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Degree Conditions for Fractional (g, f, n', m)-Critical Deleted Graphs and Fractional ID-(g, f, m)-Deleted Graphs

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Abstract A graph G is called a fractional (g, f, n', m)-critical deleted graph if after deleting any n' vertices of G the remaining graph is a fractional (g, f, m)-deleted graph. A graph G is called a fractional ID-(g, f, m)-deleted graph if after deleting any independent set I of G the remaining graph is a fractional (g, f, m)-deleted graph. In this paper, we give some sharp degree conditions for a graph to be a fractional (g, f, n', m)-critical deleted graph and a fractional ID-(g, f, m)-deleted graph. The tight degree conditions for fractional (a, b, n', m)-critical deleted graphs and fractional ID-(a, b, m)-deleted graphs are also considered.

Keywords Graph · Fractional (g, f)-factor · Fractional (g, f, m)-deleted graph · Fractional (g, f, n', m)-critical deleted graph · Fractional ID-(g, f, m)-deleted graph · Degree condition

Mathematics Subject Classification 05C70

1 Introduction

All graphs considered in this paper are finite, loopless, and without multiple edges. Let G be a graph with the vertex set $V(G)$ and the edge set $E(G)$. Let $n = |V(G)|$. For a

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vertex $x \in V(G)$, the degree and the neighborhood of x in G are denoted by $d_G(x)$ and $N_G(x)$, respectively. We use $N_G[x]$ to denote $N_G(x) \cup \{x\}$. Let $\Delta(G)$ and $\delta(G)$ denote the maximum degree and the minimum degree of G, respectively. For $S \subseteq V(G)$, we denote by $G[S]$ the subgraph of G induced by S, and let $G - S = G[V(G) \backslash S]$. For two disjoint subsets S and T of $V(G)$, we use $e_G(S, T)$ to denote the number of edges with one end in S and the other in T. Denote $\sigma_2(G) = \min\{d_G(u) + d_G(v)\}$ for each pair of non-adjacent vertices u and v of G.

Suppose that g and f are two integer-valued functions on $V(G)$ such that $0 \le g(x) \le f(x)$ for all $x \in V(G)$. A *fractional* (g, f)*-factor* is a function h that assigns to each edge of a graph G a number in [0,1] so that for each vertex x we have $g(x) \le d_G^h(x) \le f(x)$, where $d_G^h(x) = \sum_{e \in E(x)} h(e)$ is called the *fractional degree* of x in G. If $g(x) = f(x)$ for all $x \in V(G)$, then a fractional (g, f)-factor is a fractional f-factor. If $g(x) = a$, $f(x) = b$ for all $x \in V(G)$, then a fractional (g, f)-factor is a fractional $[a, b]$-factor. Moreover, if $g(x) = f(x) = k$ ($k \ge 1$ is an integer) for all $x \in V(G)$, then a fractional (g, f)-factor is just a fractional k-factor.

A graph G is called a *fractional* (g, f, m)*-deleted graph* if for each edge subset $H \subseteq E(G)$ with $|H| = m$, there exists a fractional (g, f)-factor h such that $h(e) = 0$ for all $e \in H$. That is, after removing any m edges, the resulting graph still has a fractional (g, f)-factor. A graph G is called a *fractional* (g, f, n')-critical graph if after deleting any n' vertices from G, the resulting graph still has a fractional (g, f)-factor.

The first author of this paper introduced the concept of a fractional (g, f, n', m)-critical deleted graph [2]. A graph G is called a *fractional* (g, f, n', m)*-critical deleted graph* if after deleting any n' vertices from G, the resulting graph is still a fractional (g, f, m)-deleted graph. If $g(x) = f(x)$ for all $x \in V(G)$, then fractional (g, f, m)-deleted graph, fractional (g, f, n')-critical graph, and fractional (g, f, n', m)-critical deleted graph are fractional (f, m)-deleted graph, fractional (f, n')-critical graph, and fractional (f, n', m)-critical deleted graph, respectively. If $g(x) = a$, $f(x) = b$ for all $x \in V(G)$, then fractional (g, f, m)-deleted graph, fractional (g, f, n')-critical graph, and fractional (g, f, n', m)-critical deleted graph are fractional (a, b, m)-deleted graph, fractional (a, b, n')-critical graph, and fractional (a, b, n', m)-critical deleted graph, respectively. Furthermore, if $g(x) = f(x) = k$ ($k \ge 1$ is an integer) for all $x \in V(G)$, then fractional (g, f, m)-deleted graph, fractional (g, f, n')-critical graph, and fractional (g, f, n', m)-critical deleted graph are just fractional (k, m)-deleted graph, fractional (k, n')-critical graph, and fractional (k, n', m)-critical deleted graph, respectively. Some results on fractional (g, f, n', m)-critical deleted graph were given by Gao and Wang in [4].

Yu et al. [5] studied the degree condition for fractional $k(\ge 2)$-factor and proved that G has a fractional k-factor if $n \ge 4k - 3$, $\delta(G) \ge k$, and $\max\{d_G(u), d_G(v)\} \ge n/2$ for each pair of non-adjacent vertices u and v of G. Zhou [6,7] discussed the degree conditions for (k, m)-deleted graphs. Gao and Wang [3] improved the results in [6,7] and obtained that G is a fractional (k, m)-deleted graph, with $k \ge 2$ and $m \ge 0$, if one of the following conditions holds:

(1) $n \ge 4k + 4m - 3$, $\delta(G) \ge k + m$, and $\max\{d_G(u), d_G(v)\} \ge n/2$ for each pair of non-adjacent vertices u and v of G;

(2) $\delta(G) \geq k + m$, $\sigma_2(G) \geq n$, $n \geq 4k + 4m - 5$ if $(k, m) \neq (3, 0)$ and $n \geq 8$ if $(k, m) = (3, 0)$.

Chang et al. [1] introduced the concept of *fractional ID-k-factor-critical graph* (if $G - I$ has a fractional k-factor for every independent set I of G) and proved that G is a fractional ID-k-factor-critical graph if $\delta(G) \geq 2n/3$ and $n \geq 6k - 8$. Very recently, this concept was generalized to the *fractional ID-$[a, b]$-factor-critical graph* by Zhou et al. in [8], that is, a graph G is fractional ID-$[a, b]$-factor-critical if $G - I$ admits a fractional $[a, b]$-factor for every independent set I of G. It is determined by Zhou et al. [8] that a graph G to be a fractional ID-$[a, b]$-factor-critical graph if $n \geq ((a + 2b)(a + b - 2) + 1)/b$ and $\delta(G) \geq (a + b)n/(a + 2b)$.

In this paper, we first investigate some degree conditions for a graph to be a fractional (g, f, n', m)-critical deleted graph. Our main results in the first part to be proved in the next section can be stated as follows:

Theorem 1 *Let G be a graph of order n, and let a, b, n', and m be non-negative integers such that $2 \leq a \leq b$ and $n > ((a + b)(a + b + 2m - 2) + bn')/a$. Let g, f be two integer-valued functions defined on $V(G)$ such that $a \leq g(x) \leq f(x) \leq b$ for each $x \in V(G)$. If G satisfies $\delta(G) \geq b(n + n')/(a + b)$, then G is a fractional (g, f, n', m)-critical deleted graph.*

Theorem 2 *Let G be a graph of order n, and let a, b, n', and m be non-negative integers such that $2 \leq a \leq b$, $n > ((a + b)(a + b + 2m - 1) + bn')/a$ and $\delta(G) \geq (b^2 + bn')/a + m$. Let g, f be two integer-valued functions defined on $V(G)$ such that $a \leq g(x) \leq f(x) \leq b$ for each $x \in V(G)$. If G satisfies*

$$\max\{d_G(x), d_G(y)\} \geq \frac{b(n + n')}{a + b}$$

for each pair of non-adjacent vertices x and y of G, then G is a fractional (g, f, n', m)-critical deleted graph.

Theorem 3 *Let G be a graph of order n, and let a, b, n', and m be non-negative integers such that $2 \leq a \leq b$, $n > ((a + b)(a + b + 2m - 2) + bn')/a$ and $\delta(G) \geq (b^2 + bn')/a + m$. Let g, f be two integer-valued functions defined on $V(G)$ such that $a \leq g(x) \leq f(x) \leq b$ for each $x \in V(G)$. If G satisfies $\sigma_2(G) \geq 2b(n + n')/(a + b)$, then G is a fractional (g, f, n', m)-critical deleted graph.*

Theorem 1–3 presents sufficient conditions for fractional (g, f, n', m)-critical deleted graphs from three different angles. Theorem 1 describes the minimal degree condition for fractional (g, f, n', m)-critical deleted graphs; Theorem 2 supplies the condition on the degree of non-adjacent vertices for fractional (g, f, n', m)-critical deleted graphs; Theorem 3 depicts the degree sum condition (also called fan-type condition) for fractional (g, f, n', m)-critical deleted graphs.

Let $g(x) = f(x)$ for all $x \in V(G)$ in Theorems 1, 2 and 3, we get three degree conditions for fractional (f, n', m)-critical deleted graphs. Let $m = 0$ in three results above, and the corresponding degree conditions for fractional (g, f, n')-critical graphs are given. In particular, take $n' = 0$, the following corollaries concern degree conditions for fractional (g, f, m)-deleted graphs hold, and on which the proofs of our results in the second part may reckon.

Corollary 1 *Let G be a graph of order n, and let a, b, and m be non-negative integers such that $2 \le a \le b$ and $n > (a+b)(a+b+2m-2)/a$. Let g, f be two integer-valued functions defined on $V(G)$ such that $a \le g(x) \le f(x) \le b$ for each $x \in V(G)$. If G satisfies $\delta(G) \ge bn/(a+b)$, then G is a fractional (g, f, m)-deleted graph.*

Corollary 2 *Let G be a graph of order n, and let a, b, and m be non-negative integers such that $2 \le a \le b$, $n > (a+b)(a+b+2m-1)/a$ and $\delta(G) \ge b^2/a+m$. Let g, f be two integer-valued functions defined on $V(G)$ such that $a \le g(x) \le f(x) \le b$ for each $x \in V(G)$. If G satisfies*

$$\max\{d_G(x), d_G(y)\} \ge \frac{bn}{a+b}$$

for each pair of non-adjacent vertices x and y of G, then G is a fractional (g, f, m)-deleted graph.

Corollary 3 *Let G be a graph of order n, and let a, b, and m be non-negative integers such that $2 \le a \le b$, $n > (a+b)(a+b+2m-2)/a$ and $\delta(G) \ge b^2/a+m$. Let g, f be two integer-valued functions defined on $V(G)$ such that $a \le g(x) \le f(x) \le b$ for each $x \in V(G)$. If G satisfies $\sigma_2(G) \ge 2bn/(a+b)$, then G is a fractional (g, f, m)-deleted graph.*

Some graphs will be constructed to show that the degree conditions in Theorem 1, Theorem 2, and Theorem 3 are best possible. And, the corresponding degree conditions for fractional (a, b, n', m)-critical deleted graphs will be discussed in Sect. 2.4.

The proofs of our Theorem 1, Theorem 2, and Theorem 3 are heavily based on the following lemma.

Lemma 1 (Gao [2]) *Let G be a graph, g, f be two integer-valued functions defined on $V(G)$ such that $g(x) \le f(x)$ for each $x \in V(G)$. Let n', m be two non-negative integers. Then G is fractional (g, f, n', m)-critical deleted graph if and only if*

$$f(S) - g(T) + d_{G-S}(T) \ge \max_{U\subseteq S, |U|=n', H\subseteq E(G-U), |H|=m} \times \left\{ f(U) + \sum_{x\in T} d_H(x) - e_H(T, S) \right\} \quad (1)$$

for all disjoint subsets S, T of $V(G)$ with $|S| \ge n'$.

To derive our second part results, we should extend the concept of fractional ID-$[a, b]$-factor-critical graph. A graph is called *fractional independent-set-deletable (g, f, m)-deleted graph* (in short, *fractional ID-(g, f, m)-deleted graph*) if $G - I$ is a fractional (g, f, m)-deleted graph for every independent set I of G. If $g(x) = f(x)$ for all $x \in V(G)$, then a fractional ID-(g, f, m)-deleted graph is a fractional ID-(f, m)-deleted graph. If $g(x) = a$ and $f(x) = b$ for all $x \in V(G)$, then a fractional ID-(g, f, m)-deleted graph is a fractional ID-(a, b, m)-deleted graph. If $m = 0$, then a fractional ID-(g, f, m)-deleted graph is just a fractional ID-(g, f)-factor-critical graph.

The results in [1] and [8] inspire us to think about degree conditions for fractional ID-(g, f, m)-deleted graphs. Specifically, we prove the following three results.

Theorem 4 *Let G be a graph of order n, and let a, b, and m be non-negative integers such that $2 \le a \le b$ and $n > (2a+b)(a+b+2m-2)/a$. Let g, f be two integer-valued functions defined on $V(G)$ such that $a \le g(x) \le f(x) \le b$ for each $x \in V(G)$. If G satisfies $\delta(G) \ge (a+b)n/(2a+b)$, then G is a fractional ID-(g, f, m)-deleted graph.*

Theorem 5 *Let G be a graph of order n, and let a, b, and m be non-negative integers such that $2 \le a \le b$, $n > (2a+b)(a+b+2m-1)/a$ and $\delta(G) \ge an/(2a+b)+b^2/a+m$. Let g, f be two integer-valued functions defined on $V(G)$ such that $a \le g(x) \le f(x) \le b$ for each $x \in V(G)$. If G satisfies*

$$\max\{d_G(x), d_G(y)\} \ge \frac{(a+b)n}{2a+b}$$

for each pair of non-adjacent vertices x and y of G, then G is a fractional ID-(g, f, m)-deleted graph.

Theorem 6 *Let G be a graph of order n, and let a, b, and m be non-negative integers such that $2 \le a \le b$, $n > (2a+b)(a+b+2m-2)/a$ and $\delta(G) \ge an/(2a+b)+b^2/a+m$. Let g, f be two integer-valued functions defined on $V(G)$ such that $a \le g(x) \le f(x) \le b$ for each $x \in V(G)$. If G satisfies $\sigma_2(G) \ge 2(a+b)n/(2a+b)$, then G is a fractional ID-(g, f, m)-deleted graph.*

As fractional ID-(g, f, m)-deleted graph is a special kind of fractional (g, f, n', m)-critical deleted graph when n' deleted vertices are exactly in an independent set, Theorem 4–6 describes sufficient conditions for a particular kind of fractional (g, f, n', m)-critical deleted graphs from the standpoints on minimal degree condition, non-adjacent vertices degree condition, and degree sum condition, respectively

Several examples will manifest the sharpness of Theorem 4, Theorem 5, and Theorem 6. Also, the corresponding degree conditions for fractional ID-(a, b, m)-deleted graphs will be determined later.

2 Degree Conditions for Fractional (g, f, n', m)-Critical Deleted Graphs

It is noticed that $\delta(G) \ge b(n+n')/(a+b)$ in Theorem 1 implies $\sigma_2(G) \ge 2b(n+n')/(a+b)$ and $\delta(G) \ge (b^2+bn')/a+m$ in Theorem 3. Thus, it is sufficient to prove Theorem 2 and Theorem 3 for the first part.

For completeness, we give the following result on complete graph.

Lemma 2 *Let G be a complete graph with order n, and let a, b, n', and m be non-negative integers such that $2 \le a \le b$. $n > ((a+b)(a+b+2m-2)+bn')/a$. Let g, f be two integer-valued functions defined on $V(G)$ such that $a \le g(x) \le f(x) \le b$ for each $x \in V(G)$. Then G is a fractional (g, f, n', m)-critical deleted graph.*

Proof Suppose that G satisfies the conditions of Lemma 2 but is not a fractional (g, f, n', m)-critical deleted graph. Obviously, $T \neq \emptyset$. Otherwise, (1) holds. By Lemma 1 and the fact $\sum_{x\in T} d_H(x) - e_H(T, S) \leq 2m$, there exist disjoint subsets S and T of $V(G)$ such that

$$f(S) - g(T) + d_{G-S}(T) \leq bn' + 2m - 1, \tag{2}$$

where $|S| \geq n'$. We choose S and T such that $|T|$ is minimum. Thus, for each $x \in T$, we get $d_{G-S}(x) \leq g(x) - 1 \leq b - 1$. Otherwise, if there exists some $x \in T$ such that $d_{G-S}(x) \geq g(x)$, then S and $T\backslash\{x\}$ also satisfy (2). This contradicts the choice of S and T.

For every $S \subseteq V(G)$, $G - S$ is also complete. Hence, for disjoint subsets S, T of $V(G)$, we have

$$\begin{aligned}
&f(S) - g(T) + d_{G-S}(T) - bn' - 2m\\
&\geq a|S| + \sum_{x\in T} d_{G-S}(x) - b|T| - bn' - 2m\\
&\geq a|S| - (b - n + |S| + 1)(n - |S|) - bn' - 2m\\
&= |S|^2 + (a + b - 2n + 1)|S| - bn + n^2 - n - bn' - 2m.
\end{aligned}$$

We regard it as the function of $|S|$. We consider following two cases due to the integrity of $|S|$.

Case 1. $b - a \equiv 0 \pmod 2$. Since $n > ((a + b)(a + b + 2m - 2) + bn')/a$ and $a + b \geq 4$, we obtain

$$\begin{aligned}
&|S|^2 + (a + b - 2n + 1)|S| - bn + n^2 - n - bn' - 2m\\
&\geq \left(n - \frac{a+b}{2}\right)^2 + (a + b - 2n + 1)\left(n - \frac{a+b}{2}\right) - bn + n^2 - n - bn' - 2m\\
&= an - \left(\frac{a+b}{2}\right)^2 - \frac{a+b}{2} - bn' - 2m\\
&> \left(\frac{(a+b)(a+b+2m-2) + bn'}{a}\right)a - \left(\frac{a+b}{2}\right)^2 - \frac{a+b}{2} - bn' - 2m\\
&= \frac{3}{4}(a+b)^2 - \frac{5}{2}(a+b) + (a+b-1)2m\\
&\geq \frac{3}{4} \times 16 - \frac{5}{2} \times 4 > 0,
\end{aligned}$$

which contradicts (2).

Case 2. $b - a \equiv 1 \pmod 2$. By $n > ((a+b)(a+b+2m-2)+bn')/a$ and $a+b \geq 5$, we get

$$\begin{aligned}
&|S|^2 + (a+b-2n+1)|S| - bn + n^2 - n - bn' - 2m \\
&\geq \left(n - \frac{a+b+1}{2}\right)^2 \\
&\quad + (a+b-2n+1)\left(n - \frac{a+b+1}{2}\right) - bn + n^2 - n - bn' - 2m \\
&= an - \left(\frac{a+b+1}{2}\right)^2 - bn' - 2m \\
&> \left(\frac{(a+b)(a+b+2m-2)+bn'}{a}\right)a - \left(\frac{a+b+1}{2}\right)^2 - bn' - 2m \\
&= \frac{3}{4}(a+b)^2 - \frac{5}{2}(a+b) - \frac{1}{4} + (a+b-1)2m \\
&\geq \frac{3}{4} \times 25 - \frac{5}{2} \times 5 - \frac{1}{4} > 0,
\end{aligned}$$

which is a contradiction. This completes the proof Lemma 2. □

Let $n' = 0$ in Lemma 2, we obtain the following corollary which will be used in Section 3.

Corollary 4 *Let G be a complete graph with order n, and let a, b, and m be non-negative integers such that $2 \leq a \leq b$. $n > (a+b)(a+b+2m-2)/a$. Let g, f be two integer-valued functions defined on $V(G)$ such that $a \leq g(x) \leq f(x) \leq b$ for each $x \in V(G)$. Then G is a fractional (g, f, m)-deleted graph.*

In what follows, we always assume that G is not complete. Therefore, the degree condition $\max\{d_G(x), d_G(y)\} \geq b(n+n')/(a+2b)$ for each pair of non-adjacent vertices x and y of G in Theorem 2 and $\sigma_2(G) \geq 2b(n+n')/(a+2b)$ in Theorem 3 are well-defined.

2.1 Proof of Theorem 2

Suppose that G satisfies the conditions of Theorem 2 but is not a fractional (g, f, n', m)-critical deleted graph. Obviously, $T \neq \emptyset$, and there exist disjoint subsets S and T of $V(G)$ such that (2) holds with $|S| \geq n'$. For each $x \in T$, we have $d_{G-S}(x) \leq g(x) - 1 \leq b - 1$ by choosing S and T such that $|T|$ is minimum.

Let $d_1 = \min\{d_{G-S}(x) : x \in T\}$. Then $0 \leq d_1 \leq b-1$, and

$$f(S) + d_{G-S}(T) - g(T) \geq a|S| + d_1|T| - b|T|.$$

Hence,

$$bn' + 2m - 1 \geq a|S| - (b - d_1)|T|. \tag{3}$$

We choose $x_1 \in T$ such that $d_{G-S}(x_1) = d_1$. If $T - N_T[x_1] \neq \emptyset$, let $d_2 = \min\{d_{G-S}(x) : x \in T - N_T[x_1]\}$ and choose $x_2 \in T - N_T[x_1]$ such that $d_{G-S}(x_2) = d_2$. Thus, $d_1 \leq d_2 \leq b-1$.

If $|T| \leq b$, by (3) and $|S| + d_1 \geq d_G(x_1) \geq \delta(G) \geq (b^2 + bn')/a + m$, we have

$$\begin{aligned} bn' + 2m - 1 &\geq a|S| + (d_1 - b)|T| \\ &\geq a\left(\frac{b^2 + bn'}{a} + m - d_1\right) + (d_1 - b)b \\ &= (b-a)d_1 + bn' + am \\ &\geq bn' + 2m. \end{aligned}$$

This produces a contradiction. Therefore, we get $|T| \geq b + 1 \geq a + 1$.

Since $d_{G-S}(x) \leq b - 1$ for all $x \in T$ and $|T| \geq b+1$, $T - N_T[x_1] \neq \emptyset$, hence, x_1, x_2 must be exist. In view of the degree condition of the theorem, we obtain

$$\frac{b(n+n')}{a+b} \leq \max\{d_G(x_1), d_G(x_2))\} \leq |S| + d_2,$$

which implies

$$|S| \geq \frac{b(n+n')}{a+b} - d_2. \tag{4}$$

Using $n - |S| - |T| \geq 0$, $b - d_2 > 0$ and (3), we get

$$\begin{aligned} &(n - |S| - |T|)(b - d_2) \\ &\quad\geq a|S| + \sum_{x\in T}(d_{G-S}(x) - b) - bn' - 2m + 1 \\ &\quad\geq a|S| + (d_1 - b)|N_T[x_1]| + (d_2 - b)(|T| - |N_T[x_1]|) - bn' - 2m + 1 \\ &\quad= a|S| + (d_1 - d_2)|N_T[x_1]| + (d_2 - b)|T| - bn' - 2m + 1 \\ &\quad\geq a|S| + (d_1 - d_2)(d_1 + 1) + (d_2 - b)|T| - bn' - 2m + 1. \end{aligned}$$

It follows that

$$0 \leq n(b - d_2) - (a + b - d_2)|S| + (d_2 - d_1)(d_1 + 1) + bn' + 2m - 1. \tag{5}$$

According to (4), (5), $d_1 \leq d_2 \leq b-1$ and $n > ((a+b)(a+b+2m-1)+bn')/a$, we have

$$\begin{aligned} 0 &\leq n(b-d_2) - (a+b-d_2)\left(\frac{b(n+n')}{a+b} - d_2\right) + (d_2 - d_1)(d_1+1) + bn' + 2m - 1 \\ &= -nd_2\frac{a}{a+b} + d_2\frac{bn'}{a+b} + (a+b)d_2 - d_1^2 - d_2^2 + d_1d_2 + d_2 - d_1 + 2m - 1 \\ &< -d_1^2 - d_2^2 + d_1d_2 + 2d_2 - d_1 + 2m(1-d_2) - 1. \end{aligned}$$

If $d_2 = 0$, then $d_1 = d_2 = 0$. By (4), we get $|S| \geq b(n+n')/(a+b)$ and $|T| \leq n - |S| \leq (an - bn')/(a+b)$. Since $d_{G-S}(T) \geq \sum_{x \in T} d_H(x) - e_G(T, S)$, we obtain

$$\begin{aligned} & f(S) + d_{G-S}(T) - g(T) - bn' - \left(\sum_{x \in T} d_H(x) - e_G(T, S) \right) \\ & \geq a \times \frac{b(n+n')}{a+b} - b \times \frac{an - bn'}{a+b} - bn' + \left(d_{G-S}(T) - \sum_{x \in T} d_H(x) + e_G(T, S) \right) \\ & \geq 0, \end{aligned}$$

a contradiction.

If $d_2 \geq 1$, then

$$\begin{aligned} 0 &< -d_1^2 - d_2^2 + d_1 d_2 + 2d_2 - d_1 + 2m(1 - d_2) - 1 \\ &\leq -d_2^2 + (d_1 + 2)d_2 - d_1^2 - d_1 - 1. \end{aligned}$$

Let

$$h_1(d_2) = -d_2^2 + (d_1 + 2)d_2 - d_1^2 - d_1 - 1.$$

Hence,

$$\max\{h_1(d_2)\} = h_1\left(\frac{d_1 + 2}{2}\right) = -\frac{3}{4} d_1^2 \leq 0.$$

Also a contradiction. This completes the proof of the Theorem 2. □

2.2 Proof of Theorem 3

Suppose that G satisfies the conditions of Theorem 3 but is not a fractional (g, f, n', m)-critical deleted graph. We get $T \neq \emptyset$ and there exist disjoint subsets S and T of $V(G)$ such that (2) holds with $|S| \geq n'$. By choosing S and T such that $|T|$ is minimum, we have $d_{G-S}(x) \leq g(x) - 1 \leq b - 1$ for each $x \in T$.

Let d_1, d_2, x_1, and x_2 as defined before. As discussed in Sect. 2.1, we get $d_1 \leq d_2 \leq b - 1$, $|T| \geq b + 1 \geq a + 1$ and x_1, x_2 must be exist.

In terms of the degree sum condition in Theorem 3, we obtain

$$\frac{2b(n+n')}{a+b} \leq \sigma_2(G) \leq 2|S| + d_2 + d_1,$$

which implies

$$|S| \geq \frac{b(n+n')}{a+b} - \frac{d_2 + d_1}{2}. \tag{6}$$

By the discussion in Sect. 2.1, (5) holds as well. Using (5), (6), $d_1 \le d_2 \le b-1$ and $n > ((a+b)(a+b+2m-2)+bn')/a$, we get

$$\begin{aligned}0 &\le n(b-d_2)-(a+b-d_2)\left(\frac{b(n+n')}{a+b}-\frac{d_2+d_1}{2}\right)\\&\quad+(d_2-d_1)(d_1+1)+bn'+2m-1\\&=-nd_2\frac{a}{a+b}+d_2\frac{bn'}{a+b}+(a+b)\frac{d_1+d_2}{2}-d_1^2-\frac{d_2^2}{2}+\frac{d_1d_2}{2}\\&\quad+d_2-d_1+2m-1\\&<-d_2(a+b-3)+\frac{a+b}{2}(d_1+d_2)-d_1^2-\frac{d_2^2}{2}+\frac{d_1d_2}{2}-d_1+2m(1-d_2)-1.\end{aligned}$$

The case $d_2=0$ can be proved similarly as Sect. 2.1.
If $d_2 \ge 1$ then

$$\begin{aligned}0 &< -d_2(a+b-3)+\frac{a+b}{2}(d_1+d_2)-d_1^2-\frac{d_2^2}{2}+\frac{d_1d_2}{2}-d_1+2m(1-d_2)-1\\&\le -\frac{d_2^2}{2}-d_2\left(\frac{a+b}{2}-3-\frac{d_1}{2}\right)-d_1^2+\left(\frac{a+b}{2}-1\right)d_1-1.\end{aligned}$$

Let

$$h_2(d_2)=-\frac{d_2^2}{2}-d_2\left(\frac{a+b}{2}-3-\frac{d_1}{2}\right)-d_1^2+\left(\frac{a+b}{2}-1\right)d_1-1.$$

If d_2 can reach to $3+d_1/2-(a+b)/2$ (i.e., $3+d_1/2-(a+b)/2 \ge 1$), then

$$\max\{h_2(d_2)\}=h_2\left(3+\frac{d_1}{2}-\frac{a+b}{2}\right),$$

and $d_2 \le 1$ due to $d_1 \le b-1$ and $b \ge a \ge 2$. Thus, $(d_1,d_2)=(0,1)$ or $d_1=d_2=1$. By $b \ge a \ge 2$, we verify that $h_2(d_2) \le 0$ for both $(d_1,d_2)=(1,1)$ and $(d_1,d_2)=(0,1)$, a contradiction.

If d_2 cannot take $3+d_1/2-(a+b)/2-1/(a+b)$ as its value, then

$$\begin{aligned}0 &< -\frac{d_2^2}{2}-d_2\left(\frac{a+b}{2}-3-\frac{d_1}{2}\right)-d_1^2+\left(\frac{a+b}{2}-1\right)d_1-1\\&\le -\frac{d_1^2}{2}-d_1\left(\frac{a+b}{2}-3-\frac{d_1}{2}\right)-d_1^2+\left(\frac{a+b}{2}-1\right)d_1-1\\&=-d_1^2+2d_1-1 \le 0.\end{aligned}$$

This is the final contradiction. Consequently, Theorem 3 is proved. □

2.3 Sharpness

First, the bounds on $\delta(G)$ in Theorem 2 and Theorem 3 are best in some sense. To see this, let $a = b$, and $\delta(G) = (b^2 + bn')/a + m - 1 = a + m + n' - 1$. Choose a vertex v such that $d(v) = a + m + n' - 1$. Delete n' vertices adjacent to v, then the resulting graph G_1 has $\delta(G_1) = m + a - 1$. Delete m edges incident to v in G_1, then the resulting graph G_2 has $\delta(G_2) = a - 1$, which has no fractional a-factor by the definition. Therefore, G is not a fractional (g, f, n', m)-critical deleted graph.

The degree conditions in Theorem 1, Theorem 2, and Theorem 3 are best possible. Actually, we can construct some graphs to show that the minimum degree condition in Theorem 1 cannot be weakened by $\delta(G) \geq b(n+n')/(a+b) - 1$, degree condition in Theorem 2 cannot be decreased by $\max\{d_G(x), d_G(y)\} \geq b(n + n')/(a + b) - 1$, and degree sum condition in Theorem 3 cannot be replaced by $\sigma_2(G) \geq 2b(n + n')/(a + b) - 1$.

Let $G_1 = K_{bt+n'}$ be a complete graph, $G_2 = (at + 1)K_1$ be a graph consisting of $at + 1$ isolated vertices, and $G = G_1 \vee G_2$, where t is sufficiently large (i.e., it satisfies $n > ((a + b)(a + b + 2m - 2) + bn')/a$ and $\delta(G) \geq (b^2 + bn')/a + m$). Then $n = |G_1| + |G_2| = (a + b)t + 1 + n'$. Let $S = V(G_1)$, $T = V(G_2)$, and $a = g(x) = f(x) = b$ for each $x \in V(G)$. We have

$$\frac{b(n+n')}{a+b} > \delta(G) = (bt + n') > \frac{b(n+n')}{a+b} - 1,$$
$$\frac{b(n+n')}{a+b} > \max\{d_G(x), d_G(y)\} = (bt + n') > \frac{b(n+n')}{a+b} - 1,$$
$$\frac{2b(n+n')}{a+b} > \sigma_2(G) = 2(bt + n') \geq \frac{2b(n+n')}{a+b} - 1.$$

Let $S = V(G_1)$ and $T = V(G_2)$. We verify that

$$\begin{aligned} & f(S) - g(T) + d_{G-S}(T) - \max_{U\subseteq S, |U|=n', H\subseteq E(G-U), |H|=m} \\ & \times \left\{ f(U) + \sum_{x\in T} d_H(x) - e_H(T, S) \right\} \\ & = a|S| - b|T| - bn' = -b < 0. \end{aligned}$$

By Lemma 1, G is not a fractional (g, f, n', m)-critical deleted graph.

2.4 Degree Conditions for Fractional (a, b, n', m)-Critical Deleted Graphs

Using the tricks in the proving of Lemma 2, we yield a similar result for a complete graph to be a fractional (a, b, n', m)-critical deleted graph.

Lemma 3 *Let G be a complete graph with order n, and let a, b, n', and m be non-negative integers such that $2 \leq a \leq b$. $n > (a + b)(a + b + 2m - 2)/b + n'$. Then G is a fractional (a, b, n', m)-critical deleted graph.*

Let $n' = 0$ in Lemma 3, we get following corollary which is a sufficient condition for a complete graph to be a fractional (a, b, m)-deleted graph.

Corollary 5 *Let G be a complete graph with order n, and let a, b, and m be non-negative integers such that $2 \leq a \leq b$. $n > (a+b)(a+b+2m-2)/b$. Then G is a fractional (a, b, m)-deleted graph.*

Let $g(x) = a$, $f(x) = b$ for every $x \in V(G)$. The sufficient and necessity condition for fractional (a, b, n', m)-critical deleted graph derives from Lemma 1.

Lemma 4 *Let G be a graph. Let a, b, n', m be non-negative integers such that $a \leq b$. Then G is fractional (a, b, n', m)-critical deleted graph if and only if*

$$b|S| - a|T| + d_{G-S}(T) \geq \max_{|H|=m} \left\{ bn' + \sum_{x \in T} d_H(x) - e_H(T, S) \right\} \tag{7}$$

for all disjoint subsets S, T of $V(G)$ with $|S| \geq n'$.

Based on Lemma 4, suppose that G is not a fractional (a, b, n', m)-critical deleted graph. Obviously, $T \neq \emptyset$, and there exist disjoint subsets S and T of $V(G)$ such that

$$b|S| - a|T| + d_{G-S}(T) \leq bn' + 2m - 1, \tag{8}$$

where $|S| \geq n'$. We choose S and T such that $|T|$ is minimum. Thus, $d_{G-S}(x) \leq a-1$ for each $x \in T$.

Let $d_1 = \min\{d_{G-S}(x) : x \in T\}$. Then $0 \leq d_1 \leq a-1$, and

$$bn' + 2m - 1 \geq b|S| - (a - d_1)|T|. \tag{9}$$

If $T - N_T[x_1] \neq \emptyset$, let $d_2 = \min\{d_{G-S}(x) : x \in T - N_T[x_1]\}$ and choose $x_2 \in T - N_T[x_1]$ such that $d_{G-S}(x_2) = d_2$. So, $d_1 \leq d_2 \leq a-1$.

Applying Lemma 3 and Lemma 4, using the tricks used in Sect. 2.1 and Sect. 2.2, and noticing the minor differences between (3) and (9), and $d_2 \leq a-1$ here corresponding to $d_2 \leq b-1$ in Sect. 2.1 and Sect. 2.2, we get following degree conditions for fractional (a, b, n', m)-critical deleted graphs, which correspond to Theorems 1, 2, and 3, respectively. We skip the proofs.

Theorem 7 *Let G be a graph of order n, and let a, b, n', and m be non-negative integers such that $2 \leq a \leq b$ and $n > (a+b)(a+b+2m-2)/b + n'$. If G satisfies $\delta(G) \geq (an+bn')/(a+b)$, then G is a fractional (a, b, n', m)-critical deleted graph.*

Theorem 8 *Let G be a graph of order n, and let a, b, n', and m be non-negative integers such that $2 \leq a \leq b$, $n > (a+b)(a+b+2m-1)/b+n'$ and $\delta(G) \geq a+m+n'$. If G satisfies*

$$\max\{d_G(x), d_G(y)\} \geq \frac{an + bn'}{a+b}$$

for each pair of non-adjacent vertices x and y of G, then G is a fractional (a, b, n', m)-critical deleted graph.

Theorem 9 *Let G be a graph of order n, and let a, b, n', and m be non-negative integers such that $2 \le a \le b$, $n > (a+b)(a+b+2m-2)/b+n'$ and $\delta(G) \ge a+m+n'$. If G satisfies $\sigma_2(G) \ge 2(an+bn')/(a+b)$, then G is a fractional (a, b, n', m)-critical deleted graph.*

Remark 1 Although fractional (a, b, n', m)-critical deleted graph is a special kind of fractional (g, f, n', m)-critical deleted graph when $g(x) = a$ and $f(x) = b$ for all $x \in V(G)$, Theorem 7–9 cannot be derived directly from Theorem 1–3 which are different from Corollary 1–3. Hence, clues for proving Theorem 7–9 which we present above are necessary.

The example $G = K_{bt+n'} \vee G_2 = (at+1)K_1$ in Sect. 2.3 reveals that the degree conditions in Theorem 7, Theorem 8, and Theorem 9 are sharp in some sense. Again, the restrictions on $\delta(G)$ in Theorem 8 and Theorem 9 cannot be replaced by $\delta(G) \ge a+m+n'-1$.

Let $a = b = k$ in Theorem 7, Theorem 8, and Theorem 9, and the corresponding degree conditions for fractional (k, n', m)-critical deleted graphs are given. It reveals degree conditions for fractional (a, b, n')-critical graphs by taking $m = 0$ in three results above. Especially, by taking $n' = 0$ in Theorem 7, Theorem 8, and Theorem 9, the corresponding degree conditions for fractional (a, b, m)-deleted graphs are given as follows, and on which the proofs of results in Sect. 3.3 may rely.

Corollary 6 *Let G be a graph of order n, and let a, b, and m be non-negative integers such that $2 \le a \le b$ and $n > (a+b)(a+b+2m-2)/b$. If G satisfies $\delta(G) \ge an/(a+b)$, then G is a fractional (a, b, m)-deleted graph.*

Corollary 7 *Let G be a graph of order n, and let a, b, and m be non-negative integers such that $2 \le a \le b$, $n > (a+b)(a+b+2m-1)/b$ and $\delta(G) \ge a+m$. If G satisfies*

$$\max\{d_G(x), d_G(y)\} \ge \frac{an}{a+b}$$

for each pair of non-adjacent vertices x and y of G, then G is a fractional (a, b, m)-deleted graph.

Corollary 8 *Let G be a graph of order n, and let a, b, and m be non-negative integers such that $2 \le a \le b$, $n > (a+b)(a+b+2m-2)/b$ and $\delta(G) \ge a+m$. If G satisfies $\sigma_2(G) \ge 2an/(a+b)$, then G is a fractional (a, b, m)-deleted graph.*

3 Degree Conditions for Fractional ID-(g, f, m)-Deleted Graphs

As $\delta(G) \ge (a+b)n/(2a+b)$ in Theorem 4 implies $\delta(G) \ge an/(2a+b)+b^2/a+m$ and $\sigma_2(G) \ge 2(a+b)n/(2a+b)$ in Theorem 6, it is sufficient to prove Theorem 5 and Theorem 6.

3.1 Proofs of Theorem 5 and Theorem 6

Now, we prove Theorem 5. For every independent set I, let $G' = G - I$. We yield the result by confirming that G' satisfies Corollary 2 or Corollary 4.

If G' is a complete graph, then by degree condition, we get

$$|G'| \geq \frac{(a+b)n}{2a+b} > \frac{(a+b)(a+b+2m-1)}{a} > \frac{(a+b)(a+b+2m-2)}{a}.$$

The result follows from Corollary 4.

If $|I| = 1$, then $|V(G')| > ((2a+b)(a+b+2m-1)-a)/a > (a+b)(a+b+2m-1)/a$. It is easy to verify that $\delta(G') \geq b^2/a+m$ and $\max\{d_{G'}(u), d_{G'}(v)\} \geq b|V(G')|/(a+b) = b(n-1)/(a+b)$ for each pair of non-adjacent vertices u and v of G'. Thus, the result holds from Corollary 2.

We now consider $|I| \geq 2$ and G' is not complete. By degree condition, we obtain $|V(G')| \geq (a+b)n/(2a+b) > (a+b)(a+b+2m-1)/a$. If $\max\{d_{G'}(u), d_{G'}(v)\} < b|V(G')|/(a+b)$ for some non-adjacent vertices u, v in G', then $(a+b)(|V(G')|+|I|)/(2a+b) \leq \max\{d_G(u), d_G(v)\} < b|V(G')|/(a+b)+|I|$, i.e., $|V(G')| < (a+b)/a|I| \leq ((a+b)/a) \cdot (an/(2a+b)) = (a+b)n/(2a+b)$. This contradicts $\max\{d_G(u), d_G(v)\} \geq (a+b)n/(2a+b)$ and $|I| \geq 2$. Therefore, $\max\{d_{G'}(u), d_{G'}(v)\} \geq b|V(G')|/(a+b)$ for all non-adjacent vertices u, v in G'. Furthermore, we obtain $\delta(G') \geq b^2/a+m$ by $|I| \leq an/(2a+b)$ and $\delta(G) \geq an/(2a+b)+b^2/a+m$. Then, the result follows from Corollary 2.

Thus, we complete the proof of Theorem 5. Depending on Corollary 3 and Corollary 4, Theorem 6 can be proved with the same tricks. We skip the detail proof. □

3.2 Sharpness

In order to show the sharpness of Theorems 4, 5 and 6, we rely heavily on following lemma, which is the corollary of Lemma 1 by setting $n' = 0$.

Lemma 5 *Let G be a graph, g, f be two integer-valued functions defined on $V(G)$ such that $g(x) \leq f(x)$ for each $x \in V(G)$. Let m be a non-negative integer. Then G is fractional (g, f, m)-deleted graph if and only if*

$$f(S) - g(T) + d_{G-S}(T) \geq \max_{|H|=m} \left\{ \sum_{x \in T} d_H(x) - e_H(T, S) \right\} \tag{10}$$

for all disjoint subsets S, T of $V(G)$.

Considering a graph $G = (at+1)K_1 \vee K_{bt} \vee (at+1)K_1$, where t is a sufficiently large positive integer. Clearly, $n = (2a+b)t+2$. Let $a = g(x) = f(x) = b$ for all $x \in V(G)$. We have

$$\frac{(a+b)n}{2a+b} > \delta(G) = (a+b)t+1 > \frac{(a+b)n}{2a+b} - 1,$$
$$\frac{(a+b)n}{2a+b} > \max\{d_G(u), d_G(v)\} = (a+b)t+1 > \frac{(a+b)n}{2a+b} - 1,$$
$$\frac{2(a+b)n}{2a+b} > \sigma_2(G) = 2(a+b)t+2 > \frac{2(a+b)n}{2a+b} - 1.$$

Let $I = (at+1)K_1$. For $G' = K_{bt} \vee (at+1)K_1$, let $S = K_{bt}$ and $T = (at+1)K_1$. Then we have $\sum_{x\in T} d_H(x) - e_H(T, S) = 0$ for any subset H of $E(G')$ with m edges. Therefore,

$$\begin{aligned} & f(S) - g(T) + d_{G-S}(T) - \left(\sum_{x\in T} d_H(x) - e_H(T, S)\right) \\ & = (bt) - b(at+1) \\ & = -b. \end{aligned}$$

Thus, G' is not a fractional (g, f, m)-deleted graph by Lemma 5. In conclusion, G is not a fractional ID-(g, f, m)-deleted graph.

Next, we show that the minimum degree condition in Theorem 5 and Theorem 6 is best in some sense. Let $a = b = k$. Let n be a sufficiently large integer which divided by 3. G' is such a graph with $|V(G')| = 2n/3$: a isolated vertex v adjacent to $k+m-1$ vertices in $K_{2n/3-1}$. Considering $G = ((n/3)K_1) \vee G'$. Let $I = (n/3)K_1$. Deleting I form G, we have $\delta(G') = k+m-1$. Delete m edges incident to v in G', then the resulting graph G'' has $\delta(G'') = k-1$, which has no fractional k-factor by the definition. Therefore, G' is not a fractional (k, m)-deleted graph and G is not a fractional ID-(k, m)-deleted graph.

3.3 Degree Conditions for Fractional ID-(a, b, m)-Deleted Graphs

We get the following degree conditions for fractional ID-(a, b, m)-deleted graphs using Corollarys 5, 6, 7 and 8, and the tricks in Sects. 2.4 and 3.1.

Theorem 10 *Let G be a graph of order n, and let a, b, and m be non-negative integers such that $2 \le a \le b$ and $n > (a+2b)(a+b+2m-2)/b$. If G satisfies $\delta(G) \ge (a+b)n/(a+2b)$, then G is a fractional ID-(a, b, m)-deleted graph.*

Theorem 11 *Let G be a graph of order n, and let a, b, and m be non-negative integers such that $2 \le a \le b$, $n > (a+2b)(a+b+2m-1)/b$ and $\delta(G) \ge bn/(a+2b)+a+m$. If G satisfies*

$$\max\{d_G(x), d_G(y)\} \ge \frac{(a+b)n}{a+2b}$$

for each pair of non-adjacent vertices x and y of G, then G is a fractional ID-(a, b, m)-deleted graph.

Theorem 12 *Let G be a graph of order n, and let a, b, and m be non-negative integers such that $2 \le a \le b$, $n > (a+2b)(a+b+2m-2)/b$ and $\delta(G) \ge bn/(a+2b)+a+m$. If G satisfies $\sigma_2(G) \ge 2(a+b)n/(a+2b)$, then G is a fractional ID-(a, b, m)-deleted graph.*

Remark 2 Likewise, although fractional ID-(a, b, m)-deleted graph is a special kind of fractional ID-(g, f, m)-critical deleted graph when $g(x) = a$ and $f(x) = b$ for all

$x \in V(G)$, Theorem 10–12 cannot be derived directly from Theorem 4–6. Therefore, some technologies in Sect. 2.4 and Sect. 3.1 are applied for proving Theorem 10–12.

Using the example $G = (at+1)K_1 \vee K_{bt} \vee (at+1)K_1$ in Sect. 3.2, we verify that the degree conditions in Theorem 10, Theorem 11, and Theorem 12 are also sharp in some sense. Again, the restrictions on $\delta(G)$ in Theorem 11 and Theorem 12 cannot be weaken.

We get three degree conditions for fractional ID-(k, m)-deleted graphs from Theorem 10, Theorem 11, and Theorem 12 by taking $a = b = k$. Let $m = 0$ in three results above, and the corresponding degree conditions for fractional ID-$[a, b]$-factor-critical graphs are determined.

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