty}^{t_1 - \widetilde{\epsilon}} [U(t_2, s) - U(t_1, s)] f(s, x(s))\, \mathrm{d}s \right\| \\
&\quad + \left\| \int_{t_1 - \widetilde{\epsilon}}^{t_1} [U(t_2, s) - U(t_1, s)] f(s, x(s))\, \mathrm{d}s \right\| \\
&\quad + \left\| \int_{t_1}^{t_2} U(t_2, s) f(s, x(s))\, \mathrm{d}s \right\| \\
&\le \left\| \int_{\widetilde{\epsilon}}^{\infty} [U(t_2, t_1 - s) - U(t_1, t_1 - s)] f(t_1 - s, x(t_1 - s))\, \mathrm{d}s \right\| \\
&\quad + K \int_{t_1 - \widetilde{\epsilon}}^{t_1} [\mathrm{e}^{-\omega(t_2 - s)} + \mathrm{e}^{-\omega(t_1 - s)}] \|f(s, x(s))\|\, \mathrm{d}s \\
&\quad + K \int_{t_1}^{t_2} \mathrm{e}^{-\omega(t_2 - s)} \|f(s, x(s))\|\, \mathrm{d}s \\
&\le \left\| \int_{\widetilde{\epsilon}}^{\infty} \left[U\left(t_2, t_1 - \frac{s}{2}\right) - U\left(t_1, t_1 - \frac{s}{2}\right) \right] U\left(t_1 - \frac{s}{2}, t_1 - s\right) f(t_1 - s, x(t_1 - s))\, \mathrm{d}s \right\| \\
&\quad + K \left(\int_{t_1 - \widetilde{\epsilon}}^{t_1} [\mathrm{e}^{-\omega(t_2 - s)} + \mathrm{e}^{-\omega(t_1 - s)}]^q\, \mathrm{d}s \right)^{\frac{1}{q}} \left(\int_{t_1 - \widetilde{\epsilon}}^{t_1} \|f(s, x(s))\|^p\, \mathrm{d}s \right)^{\frac{1}{p}} \\
&\quad + K \left(\int_{t_1}^{t_2} \mathrm{e}^{-q\omega(t_2 - s)}\, \mathrm{d}s \right)^{\frac{1}{q}} \left(\int_{t_1}^{t_2} \|f(s, x(s))\|^p\, \mathrm{d}s \right)^{\frac{1}{p}} \\
&\le \frac{\epsilon}{\gamma} K \int_{\widetilde{\epsilon}}^{\infty} \mathrm{e}^{-\frac{\omega s}{2}} \|f(t_1 - s, x(t_1 - s))\|\, \mathrm{d}s + 2K \widetilde{\epsilon}^{\frac{1}{q}} M_f + K \delta^{\frac{1}{q}} M_f \\
&\le \frac{\epsilon}{\gamma} K \sum_{n=1}^{\infty} \int_{\widetilde{\epsilon}+n-1}^{\widetilde{\epsilon}+n} \mathrm{e}^{-\frac{\omega s}{2}} \|f(t_1 - s, x(t_1 - s))\|\, \mathrm{d}s + 2K \left[\left(\frac{\epsilon}{6KM_f} \right)^q \right]^{\frac{1}{q}} M_f + K \widetilde{\epsilon}^{\frac{1}{q}} M_f \\
&\le \frac{\epsilon}{\gamma} K \sum_{n=1}^{\infty} \left(\int_{\widetilde{\epsilon}+n-1}^{\widetilde{\epsilon}+n} \mathrm{e}^{-\frac{q\omega s}{2}}\, \mathrm{d}s \right)^{\frac{1}{q}} \left(\int_{\widetilde{\epsilon}+n-1}^{\widetilde{\epsilon}+n} \|f(t_1 - s, x(t_1 - s))\|^p\, \mathrm{d}s \right)^{\frac{1}{p}} + \frac{\epsilon}{3} + \frac{\epsilon}{6}
\end{aligned}$$

$$\leq \frac{\epsilon}{\gamma} K M_f \sqrt[q]{\frac{2(e^{\frac{q\omega}{2}}-1)}{q\omega}} \sum_{n=1}^{\infty} e^{-\frac{\omega(\tilde{\epsilon}+n)}{2}} + \frac{\epsilon}{3} + \frac{\epsilon}{6}$$
$$\leq \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{6} < \epsilon.$$

This implies that the set $\{\Gamma x : x \in \mathbb{B}\}$ is equicontinuous.

Now we denote the closed convex hull of $\Gamma\mathbb{B}$ by $\overline{co}\Gamma\mathbb{B}$. Since $\Gamma\mathbb{B} \subset \mathbb{B}$ and $\mathbb{B}$ is closed convex, $\overline{co}\Gamma\mathbb{B} \subset \mathbb{B}$. Thus, $\Gamma(\overline{co}\Gamma\mathbb{B}) \subset \Gamma\mathbb{B} \subset \overline{co}\Gamma\mathbb{B}$. This implies that Γ is a continuous mapping from $\overline{co}\Gamma\mathbb{B}$ to $\overline{co}\Gamma\mathbb{B}$. It is easy to verify that $\overline{co}\Gamma\mathbb{B}$ has the properties (B1) and (B2). More explicitly, $\{x(t) : x \in \overline{co}\Gamma\mathbb{B}\}$ is relatively compact in $\mathbb{X}$ for each $t \in \mathbb{R}$, and $\overline{co}\Gamma\mathbb{B} \subset BC(\mathbb{R}, \mathbb{X})$ is uniformly bounded and equicontinuous. By the Ascoli–Arzelà theorem, the restriction of $\overline{co}\Gamma\mathbb{B}$ to every compact subset K_3 of $\mathbb{R}$, namely $\{x(t) : x \in \overline{co}\Gamma\mathbb{B}\}_{x\in K_3}$, is relatively compact in $C(K_3, \mathbb{X})$. Thus, by the conditions (A5) (II) and that Γ is well defined and continuous, we deduce that $\Gamma : \overline{co}\Gamma\mathbb{B} \to \overline{co}\Gamma\mathbb{B}$ is a compact operator. Noting the continuity of Γ, it follows from Schauder's fixed point theorem that there is a fixed point $x(\cdot)$ for Γ in $\overline{co}\Gamma\mathbb{B}$. That is, Eq. (1.1) has at least one μ-pseudo almost automorphic mild solution $x \in \mathbb{B}$. This completes the proof. □

The following existence result is based upon nonlinear Leray–Schauder alternative theorem. For that, we require the following assumption:

(A8) There exists a continuous nondecreasing function $W : [0, \infty) \to (0, \infty)$ such that

$$\|f(t, x)\| \leq W(\|x\|) \quad \text{forall } t \in \mathbb{R} \textit{ and } x \in \mathbb{X}.$$

Theorem 4.4 *Let $\mu \in \mathfrak{M}$. Assume that the conditions (H0), (A1)–(A2) are satisfied. Let $f : \mathbb{R} \times \mathbb{X} \to \mathbb{X}$ be a function that satisfies assumptions (A6)–(A8), and the following additional conditions:*

(i) *For each $z \geq 0$, the function $t \to \int_{-\infty}^{t} e^{-\omega(t-s)} W(zh(s))\, ds$ belongs to $BC(\mathbb{R})$. We set*
$$\beta(z) = K \left\| \int_{-\infty}^{t} e^{-\omega(t-s)} W(zh(s))\, ds \right\|_h .$$
(ii) *For each $\epsilon > 0$, there is $\delta > 0$ such that for every $u, v \in C_h(\mathbb{X})$, $\|u - v\|_h \leq \delta$ implies that*
$$\int_{-\infty}^{t} e^{-\omega(t-s)} \|f(s, u(s)) - f(s, v(s))\|\, ds \leq \epsilon,$$
for all $t \in \mathbb{R}$.

(iii) $\liminf_{\xi\to\infty} \frac{\xi}{\beta(\xi)} > 1$.

(iv) *For all $a, b \in \mathbb{R}$, $a < b$, and $z > 0$, the set $\{f(s, h(s)x) : a \leq s \leq b, x \in C_h(\mathbb{X}), \|x\|_h \leq z\}$ is relatively compact in $\mathbb{X}$.*

Then Eq. (1.1) *has a μ-pseudo almost automorphic mild solution.*

Proof We define the nonlinear operator $\Gamma : C_h(\mathbb{X}) \to C_h(\mathbb{X})$ by

$$(\Gamma u)(t) := \int_{-\infty}^{t} U(t,s) f(s,u(s))\,\mathrm{d}s, \quad t \in \mathbb{R}.$$

We will show that Γ has a fixed point in $PAA(\mathbb{R}, \mathbb{X}, \mu)$. For the sake of convenience, we divide the proof into several steps.

(I) For $u \in C_h(\mathbb{X})$, we have that

$$\|(\Gamma u)(t)\| \le K \int_{-\infty}^{t} \mathrm{e}^{-\omega(t-s)} W(\|u(s)\|)\,\mathrm{d}s \le K \int_{-\infty}^{t} \mathrm{e}^{-\omega(t-s)} W\left(\|u\|_h h(s)\right)\,\mathrm{d}s.$$

It follows from condition (i) that Γ is well defined.

(II) The operator Γ is continuous. In fact, for any $\epsilon > 0$, we take $\delta > 0$ involved in condition (ii). If $u, v \in C_h(\mathbb{X})$ and $\|u - v\|_h \le \delta$, then

$$\|(\Gamma u)(t) - (\Gamma v)(t)\| \le K \int_{-\infty}^{t} \mathrm{e}^{-\omega(t-s)} \|f(s,u(s)) - f(s,v(s))\|\,\mathrm{d}s \le \epsilon,$$

which shows the assertion.

(III) We will show that Γ is completely continuous. We set $B_z(\mathbb{X})$ for the closed ball with center at 0 and radius z in the space $\mathbb{X}$. Let $V = \Gamma(B_z(C_h(\mathbb{X})))$ and $v = \Gamma(u)$ for $u \in B_z(C_h(\mathbb{X}))$. First, we will prove that $V(t)$ is a relatively compact subset of $\mathbb{X}$ for each $t \in \mathbb{R}$. It follows from condition (i) that the function $s \to K\mathrm{e}^{-\omega s} W(zh(t-s))$ is integrable on $[0, \infty)$. Hence, for $\epsilon > 0$, we can choose $a \ge 0$ such that $K \int_a^\infty \mathrm{e}^{-\omega s} W(zh(t-s))\,\mathrm{d}s \le \epsilon$. Since

$$v(t) = \int_0^a U(t,t-s) f(t-s, u(t-s))\,\mathrm{d}s + \int_a^\infty U(t,t-s) f(t-s,u(t-s))\,\mathrm{d}s$$

and

$$\left\| \int_a^\infty U(t,t-s) f(t-s,u(t-s))\,\mathrm{d}s \right\| \le K \int_a^\infty \mathrm{e}^{-\omega s} W(zh(t-s))\,\mathrm{d}s \le \epsilon,$$

we get $v(t) \in \overline{ac_0(N)} + B_\epsilon(\mathbb{X})$, where $c_0(N)$ denotes the convex hull of N and $N = \{U(t,t-s) f(\xi, h(\xi)x) : 0 \le s \le a, t-a \le \xi \le t, \|x\|_h \le z\}$. Using the strong continuity of $U(t,s)$ and property (iv) of f, we infer that N is a relatively compact set, and $V(t) \subseteq \overline{ac_0(N)} + B_\epsilon(\mathbb{X})$, which establishes our assertion. Second, we show that the set V is equicontinuous. In fact, we can decompose

$$\begin{aligned} v(t+s) - v(t) &= \int_0^s U(t,t-\sigma) f(t+s-\sigma, u(t+s-\sigma))\,\mathrm{d}\sigma \\ &\quad + \int_0^a [U(t,t-\sigma-s) - U(t,t-\sigma)] f(t-\sigma, u(t-\sigma))\,\mathrm{d}\sigma \\ &\quad + \int_a^\infty [U(t,t-\sigma-s) - U(t,t-\sigma)] f(t-\sigma, u(t-\sigma))\,\mathrm{d}\sigma. \end{aligned}$$

For each $\epsilon > 0$, we can choose $a > 0$ and $\delta_1 > 0$ such that

$$\begin{aligned}
&\left\| \int_0^s U(t, t-\sigma) f(t+s-\sigma, u(t+s-\sigma))\, \mathrm{d}\sigma \right. \\
&\quad \left. + \int_a^\infty [U(t, t-\sigma-s) - U(t, t-\sigma)] f(t-\sigma, u(t-\sigma))\, \mathrm{d}\sigma \right\| \\
&\leq K \int_0^s \mathrm{e}^{-\omega\sigma} W(zh(t+s-\sigma))\, \mathrm{d}\sigma \\
&\quad + K \int_a^\infty [\mathrm{e}^{-\omega(\sigma+s)} + \mathrm{e}^{-\omega\sigma}] W(zh(t-\sigma))\, \mathrm{d}\sigma \\
&\leq \frac{\epsilon}{2}
\end{aligned}$$

for $s \leq \delta_1$. Moreover, since $\{f(t-\sigma, u(t-\sigma)) : 0 \leq \sigma \leq a, u \in B_z(C_h(\mathbb{X}))\}$ is a relatively compact set and $U(t, s)$ is strongly continuous, we can choose $\delta_2 > 0$ such that $\|[U(t, t-\sigma-s) - U(t, t-\sigma)] f(t-\sigma, u(t-\sigma))\| \leq \frac{\epsilon}{2a}$ for $s \leq \delta_2$. Combining these estimates, we get $\|v(t+s) - v(t)\| \leq \epsilon$ for s small enough and independent of $u \in B_z(C_h(\mathbb{X}))$.
Finally, applying condition (i), we can see that

$$\frac{\|v(t)\|}{h(t)} \leq \frac{K}{h(t)} \int_{-\infty}^t \mathrm{e}^{-\omega(t-s)} W(zh(s))\, \mathrm{d}s \to 0, \quad |t| \to \infty,$$

and this convergence is independent of $u \in B_z(C_h(\mathbb{X}))$. Hence, by Lemma 2.10, V is a relatively compact set in $(C_h(\mathbb{X}))$.

(IV) Let us show that $u^\lambda(\cdot)$ is a solution of equation $u^\lambda = \lambda\Gamma(u^\lambda)$ for some $0 < \lambda < 1$. We can estimate

$$\begin{aligned}
\left\| u^\lambda(t) \right\| &= \lambda \left\| \int_{-\infty}^t U(t, s) f(s, u^\lambda(s))\, \mathrm{d}s \right\| \\
&\leq K \int_{-\infty}^t \mathrm{e}^{-\omega(t-s)} W(\|u^\lambda\|_h h(s))\, \mathrm{d}s \\
&\leq \beta(\|u^\lambda\|_h) h(t).
\end{aligned}$$

Hence, we get

$$\frac{\|u^\lambda\|_h}{\beta(\|u^\lambda\|_h)} \leq 1$$

and combining with condition (iii), we conclude that the set $\{u^\lambda : u^\lambda = \lambda\Gamma(u^\lambda), \lambda \in (0, 1)\}$ is bounded.

(V) It follows from Theorem 2.4, (A6)–(A7), and Theorem 3.3 that the function $t \to f(t, u(t))$ belongs to $PAA^p(\mathbb{R}, \mathbb{X}, \mu)$, whenever $u \in PAA(\mathbb{R}, \mathbb{X}, \mu)$. Moreover, from Lemma 4.1 we infer that $\Gamma(PAA(\mathbb{R}, \mathbb{X}, \mu)) \subset PAA(\mathbb{R}, \mathbb{X}, \mu)$, and noting that $PAA(\mathbb{R}, \mathbb{X}, \mu)$ is a closed subspace of $C_h(\mathbb{X})$, consequently, we can consider $\Gamma : PAA(\mathbb{R}, \mathbb{X}, \mu) \to PAA(\mathbb{R}, \mathbb{X}, \mu)$. Using properties (I)–(III), we deduce that this map is completely continuous. Applying Lemma 2.11, we infer that Γ has a fixed point $u \in PAA(\mathbb{R}, \mathbb{X}, \mu)$, which completes the proof. □

Corollary 4.1 *Let $\mu \in \mathfrak{M}$. Assume that (H0), (A1)–(A2) are satisfied. Let $f : \mathbb{R} \times \mathbb{X} \to \mathbb{X}$ be a function that satisfies assumptions (A6)–(A7) and the Hölder type condition:*

$$\|f(t,u) - f(t,v)\| \le \varrho \|u - v\|^{\alpha}, \quad 0 < \alpha < 1,$$

for all $t \in \mathbb{R}$ and $u, v \in \mathbb{X}$, where $\varrho > 0$ is a constant. Moreover, assume the following conditions:

(a) $f(t, 0) = q$.
(b) $\sup_{t \in \mathbb{R}} K \int_{-\infty}^{t} \mathrm{e}^{-\omega(t-s)} h(s)^{\alpha} \, \mathrm{d}s = \varrho_2 < \infty$.
(c) *For all $a, b \in \mathbb{R}$, $a < b$, and $z > 0$, the set $\{f(s, h(s)x) : a \le s \le b, x \in C_h(\mathbb{X}), \|x\|_h \le z\}$ is relatively compact in $\mathbb{X}$.*

Then Eq. (1.1) *has a μ-pseudo almost automorphic mild solution.*

Proof Let $\varrho_0 = \|q\|, \varrho_1 = \varrho$. We take $W(\xi) = \varrho_0 + \varrho_1 \xi^{\alpha}$. Then condition (A8) is satisfied. It follows from (b) that function f satisfies (i) in Theorem 4.4. Note that for each $\epsilon > 0$ there is $0 < \delta^{\alpha} < \frac{\epsilon}{\varrho_1 \varrho_2}$ such that for every $u, v \in C_h(\mathbb{X})$, $\|u - v\|_h \le \delta$ implies that $K \int_{-\infty}^{t} \mathrm{e}^{-\omega(t-s)} \|f(s, u(s) - f(s, v(s))\| \, \mathrm{d}s \le \epsilon$ for all $t \in \mathbb{R}$. The hypothesis (iii) in the statement of Theorem 4.4 can be easily verified using the definition of W. So by Theorem 4.4 we can prove that Eq. (1.1) has a μ-pseudo almost automorphic mild solution. □

Acknowledgments The authors are grateful to the anonymous referees for the valuable comments to improve this paper. This work was supported by NSF of China (11361032), Program for New Century Excellent Talents in University (NCET-10-0022).

References

1. Bochner, S.: A new approach to almost periodicity. Proc. Natl. Acad. Sci. USA **48**, 2039–2043 (1962)
2. Bochner, S.: Continuous mappings of almost automorphic and almost periodic functions. Proc. Natl. Acad. Sci. USA **52**, 907–910 (1964)
3. N'Guérékata, G.M.: Almost Automorphic and Almost Periodic Functions in Abstract Spaces. Kluwer Academic, New York (2001)
4. N'Guérékata, G.M.: Topics in Almost Automorphy. Springer, New York (2005)
5. Diagana, T.: Almost Automorphic Type and Almost Periodic Type Functions in Abstract Spaces. Springer, New York (2013)
6. N'Guérékata, G.M., Pankov, A.: Stepanov-like almost automorphic functions and monotone evolution equations. Nonlinear Anal. Theory Method. Appl. **68**, 2658–2667 (2008)
7. Blot, J., Mophou, G.M., N'Guérékata, G.M., Pennequin, D.: Weighted pseudo almost automorphic functions and applications to abstract differential equations. Nonlinear Anal. Theory Method. Appl. **71**, 903–909 (2009)
8. Mophou, G.M.: Weighted pseudo almost automorphic mild solutions to semilinear fractional differential equations. Appl. Math. Comput. **217**, 7579–7587 (2011)

9. Xia, Z.N., Fan, M.: Weighted Stepanov-like pseudo almost automorphy and applications. Nonlinear Anal. Theor. Method. Appl. **75**, 2378–2397 (2012)
10. Zhang, R., Chang, Y.K., N'Guérékata, G.M.: New composition theorems of Stepanov-like weighted pseudo almost automorphic functions and applications to nonautonomous evolution equations. Nonlinear Anal. Real World Appl. **13**, 2866–2879 (2012)
11. Zhang, R., Chang, Y.K., N'Guérékata, G.M.: Weighted pseudo almost automorphic solutions for non-autonomous neutral functional differential equations with infinite delay. Sci. Sin. Math. **43**, 273–292 (2013). doi:10.1360/012013-9 (**in Chinese**)
12. Chang, Y.K., Zhang, R., N'Guérékata, G.M.: Weighted pseudo almost automorphic mild solutions to semilinear fractional differential equations. Comput. Math. Appl. **64**, 3160–3170 (2012)
13. Zhang, R., Chang, Y.K., N'Guérékata, G.M.: Weighted pseudo almost automorphic solutions to nonautonomous semilinear evolution equations with delay and S^p-weighted pseudo almost automorphic coefficients, Topol. Method Nonlinear Anal. (accepted)
14. Zhang, R., Chang, Y.K., N'Guérékata, G.M.: Existence of weighted pseudo almost automorphic mild solutions to semilinear integral equations with S^p-weighted pseudo almost automorphic coefficients. Discret. Contin. Dyn. Syst. **A 33**, 5525–5537 (2013)
15. Blot, J., Cieutat, P., Ezzinbi, K.: Measure theory and pseudo almost automorphic functions: new developments and applications. Nonlinear Anal. Theory Method. Appl. **75**, 2426–2447 (2012)
16. Acquistapace, P., Terreni, B.: A unified approach to abstract linear parabolic equations. Rend. Semin. Mat. Univ. Padova **78**, 47–107 (1987)
17. Liang, J., N'Guérékata, G.M., Xiao, T.J., et al.: Some properties of pseudo almost automorphic functions and applications to abstract differential equations. Nonlinear Anal. Theory Method. Appl. **70**, 2731–2735 (2009)
18. Diagana, T., Mophou, G.M., N'Guérékata, G.M.: Existence of weighted pseudo almost periodic solutions to some classes of differential equations with S^p-weighted pseudo almost periodic coefficients. Nonlinear Anal. Theory Method. Appl. **72**(1), 430–438 (2010)
19. Lee, H., Alkahby, H.: Stepanov-like almost automorphic solutions of nonautonomous semilinear evolution equations with delay. Nonlinear Anal. Theory Method. Appl. **69**, 2158–2166 (2008)
20. Diagana, T.: Existence of pseudo-almost automorphic solutions to some abstract differential equations with S^p-pseudo-almost automorphic coefficients. Nonlinear Anal. Theory Method. Appl. **70**, 3781–3790 (2009)
21. Fan, Z.B., Liang, J., Xiao, T.J.: On Stepanov-like (pseudo) almost automorphic functions. Nonlinear Anal. Theory Method. Appl. **74**, 2853–2861 (2011)
22. Henríquez, H.R., Lizama, C.: Compact almost automorphic solutions to integral equations with infinite delay. Nonlinear Anal. Theory Methods Appl. **71**, 6029–6037 (2009)
23. Granas, A., Dugundji, J.: Fixed Point Theory. Springer, New York (2003)
24. Engel, K.J., Nagel, R.: One-Parameter Semigroups for Linear Evolution Equations. Springer, New York (2000)
25. Ding, H.S., Ling, J., N'Guérékata, G.M., Xiao, T.J.: Pseudo-almost periodicity of some nonautonomous evolution equations with delay. Nonlinear Anal. Theory Method. Appl. **67**, 1412–1418 (2007)