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A Best Proximity Point Theorem for Generalized Geraghty–Suzuki Contractions

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Abstract We give a new type of contractive condition that ensures the existence and uniqueness of fixed points and best proximity points in complete metric spaces. We provide an example to validate our best proximity point theorem. This result extends and complements some known results from the literature.

Keywords Best proximity point · Fixed point · Generalized Geraghty–Suzuki contraction

1 Introduction and Preliminaries

The Banach contraction mapping principle is a crucial theorem in fixed point theory, which asserts that every contraction on a complete metric space has a unique fixed point. Consequently, a number of extensions of this result appeared in the literature (see [28] and references therein); in particular, one of the most interesting generalizations was given by Geraghty [8] as follows.

Theorem 1 (Geraghty [8]) *Let* (X, d) *be a complete metric space and* $T : X \to X$ *be an operator. Suppose that there exists* $\beta : [0, +\infty) \to [0, 1)$ *satisfying the condition*

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$$\beta(t_n) \to 1 \textit{ implies } t_n \to 0, \ \textit{as } n \to +\infty.$$

If T satisfies the following inequality

$$d(Tx, Ty) \leq \beta(d(x, y))\, d(x, y), \quad \textit{for all } x, y \in X,$$

then T has a unique fixed point.

On the other hand, Kirk [13] explored several significant generalizations of the Banach contraction mapping principle to the case of non-self-mappings. Let A and B be nonempty subsets of a metric space (X, d). A mapping $T : A \to B$ is called a k-contraction if there exists $k \in [0, 1)$ such that $d(Tx, Ty) \leq kd(x, y)$, for all $x, y \in A$. Notice that k-contraction coincides with Banach contraction mapping if one take $A = B$.
Moreover, a contraction non-self-mapping may not have a fixed point. In this case, it is quite natural to find an element x such that $d(x, Tx)$ is minimum, which implies that x and Tx are in close proximity to each other. Precisely, in light of the fact that $d(x, Tx)$ is at least $d(A, B) := \inf\{d(x, y) : x \in A, y \in B\}$, we are interested in establishing the existence of an element x for which $d(x, Tx) = d(A, B)$, such an element is designated as a best proximity point of the non-self-mapping T. Obviously, a best proximity point reduces to a fixed point if the considered mapping is a self-mapping.

This research subject has attracted attention of many authors, as confirmed referring to [1–30]. It should be noted that best proximity point theorems furnish an approximate solution to the equation $Tx = x$, when T has no fixed point.

Here, we collect some notions and notations which will be used throughout the rest of this work. We denote by A_0 and B_0 the following sets:

$$A_0 = \{x \in A : d(x, y) = d(A, B) \text{ for some } y \in B\},$$
$$B_0 = \{y \in B : d(x, y) = d(A, B) \text{ for some } x \in A\}.$$

In 2003, Kirk et al. [14] presented sufficient conditions for determining when the sets A_0 and B_0 are nonempty.

Let $\mathcal{F}$ denote the class of all functions $\beta : [0, +\infty) \to [0, 1)$ satisfying the following condition:

$$\beta(t_n) \to 1 \text{ implies } t_n \to 0, \text{ as } n \to +\infty.$$

Definition 1 ([8]) Let (A, B) be a pair of nonempty subsets of a metric space (X, d). A mapping $T : A \to B$ is said to be a Geraghty-contraction if there exists $\beta \in \mathcal{F}$ such that

$$d(Tx, Ty) \leq \beta((d(x, y))\, d(x, y), \quad \text{for all } x, y \in A.$$

In [24], Raj introduced the following definition.

Definition 2 Let (A, B) be a pair of nonempty subsets of a metric space (X, d) with $A_0 \neq \emptyset$. Then the pair (A, B) is said to have the P-property if and only if for all $x_1, x_2 \in A_0$ and $y_1, y_2 \in B_0$,

$$\left.\begin{aligned} d(x_1, y_1) = d(A, B) \\ d(x_2, y_2) = d(A, B) \end{aligned}\right\} \Rightarrow d(x_1, x_2) = d(y_1, y_2).$$

Also in [24], the author showed that any pair (A, B) of nonempty closed convex subsets of a real Hilbert space satisfies the P-property. Moreover, it is easily seen that, for any nonempty subset A of (X, d), the pair (A, A) has the P-property.

Finally we recall the result obtained by Caballero et al. [4].

Theorem 2 *Let (A, B) be a pair of nonempty closed subsets of a complete metric space (X, d) such that A_0 is nonempty. Let $T : A \to B$ be a Geraghty-contraction satisfying $T(A_0) \subseteq B_0$. Suppose that the pair (A, B) has the P-property. Then there exists a unique x^* in A such that $d(x^*, Tx^*) = d(A, B)$.*

In this paper, motivated by Caballero et al. [4] and Salimi and Karapinar [25], we give a new type of contractive condition that ensures the existence and uniqueness of fixed points and best proximity points in complete metric spaces. The presented results are independent from the analogous results in [4], as shown with a simple example.

2 Main Results

In this section, we introduce the notion of generalized Geraghty–Suzuki contraction and use this notion for proving our main result.

Definition 3 Let (A, B) be a pair of nonempty subsets of a metric space (X, d). A mapping $T : A \to B$ is said to be a generalized Geraghty–Suzuki contraction if there exists $\beta \in \mathcal{F}$ such that

$$\frac{1}{2}d^*(x, Tx) \leq d(x, y) \Rightarrow d(Tx, Ty) \leq \beta(M(x, y))[M(x, y) - d(A, B)], \quad (2.1)$$

for all $x, y \in A$, where $d^*(x, y) = d(x, y) - d(A, B)$ and

$$M(x, y) = \max\left\{d(x, y), d(x, Tx), d(y, Ty)\right\}.$$

Thus, we state and prove the following result of existence and uniqueness.

Theorem 3 *Let (A, B) be a pair of nonempty closed subsets of a complete metric space (X, d) such that A_0 is nonempty. Let $T : A \to B$ be a generalized Geraghty–Suzuki contraction such that $T(A_0) \subseteq B_0$. Suppose that the pair (A, B) has the P-property. Then there exists a unique x^* in A such that $d(x^*, Tx^*) = d(A, B)$.*

Proof Let us select an element $x_0 \in A_0$; since $Tx_0 \in T(A_0) \subseteq B_0$, we can find $x_1 \in A_0$ such that $d(x_1, Tx_0) = d(A, B)$. Further, since $Tx_1 \in T(A_0) \subseteq B_0$, it follows that there is an element x_2 in A_0 such that $d(x_2, Tx_1) = d(A, B)$. Recursively, we obtain a sequence $\{x_n\}$ in A_0 such that

$$d(x_{n+1}, Tx_n) = d(A, B), \text{ for any } n \in \mathbb{N} \cup \{0\}. \quad (2.2)$$

Since (A, B) has the P-property, we derive that

$$d(x_n, x_{n+1}) = d(Tx_{n-1}, Tx_n), \text{ for any } n \in \mathbb{N}. \quad (2.3)$$

Now, by (2.2) we get

$$d(x_{n-1}, Tx_{n-1}) \leq d(x_{n-1}, x_n) + d(x_n, Tx_{n-1}) = d(x_{n-1}, x_n) + d(A, B) \quad (2.4)$$

and by (2.2) and (2.3) we obtain

$$d(x_n, Tx_n) \leq d(x_n, Tx_{n-1}) + d(Tx_{n-1}, Tx_n) = d(x_n, x_{n+1}) + d(A, B).$$

Therefore, we have

$$\begin{aligned} M(x_{n-1}, x_n) &= \max\{d(x_{n-1}, x_n), d(x_{n-1}, Tx_{n-1}), d(x_n, Tx_n)\} \\ &\leq \max\{d(x_{n-1}, x_n), d(x_n, x_{n+1})\} + d(A, B). \end{aligned} \quad (2.5)$$

Clearly, if there exists $n_0 \in \mathbb{N}$ such that $d(x_{n_0}, x_{n_0+1}) = 0$, then we have nothing to prove, the conclusion is immediate. In fact,

$$0 = d(x_{n_0}, x_{n_0+1}) = d(Tx_{n_0-1}, Tx_{n_0}),$$

and consequently, $Tx_{n_0-1} = Tx_{n_0}$. Thus, we conclude that

$$d(A, B) = d(x_{n_0}, Tx_{n_0-1}) = d(x_{n_0}, Tx_{n_0}).$$

For the rest of the proof, we suppose that $d(x_n, x_{n+1}) > 0$ for any $n \in \mathbb{N} \cup \{0\}$. Now from (2.4), we deduce that

$$\frac{1}{2} d^*(x_{n-1}, Tx_{n-1}) \leq d^*(x_{n-1}, Tx_{n-1}) \leq d(x_n, x_{n-1})$$

and by (2.1), we get

$$\begin{aligned} d(x_n, x_{n+1}) &= d(Tx_{n-1}, Tx_n) \\ &\leq \beta(M(x_{n-1}, x_n))[M(x_{n-1}, x_n) - d(A, B)] \\ &< M(x_{n-1}, x_n) - d(A, B). \end{aligned} \quad (2.6)$$

By (2.5) and (2.6), we obtain

$$d(x_n, x_{n+1}) < M(x_{n-1}, x_n) - d(A, B) \leq \max\{d(x_{n-1}, x_n), d(x_n, x_{n+1})\}.$$

Now, if $\max\{d(x_{n-1}, x_n), d(x_n, x_{n+1})\} = d(x_n, x_{n+1})$, then

$$d(x_n, x_{n+1}) < d(x_n, x_{n+1}),$$

which is a contradiction and hence

$$M(x_{n-1}, x_n) \leq \max\{d(x_{n-1}, x_n), d(x_n, x_{n+1})\} + d(A, B) = d(x_{n-1}, x_n) + d(A, B).$$

Therefore, by (2.6) we get

$$\begin{aligned} d(x_n, x_{n+1}) &= d(Tx_{n-1}, Tx_n) \\ &\le \beta(M(x_{n-1}, x_n))d(x_{n-1}, x_n) \\ &< d(x_{n-1}, x_n), \end{aligned} \tag{2.7}$$

for all $n \in \mathbb{N}$. Consequently, $\{d(x_n, x_{n+1})\}$ is a decreasing sequence and bounded below and so $\lim_{n\to+\infty} d(x_n, x_{n+1}) := L$ exists. Suppose $L > 0$ and then, from (2.7), we have

$$\frac{d(x_{n+1}, x_{n+2})}{d(x_n, x_{n+1})} \le \beta(M(x_n, x_{n+1})) \le 1,$$

for any $n \ge 0$, which implies that

$$\lim_{n\to+\infty} \beta(M(x_n, x_{n+1})) = 1.$$

On the other hand, since $\beta \in \mathcal{F}$, we get $\lim_{n\to+\infty} M(x_n, x_{n+1}) = 0$, that is,

$$L = \lim_{n\to+\infty} d(x_n, x_{n+1}) = 0.$$

Since, $d(x_n, Tx_{n-1}) = d(A, B)$ holds for all $n \in \mathbb{N}$ and the pair (A, B) satisfies the P-property, then for all $m, n \in \mathbb{N}$, we can write $d(x_m, x_n) = d(Tx_{m-1}, Tx_{n-1})$. Using the fact that

$$d(x_l, Tx_l) \le d(x_l, x_{l+1}) + d(x_{l+1}, Tx_l) = d(x_l, x_{l+1}) + d(A, B)$$

for all $l \in \mathbb{N}$, we deduce easily

$$\begin{aligned} M(x_m, x_n) &= \max\{d(x_m, x_n), d(x_m, Tx_m), d(x_n, Tx_n)\} \\ &\le \max\{d(x_m, x_n), d(x_m, x_{m+1}), d(x_n, x_{n+1})\} + d(A, B). \end{aligned}$$

Since $\lim\limits_{n\to+\infty} d(x_n, x_{n+1}) = 0$, then we have

$$\lim_{m,n\to+\infty} M(x_m, x_n) \le \lim_{m,n\to+\infty} d(x_m, x_n) + d(A, B). \tag{2.8}$$

We shall show that $\{x_n\}$ is a Cauchy sequence. If not, then we get

$$\left[\lim_{m,n\to+\infty} d(x_n, x_m) > 0.\right]$$

Thus, without loss of generality, we can assume

$$\varepsilon = \lim_{m,n\to+\infty} d(x_n, x_m) > 0. \tag{2.9}$$

By using the triangular inequality, we have

$$d(x_n, x_m) \leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{m+1}) + d(x_{m+1}, x_m). \tag{2.10}$$

Now, since $\lim_{n\to+\infty} d(x_n, x_{n+1}) = 0$, then

$$\begin{aligned} d(A, B) &\leq \lim_{m\to+\infty} d(x_m, Tx_m) \\ &\leq \lim_{m\to+\infty} [d(x_m, x_{m+1}) + d(x_{m+1}, Tx_m)] \\ &= \lim_{m\to+\infty} [d(x_m, x_{m+1}) + d(A, B)] = d(A, B), \end{aligned}$$

which implies $\lim_{m\to+\infty} d(x_m, Tx_m) = d(A, B)$, that is

$$\lim_{m\to+\infty} \frac{1}{2} d^*(x_m, Tx_m) = \lim_{m\to+\infty} \frac{1}{2}[d(x_m, Tx_m) - d(A, B)] = 0.$$

On the other hand, from (2.9) it follows that there exists $N \in \mathbb{N}$ such that, for all $m, n \geq N$, we have

$$\frac{1}{2} d^*(x_m, Tx_m) \leq d(x_n, x_m).$$

Now, from (2.1) and (2.10) we have

$$\begin{aligned} d(x_n, x_m) &\leq d(x_n, x_{n+1}) + d(Tx_n, Tx_m) + d(x_{m+1}, x_m) \\ &\leq d(x_n, x_{n+1}) + \beta(M(x_n, x_m))[M(x_n, x_m) - d(A, B)] + d(x_{m+1}, x_m). \end{aligned} \tag{2.11}$$

Then from (2.8), (2.11) and $\lim_{n\to+\infty} d(x_n, x_{n+1}) = 0$, we have

$$\begin{aligned} \lim_{m,n\to+\infty} d(x_n, x_m) &\leq \lim_{m,n\to+\infty} \beta(M(x_n, x_m)) \lim_{m,n\to+\infty} [M(x_m, x_n) - d(A, B)] \\ &\leq \lim_{m,n\to+\infty} \beta(M(x_n, x_m)) \lim_{m,n\to+\infty} d(x_m, x_n) \end{aligned}$$

and so, by (2.9), we get

$$1 \leq \lim_{m,n\to+\infty} \beta(M(x_n, x_m)),$$

that is $\lim_{m,n\to+\infty} \beta(M(x_n, x_m)) = 1$. Therefore, $\lim_{m,n\to+\infty} M(x_n, x_m) = 0$ and consequently $\lim_{m,n\to+\infty} d(x_n, x_m) = 0$, which is a contradiction. Thus, $\{x_n\}$ is a Cauchy sequence. Since $\{x_n\} \subset A$ and A is a closed subset of the complete metric space (X, d), we can find $x^* \in A$ such that $x_n \to x^*$, as $n \to +\infty$. We shall show that $d(x^*, Tx^*) = d(A, B)$. Suppose to the contrary that $d(x^*, Tx^*) > d(A, B)$. At first, we have

$$\begin{aligned} d(x^*, Tx^*) &\leq d(x^*, Tx_n) + d(Tx_n, Tx^*) \\ &\leq d(x^*, x_{n+1}) + d(x_{n+1}, Tx_n) + d(Tx_n, Tx^*) \\ &\leq d(x^*, x_{n+1}) + d(A, B) + d(Tx_n, Tx^*), \end{aligned}$$

and taking limit as $n \to +\infty$, we get

$$d(x^*, Tx^*) - d(A, B) \leq \lim_{n\to+\infty} d(Tx_n, Tx^*). \tag{2.12}$$

Also, we have

$$d(x_n, Tx_n) \leq d(x_n, x_{n+1}) + d(x_{n+1}, Tx_n) = d(x_n, x_{n+1}) + d(A, B).$$

Taking limit as $n \to +\infty$ in the above inequality, we obtain

$$\lim_{n\to+\infty} d(x_n, Tx_n) \leq d(A, B),$$

that is, $\lim_{n\to+\infty} d(x_n, Tx_n) = d(A, B)$. Then, we get

$$\begin{aligned} \lim_{n\to+\infty} M(x_n, x^*) &= \max\{\lim_{n\to+\infty} d(x^*, x_n), \lim_{n\to+\infty} d(x_n, Tx_n), d(x^*, Tx^*)\} \\ &= d(x^*, Tx^*) \end{aligned}$$

and hence

$$\lim_{n\to+\infty} M(x_n, x^*) - d(A, B) = d(x^*, Tx^*) - d(A, B). \tag{2.13}$$

Next, we have

$$\begin{aligned} d^*(x_n, Tx_n) &= d(x_n, Tx_n) - d(A, B) \\ &\leq d(x_n, x_{n+1}) + d(x_{n+1}, Tx_n) - d(A, B) \\ &= d(x_n, x_{n+1}) \end{aligned} \tag{2.14}$$

and

$$\begin{aligned} d^*(x_{n+1}, Tx_{n+1}) &= d(x_{n+1}, Tx_{n+1}) - d(A, B) \\ &\leq d(Tx_n, Tx_{n+1}) + d(x_{n+1}, Tx_n) - d(A, B) \\ &= d(Tx_n, Tx_{n+1}) \\ &= d(x_{n+1}, x_{n+2}) \\ &< d(x_n, x_{n+1}), \end{aligned} \tag{2.15}$$

and so (2.14) and (2.15) imply that

$$\frac{1}{2}[d^*(x_n, Tx_n) + d^*(x_{n+1}, Tx_{n+1})] \leq d(x_n, x_{n+1}). \tag{2.16}$$

Now, we suppose that the following inequalities hold

$$\frac{1}{2}d^*(x_n, Tx_n) > d(x_n, x^*) \quad \text{and} \quad \frac{1}{2}d^*(x_{n+1}, Tx_{n+1}) > d(x_{n+1}, x^*),$$

for some $n \in \mathbb{N} \cup \{0\}$. Hence, by using (2.16), we can write

$$\begin{aligned} d(x_n, x_{n+1}) &\leq d(x_n, x^*) + d(x_{n+1}, x^*) \\ &< \frac{1}{2}[d^*(x_n, Tx_n) + d^*(x_{n+1}, Tx_{n+1})] \\ &\leq d(x_n, x_{n+1}), \end{aligned}$$

which is a contradiction. Then, for any $n \in \mathbb{N} \cup \{0\}$, either

$$\frac{1}{2}d^*(x_n, Tx_n) \leq d(x_n, x^*) \quad \text{or} \quad \frac{1}{2}d^*(x_{n+1}, Tx_{n+1}) \leq d(x_{n+1}, x^*)$$

holds. Therefore, by (2.1), (2.12) and (2.13) we deduce

$$\begin{aligned} d(x^*, Tx^*) - d(A, B) &\leq \lim_{n\to+\infty} d(Tx_n, Tx^*) \\ &\leq \lim_{n\to+\infty} \beta(M(x_n, x^*)) \lim_{n\to+\infty} [M(x_n, x^*) - d(A, B)] \\ &= \lim_{n\to+\infty} \beta(M(x_n, x^*))[d(x^*, Tx^*) - d(A, B)]. \end{aligned} \tag{2.17}$$

Since $d(x^*, Tx^*) - d(A, B) > 0$, then from (2.17) we get

$$1 \leq \lim_{n\to+\infty} \beta(M(x_n, x^*)),$$

that is,

$$\lim_{n\to+\infty} \beta(M(x_n, x^*)) = 1,$$

which implies

$$\lim_{n\to+\infty} M(x_n, x^*) = d(x^*, Tx^*) = 0$$

and so $d(x^*, Tx^*) = 0 > d(A, B)$, a contradiction. Therefore, $d(x^*, Tx^*) \leq d(A, B)$, that is, $d(x^*, Tx^*) = d(A, B)$. This means that x^* is a best proximity point of T and so the existence of a best proximity point is proved.

We shall show the uniqueness of the best proximity point of T. Suppose that x^* and y^* are two distinct best proximity points of T, that is, $x^* \neq y^*$. This implies that

$$d(x^*, Tx^*) = d(A, B) = d(y^*, Ty^*).$$

Using the P-property, we have

$$d(x^*, y^*) = d(Tx^*, Ty^*)$$

and so

$$\begin{aligned} M(x^*, y^*) &= \max\{d(x^*, y^*), d(x^*, Tx^*), d(y^*, Ty^*)\} \\ &= \max\{d(x^*, y^*), d(A, B)\}. \end{aligned}$$

Also, we have

$$\frac{1}{2}d^*(x^*, Tx^*) = \frac{1}{2}[d(x^*, Tx^*) - d(A, B)] = 0 \le d(x^*, y^*).$$

Since $M(x^*, y^*) - d(A, B) \le d(x^*, y^*)$, by (2.1), we have

$$\begin{aligned} d(x^*, y^*) &= d(Tx^*, Ty^*) \\ &\le \beta(M(x^*, y^*))[M(x^*, y^*) - d(A, B)] \\ &< d(x^*, y^*), \end{aligned}$$

which is a contradiction. This completes the proof. □

In order to demonstrate the independence of our result from Theorem 2, we give the following example.

Example 1 Consider the space $X = \mathbb{R}^2$ endowed with the metric $d : X \times X \to [0, +\infty)$ given by

$$d((x_1, x_2), (y_1, y_2)) = |x_1 - y_1| + |x_2 - y_2|,$$

for all $(x_1, x_2), (y_1, y_2) \in X$. Define the sets

$$A = \{(1, 0), (4, 5), (5, 4)\} \text{ and } B = \{(0, 0), (0, 4), (4, 0)\},$$

so that $d(A, B) = 1$, $A_0 = \{(1, 0)\}$, $B_0 = \{(0, 0)\}$ and the pair (A, B) has the P-property. Also define $T : A \to B$ by

$$T(x_1, x_2) = \begin{cases} (x_1, 0) & \text{if } x_1 \le x_2, \\ (0, x_2) & \text{if } x_1 > x_2. \end{cases}$$

Notice that $T(A_0) \subseteq B_0$. Now, consider the function $\beta : [0, +\infty) \to [0, 1)$ given by

$$\beta(t) = \begin{cases} 0 & \text{if } t = 0, \\ \frac{\ln(1+t)}{t} & \text{if } 0 < t \le 1, \\ \frac{t}{1+t} & \text{if } 1 < t \le 10, \\ \frac{10}{11} & \text{if } t > 10, \end{cases}$$

and note that $\beta \in \mathcal{F}$.
Assume that $\frac{1}{2}d^*(x, Tx) \leq d(x, y)$, for some $x, y \in A$. Then,

$$\begin{cases} x = (1, 0), & y = (4, 5) \text{ or} \\ x = (1, 0), & y = (5, 4) \text{ or} \\ y = (1, 0), & x = (4, 5) \text{ or} \\ y = (1, 0), & x = (5, 4). \end{cases}$$

Since $d(Tx, Ty) = d(Ty, Tx)$ and $M(x, y) = M(y, x)$ for all $x, y \in A$, hence without loss of generality, we can assume that

$$(x, y) = ((1, 0), (4, 5)) \text{ or } (x, y) = ((1, 0), (5, 4)).$$

Now, we distinguish the following cases:

(i) if $(x, y) = ((1, 0), (4, 5))$, then

$$\begin{aligned} d(T(1, 0), T(4, 5)) = 4 &\leq \frac{8}{1+8} \cdot (8-1) \\ &= \beta(M((1, 0), (4, 5)))[M((1, 0), (4, 5)) - 1]; \end{aligned}$$

(ii) if $(x, y) = ((1, 0), (5, 4))$, then

$$\begin{aligned} d(T(1, 0), T(5, 4)) = 4 &\leq \frac{8}{1+8} \cdot (8-1) \\ &= \beta(M((1, 0), (5, 4)))[M((1, 0), (5, 4)) - 1]. \end{aligned}$$

Consequently, we have

$$\frac{1}{2}d^*(x, Tx) \leq d(x, y) \Rightarrow d(Tx, Ty) \leq \beta(M(x, y))[M(x, y) - d(A, B)]$$

and hence all the conditions of Theorem 3 hold and T has a unique best proximity point. Here, $x = (1, 0)$ is a unique best proximity point of T. On the other hand, if $(x, y) = ((4, 5), (5, 4))$, then we have

$$d(T(4, 5), T(5, 4)) = 8 > 4/3 = \beta(d((4, 5), (5, 4)))d((4, 5), (5, 4)),$$

that is, Theorem 2 cannot be applied in this case.

If in Theorem 3 we take $\beta(t) = r$, where $r \in [0, 1)$, then we have the following consequence.

Corollary 1 *Let (A, B) be a pair of nonempty closed subsets of a complete metric space (X, d) such that A_0 is nonempty. Let $T : A \to B$ be a non-self-mapping such that $T(A_0) \subseteq B_0$ and*

$$\frac{1}{2}d^*(x, Tx) \leq d(x, y) \Rightarrow d(Tx, Ty) \leq r[\max\left\{d(x, y), d(x, Tx), d(y, Ty)\right\} - d(A, B)],$$

for all $x, y \in A$*, where* $d^*(x, y) = d(x, y) - d(A, B)$*. Suppose that the pair* (A, B) *has the* P*-property. Then there exists a unique* x^* *in* A *such that* $d(x^*, Tx^*) = d(A, B)$*.*

If in Theorem 3 we take $A = B = X$, then we deduce the following fixed point result.

Corollary 2 *Let* (X, d) *be a complete metric space and* $T : X \to X$ *be a self-mapping. Assume that there exists* $\beta \in \mathcal{F}$ *such that*

$$\frac{1}{2}d(x, Tx) \leq d(x, y) \Rightarrow d(Tx, Ty) \leq \beta(M(x, y))M(x, y),$$

for all $x, y \in A$*, where*

$$M(x, y) = \max\left\{d(x, y), d(x, Tx), d(y, Ty)\right\}.$$

Then T *has a unique fixed point.*

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