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Normal Families and Shared Values of Meromorphic Functions

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Abstract Let a and b be two given meromorphic functions on a domain D. We study normality of the family $\mathcal{F}$ of meromorphic functions that satisfy $f(z)f^{(k)}(z) = a(z) \Leftrightarrow f^{(k)}(z) = b(z)$ for every $f \in \mathcal{F}$ on D. Examples are also given to show the necessity of the conditions in our results.

Keywords Meromorphic functions · Normal families · Shared values · Shared functions

Mathematics Subject Classification 30D45

1 Introduction and Main Result

Let $\mathcal{F}$ be a family of meromorphic functions on a domain $D \subset \mathbb{C}$. Then $\mathcal{F}$ is said to be normal on D in the sense of Montel, if each sequence of $\mathcal{F}$ contains a subsequence which converges spherically uniformly on each compact subset of D to a meromorphic function which may be ∞ identically. See [1–3].

For two functions f and g meromorphic on D, and two complex numbers or meromorphic functions a and b, we write $f(z) = a(z) \Rightarrow g(z) = b(z)$ if $g(z) = b(z)$ whenever $f(z) = a(z)$, and write $f(z) = a(z) \Leftrightarrow g(z) = b(z)$ if $f(z) = a(z)$ if and only if $g(z) = b(z)$. When a is a complex value and $f(z) = a \Leftrightarrow g(z) = a$, we also say that f and g share the value a or a is a shared value of f and g. For families

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of meromorphic functions, the connection between normality and shared values has been studied frequently following Schwick's initial paper [4]. Some recent results in this area appear in [5–9].

The starting point of this paper is the following result.

Theorem A [10, Theorem 2] *Let $\mathcal{F}$ be a family of meromorphic functions on a domain D, k be a positive integer, and let $a \neq 0$ and b be two finite values. If, for every $f \in \mathcal{F}$, all zeros of f have multiplicity at least k and $f(z)f^{(k)}(z) = a \Leftrightarrow f^{(k)}(z) = b$, then the family $\mathcal{F}$ is normal on D.*

In this paper, we prove the following result.

Theorem 1.1 *Let k be a positive integer, and let $a(z)(\not\equiv 0)$ and $b(z)$ be two functions meromorphic on D such that*

(i) *all zeros of a have multiplicity at most $k-1$ and all poles of a have multiplicity at most k;*
(ii) *each pole of b that is not a zero of a has multiplicity at most $\lceil \frac{k}{2} \rceil - 1$; and each pole of b that is a zero of a with multiplicity m has multiplicity at most $\lceil \frac{k-m}{2} \rceil - 1$.*

Then the family $\mathcal{F}$ of meromorphic functions on a domain D, all of whose zeros have multiplicity at least k, such that $f(z)f^{(k)}(z) = a(z) \Leftrightarrow f^{(k)}(z) = b(z)$ for every $f \in \mathcal{F}$, is normal on D.

Here, $\lceil x \rceil$ denotes the smallest integer that is not less than x. For example, $\lceil 2.1 \rceil = 3$ and $\lceil 2 \rceil = 2$.

Example 1.1 Let $D = \{z : |z| < 1\}$ and $\mathcal{F} = \{f_n\}$, where

$$f_n(z) = e^{nz} - \frac{1}{n}.$$

Then $f_n'(z) = ne^{nz}$, and $f_n(z)f_n'(z) = n\left(e^{nz} - \frac{1}{n}\right)e^{nz}$. It follows that $f_n(z)f_n'(z) = 0 \Leftrightarrow f_n'(z) = 1$, but $\mathcal{F}$ is not normal at 0. This shows that the condition $a(z) \not\equiv 0$ is necessary in Theorem 1.1.

Example 1.2 Let $D = \{z : |z| < 1\}$ and $\mathcal{F} = \{f_n\}$, where

$$f_n(z) = z + \frac{1}{nz},$$

and let $a(z) = z$ and $b = 1$. We see that $f_n'(z) = 1 - \frac{1}{nz^2} \neq 1$ and $f_n(z)f_n'(z) = z\left(1 - \frac{1}{n^2z^4}\right) \neq z$. So for every $f_n \in \{f_n\}$ satisfies that $f_n(z)f_n'(z) = a(z) \Leftrightarrow f_n'(z) = b(z)$. But $\mathcal{F}$ is not normal at 0. This shows that the condition that every zero of a has multiplicity at most $k-1$ (at least for $k=1$) is sharp in Theorem 1.1.

Example 1.3 Let $D = \{z : |z| < 1\}$ and $\mathcal{F} = \{f_n\}$, where $f_n(z) = nz^{k+1}$, and let $a(z) = z^{k+2}$ and $b(z) = z$. We see that $f_n^{(k)}(z) = n(k+1)!z$ and $f_n(z)f_n^{(k)}(z) = n^2(k+1)!z^{k+2}$. So for every $f_n \in \{f_n\}$ satisfies that $f_n(z)f_n^{(k)}(z) = a(z) \Leftrightarrow f_n^{(k)}(z) = b(z)$. But $\mathcal{F}$ is not normal at 0. This shows that the condition that every zero of a has multiplicity at most $k-1$ is necessary in Theorem 1.1.

Example 1.4 Let $D = \{z : |z| < 1\}$ and $\mathcal{F} = \{f_n\}$, where $f_n(z) = 1/nz$, and let $a(z) = 1/z^{k+2}$ and $b = 1/z^{k+1}$. We see that $f_n^{(k)}(z) = (-1)^k k!/nz^{k+1}$ and $f_n(z)f_n'(z) = (-1)^k k!/n^2 z^{k+2}$. So for every $f_n \in \{f_n\}$ satisfies that $f_n(z)f_n'(z) = a(z) \Leftrightarrow f_n'(z) = b(z)$. But $\mathcal{F}$ is not normal at 0. This shows that the condition that every pole of a has multiplicity at most k is necessary in Theorem 1.1.

2 Some Lemmas

In order to prove our theorem, we require the following results. We assume the standard notations of value distribution theory, as presented and used in [2]. In particular, we write $f_n \xrightarrow{\chi} f$ on D to denote that the sequence $\{f_n\}$ converges spherically locally uniformly to f on D and denote $f_n \to f$ on D if the convergence is in Euclidean metric.

Lemma 2.1 ([11, Theorem 2]; [12, Lemma 2]) *Let $\mathcal{F}$ be a family of functions meromorphic on D, all of whose zeros have multiplicity at least k. Then if $\mathcal{F}$ is not normal at some point z_0 in D, there exist, for each $0 \le \alpha < k$, points z_n in D with $z_n \to z_0$, positive numbers $\rho_n \to 0$ and functions $f_n \in \mathcal{F}$ such that $g_n(\zeta) = \rho_n^{-\alpha} f_n(z_n + \rho_n\zeta) \xrightarrow{\chi} g(\zeta)$ on $\mathbb{C}$, where g is a nonconstant meromorphic function on $\mathbb{C}$, all of whose zeros have multiplicity at least k, such that $g^\sharp(\zeta) \le g^\sharp(0) = 1$. In particular, g has order at most two.*

Here, as usual, $g^\sharp(\zeta) = |g'(\zeta)|/(1 + |g(\zeta)|^2)$ is the spherical derivative.

Lemma 2.2 [10, Lemmas 9 and 10] *Let g be a nonconstant meromorphic function in $\mathbb{C}$, and a be a nonzero constant. If all zeros of g have multiplicity at least k and $g^{(k)} \not\equiv 0$, then the equation $gg^{(k)} = a$ has solutions on $\mathbb{C}$, where k is a positive integer.*

Lemma 2.3 [13, Lemma 8] *Let f be a nonpolynomial rational function such that $f'(z) \neq 1$ for $z \in \mathbb{C}$. Then*

$$f(z) = z + c + \frac{a}{(z+b)^m},$$

where $a \neq 0, b, c$ are constants and m is a positive integer.

Lemma 2.4 [14, Theorem 1.1] *Let g be a transcendental meromorphic function on $\mathbb{C}$, and $R \not\equiv 0$ be a rational function. If all zeros and poles of g are multiple except possibly finitely many, then $g' - R$ has infinitely many zeros on $\mathbb{C}$.*

Lemma 2.5 *Let $k \ge 2$ and m be two integers, and let g be a meromorphic function on $\mathbb{C}$, all of whose zeros have multiplicity at least k. If $g(\zeta)g^{(k)}(\zeta) \neq \gamma\zeta^m$ on $\mathbb{C} \setminus \{0\}$ and $g^{(k)}(\zeta) \neq 0$ on $\mathbb{C} \setminus \{0\}$, where γ is a given nonzero constant, then $m \ge k$ or $m \le -(k+2)$, and g must be a rational function of the form $g(\zeta) = C\zeta^{\frac{m+k}{2}}$ for some nonzero constant C.*

Proof Without loss of generality, we may assume that $\gamma = 1$. If not, we can use $G(\zeta) = \gamma^{-\frac{1}{2}} g$ to replace g. The conditions guarantee that all zeros of g, possibly except $\zeta = 0$, have multiplicity k exactly.

Suppose first that g is transcendental. Then by Nevanlinna's second fundamental theorem, we have

$$\begin{aligned} T\left(r, \frac{gg^{(k)}}{\zeta^m}\right) &\le \overline{N}\left(r, \frac{gg^{(k)}}{\zeta^m}\right) + \overline{N}\left(r, \frac{1}{\frac{gg^{(k)}}{\zeta^m}}\right) + \overline{N}\left(r, \frac{1}{\frac{gg^{(k)}}{\zeta^m} - 1}\right) + S(r, g) \\ &= \overline{N}(r, g) + \overline{N}\left(r, \frac{1}{g}\right) + S(r, g). \end{aligned} \tag{2.1}$$

where $S(r, g) = o(T(r, g))$ as $r \to \infty$, possibly outside a set of finite measure. On the other hand, we have by Nevanlinna's first fundamental theorem

$$\begin{aligned} T\left(r, \frac{gg^{(k)}}{\zeta^m}\right) \ge N\left(r, \frac{gg^{(k)}}{\zeta^m}\right) &\ge N(r, g) + N(r, g^{(k)}) + S(r, g) \\ &= 2N(r, g) + k\overline{N}(r, g) + S(r, g) \\ &\ge (k + 2)\overline{N}(r, g) + S(r, g) \end{aligned} \tag{2.2}$$

and

$$\begin{aligned} T\left(r, \frac{gg^{(k)}}{\zeta^m}\right) \ge N\left(r, \frac{\zeta^m}{gg^{(k)}}\right) &\ge N\left(r, \frac{1}{g}\right) + N\left(r, \frac{1}{g^{(k)}}\right) + S(r, g) \\ &= k\overline{N}\left(r, \frac{1}{g}\right) + S(r, g). \end{aligned} \tag{2.3}$$

Then by (2.1)–(2.3), we have

$$T\left(r, \frac{gg^{(k)}}{\zeta^m}\right) \le \left(\frac{1}{k+2} + \frac{1}{k}\right) T\left(r, \frac{gg^{(k)}}{\zeta^m}\right) + S(r, g). \tag{2.4}$$

Since $k \ge 2$, we see from (2.4) that

$$T\left(r, \frac{gg^{(k)}}{\zeta^m}\right) = S(r, g). \tag{2.5}$$

Then by (2.2) and (2.3), we have

$$N(r, g) = S(r, g), \; N\left(r, \frac{1}{g}\right) = S(r, g).$$

Thus

$$T\left(r, \frac{g}{g^{(k)}}\right) = T\left(r, \frac{g^{(k)}}{g}\right) + O(1) = N\left(r, \frac{g^{(k)}}{g}\right) + S(r, g) = S(r, g), \quad (2.6)$$

and hence by (2.5) and (2.6),

$$2T(r, g) = T\left(r, \zeta^m \cdot \frac{gg^{(k)}}{\zeta^m} \cdot \frac{g}{g^{(k)}}\right) = S(r, g).$$

This is a contradiction. Hence there is no transcendental function that satisfies the conditions of the lemma. □

Now we consider the case that g is a rational function.

Case 1. g has at least one nonzero pole. We denote by $\zeta_i (i = 1, 2, \ldots, n)$ all distinct poles of g on $\mathbb{C} \setminus \{0\}$, and $p_i (i = 1, 2, \ldots, n)$ their corresponding multiplicities. Since $g^{(k)}(\zeta) \neq 0$ on $\mathbb{C} \setminus \{0\}$, $g^{(Ak)}$ has the form

$$g^{(k)}(\zeta) = \frac{\lambda\zeta^s}{\prod_{i=1}^{n}(\zeta - \zeta_i)^{p_i+k}}, \quad (2.7)$$

where $s \in \mathbb{Z}$ is an integer and λ is a nonzero constant. And since $g(\zeta)g^{(k)}(\zeta) \neq \zeta^m$ on $\mathbb{C} \setminus \{0\}$, we have

$$g(\zeta)g^{(k)}(\zeta) = \zeta^m + \frac{\mu\zeta^l}{\prod_{i=1}^{n}(\zeta - \zeta_i)^{2p_i+k}} = \frac{\zeta^m \prod_{i=1}^{n}(\zeta - \zeta_i)^{2p_i+k} + \mu\zeta^l}{\prod_{i=1}^{n}(\zeta - \zeta_i)^{2p_i+k}} \quad (2.8)$$

for some integer $l \in \mathbb{Z}$ and nonzero constant μ. So, by (2.7) and (2.8),

$$g(\zeta) = \frac{\zeta^m \prod_{i=1}^{n}(\zeta - \zeta_i)^{2p_i+k} + \mu\zeta^l}{\lambda\zeta^s \prod_{i=1}^{n}(\zeta - \zeta_i)^{p_i}}. \quad (2.9)$$

Next, we consider three cases according to $m > l$, $m = l$ and $m < l$.

Case 1.1. Suppose that $m > l$. Then as all zeros of g, possibly except $\zeta = 0$, have multiplicity k exactly, we see from (2.9) that all zeros of the polynomial

$$P_1(\zeta) = \zeta^{m-l} \prod_{i=1}^{n}(\zeta - \zeta_i)^{2p_i+k} + \mu$$

on $\mathbb{C} \setminus \{0\}$, and hence on $\mathbb{C}$ since $P_1(0) = \mu \neq 0$, have exact multiplicity $k \geq 2$. This shows that P_1 has

$$\tau_1 = \frac{\deg P_1}{k} = \frac{m - l + \sum_{i=1}^{n}(2p_i + k)}{k} > n$$

distinct zeros, and each zero of P_1 is a zero of P_1' with multiplicity $k-1$. By computation, we have

$$\begin{aligned} P_1'(\zeta) &= \zeta^{m-l-1}\prod_{i=1}^{n}(\zeta-\zeta_i)^{2p_i+k-1}\Big[(m-l)\prod_{i=1}^{n}(\zeta-\zeta_i)+\zeta\sum_{i=1}^{n}(2p_i+k)\\ &\quad\times\prod_{j\neq i}(\zeta-\zeta_j)\Big]. \end{aligned}$$

Since $P_1(\zeta_i)\neq 0$ and $P_1(0)\neq 0$, it follows that the polynomial

$$Q_1(\zeta)=(m-l)\prod_{i=1}^{n}(\zeta-\zeta_i)+\zeta\sum_{i=1}^{n}(2p_i+k)\prod_{j\neq i}(\zeta-\zeta_j)$$

has at least τ_1 distinct zeros with multiplicity $k-1$. Thus,

$$n=\deg Q_1\geq (k-1)\tau_1>(k-1)n.$$

This is impossible, since $k\geq 2$.

Case 1.2. Suppose that $m=l$. Then as showed in Case 1.1, all zeros of the polynomial

$$P_2(\zeta)=\prod_{i=1}^{n}(\zeta-\zeta_i)^{2p_i+k}+\mu$$

on $\mathbb{C}\setminus\{0\}$ have exact multiplicity $k\geq 2$. Denote by α the multiplicity if 0 is a zero of P_2, and say $\alpha=0$ if $P_2(0)\neq 0$. This shows that P_2 has

$$\tau_2=\frac{\deg P_2-\alpha}{k}=\frac{\sum_{i=1}^{n}(2p_i+k)-\alpha}{k}$$

distinct zeros on $\mathbb{C}\setminus\{0\}$, and each zero of P_2 on $\mathbb{C}\setminus\{0\}$ is a zero of P_2' with multiplicity $k-1$. By computation, we have

$$P_2'(\zeta)=\prod_{i=1}^{n}(\zeta-\zeta_i)^{2p_i+k-1}Q_2(\zeta),\ \text{where } Q_2(\zeta)=\sum_{i=1}^{n}(2p_i+k)\prod_{j\neq i}(\zeta-\zeta_j).$$

Since $P_2(\zeta_i)\neq 0$, the polynomial Q_2 has at least τ_2 distinct zeros on $\mathbb{C}\setminus\{0\}$ with multiplicity $k-1$. Further, if $\alpha\geq 2$, then 0 is a zero of Q_2 with multiplicity $\alpha-1$. Let $\beta=\alpha-1$ if $\alpha\geq 2$, and $\beta=0$ if $\alpha=0$ or $\alpha=1$. Thus, we see that

$$n-1=\deg Q_2\geq (k-1)\tau_2+\beta=\frac{k-1}{k}\sum_{i=1}^{n}(2p_i+k)+\beta-\frac{k-1}{k}\alpha$$

$$\geq \frac{(k-1)(k+2)n}{k} + \beta - \alpha.$$

Then we have

$$\alpha - 1 \geq \frac{k^2-2}{k} n + \beta > \beta,$$

which is a contradiction.

Case 1.3. Suppose that $m < l$. Then as showed in Case 1.1, all zeros of the polynomial

$$P_3(\zeta) = \prod_{i=1}^{n} (\zeta - \zeta_i)^{2p_i+k} + \mu \zeta^{l-m}$$

on $\mathbb{C} \setminus \{0\}$ have exact multiplicity $k \geq 2$. Note that $P_3(0) \neq 0$. This shows that P_3 has

$$\tau_3 = \frac{\deg P_3}{k}$$

distinct zeros on $\mathbb{C} \setminus \{0\}$, and each zero of P_3 is a zero of P_3' with multiplicity $k-1$. By computation, we have

$$\left(\zeta^{m-l} P_3(\zeta)\right)' = \zeta^{m-l-1} \prod_{i=1}^{n} (\zeta - \zeta_i)^{2p_i+k-1} Q_3(\zeta),$$

where

$$Q_3(\zeta) = (m-l) \prod_{i=1}^{n} (\zeta - \zeta_i) + \zeta \sum_{i=1}^{n} (2p_i + k) \prod_{j \neq i} (\zeta - \zeta_j).$$

Since $P_3(\zeta_i) \neq 0$ and $P_3(0) \neq 0$, it follows that the polynomial Q_3 has at least τ_3 distinct zeros with multiplicity $k-1$. Thus,

$$\deg Q_3 \geq (k-1)\tau_3. \tag{2.10}$$

If $\deg P_3 \geq \sum_{i=1}^{n} (2p_i + k)$, then $\tau_3 \geq \sum_{i=1}^{n} (2p_i + k)/k \geq (k+2)n/k$. This, together with (2.10) and the fact $\deg Q_3 \leq n$, leads to a contradiction.

Thus $\deg P_3 < \sum_{i=1}^{n} (2p_i + k)$. Since $\deg P_3 = \max\{\sum_{i=1}^{n} (2p_i + k),\ l-m\}$ if $\sum_{i=1}^{n} (2p_i + k) \neq l - m$, we see that

$$\sum_{i=1}^{n} (2p_i + k) = l - m \tag{2.11}$$

and $\mu = -1$. Hence deg $Q_3 \leq n-1$, so that by (2.10)

$$\tau_3 \leq \frac{n-1}{k-1}. \tag{2.12}$$

Now since P_3 has τ_3 distinct zeros with exact multiplicity k, we can obtain that

$$\prod_{i=1}^{n}(\zeta-\zeta_i)^{2p_i+k} - \zeta^{l-m} = c\left[\prod_{i=1}^{\tau_3}(\zeta-w_i)\right]^k \tag{2.13}$$

for some nonzero constant c and τ_3 distinct nonzero points w_i. It follows from (2.13) with the transformation $\zeta \to 1/z$ that

$$R(z) := \prod_{i=1}^{n}(1-\zeta_i z)^{2p_i+k} - 1 = cz^{l-m-\tau_3 k}\left[\prod_{i=1}^{\tau_3}(1-w_i z)\right]^k.$$

Thus 0 is a zero of R with multiplicity $l-m-\tau_3 k$. Since

$$R'(z) = \prod_{i=1}^{n}(1-\zeta_i z)^{2p_i+k-1}\left[\sum_{i=1}^{n}(2p_i+k)(-\zeta_i)\prod_{j\neq i}(1-\zeta_j z)\right],$$

we see that 0 is a zero of R' with multiplicity at most $n-1$. Hence

$$l-m-\tau_3 k \leq n.$$

This with (2.11) and (2.12) shows that

$$(k+2)n \leq \sum_{i=1}^{n}(2p_i+k) = l-m \leq \tau_3 k + n \leq \frac{k(n-1)}{k-1} + n,$$

which is impossible.

Case 2. g has no nonzero poles. Then as $g^{(k)}(\zeta) \neq 0$ on $\mathbb{C}\setminus\{0\}$, we have $g^{(k)}(\zeta) = c\zeta^s$ for some constant $c \neq 0$ and integer $s \in \mathbb{Z}$.

If $s \geq 0$, then g is a polynomial with deg $g = s+k$. And since $g(\zeta)g^{(k)}(\zeta) \neq \zeta^m$ on $\mathbb{C}\setminus\{0\}$, we also have $g(\zeta)g^{(k)}(\zeta) = \zeta^m + \lambda\zeta^t$ for some constant $\lambda \neq 0$ and integer t. Thus

$$g(\zeta) = \frac{1}{c}\zeta^{m-s} + \frac{\lambda}{c}\zeta^{t-s}.$$

If $m \neq t$, then it can be seen that g has at least one simple zero on $\mathbb{C}\setminus\{0\}$, which contradicts that all zeros of g on $\mathbb{C}\setminus\{0\}$ have multiplicity $k \geq 2$. Thus $m = t$, then

$\lambda + 1 \neq 0$ and $g(\zeta) = (\lambda + 1)\zeta^{m-s}/c$. Thus $m - s = \deg g = s + k$, and hence $m - s = (m + k)/2$, so that $g(\zeta) = C\zeta^{\frac{m+k}{2}}$ for some nonzero constant C and $m \geq k$.

If $s < 0$, then 0 is the pole of g with multiplicity $-s - k > 0$. And since $g(\zeta)g^{(k)}(\zeta) \neq \zeta^m$ on $\mathbb{C} \setminus \{0\}$, we also have $g(\zeta)g^{(k)}(\zeta) = \zeta^m + \lambda\zeta^t$ for some constant $\lambda \neq 0$ and integer t. Thus

$$g(\zeta) = \frac{1}{c}\zeta^{m-s} + \frac{\lambda}{c}\zeta^{t-s}.$$

If $m \neq t$, then it can be seen that g has at least one simple zero on $\mathbb{C} \setminus \{0\}$, which contradicts that all zeros of g on $\mathbb{C} \setminus \{0\}$ have multiplicity $k \geq 2$. Thus $m = t$, then $\lambda + 1 \neq 0$ and $g(\zeta) = (\lambda + 1)\zeta^{m-s}/c$. Thus $-m + s = -s - k$, and hence $m - s = (m + k)/2 < 0$, so that $g(\zeta) = C\zeta^{\frac{m+k}{2}}$ for some nonzero constant C. Note, $m = 2s + k \leq -2(k + 1) + k \leq -(k + 2)$.

The lemma is proved. □

Lemma 2.6 *Let g be a meromorphic function on $\mathbb{C}$. If $g'(\zeta) \neq 0$ on $\mathbb{C} \setminus \{0\}$, then the equation $g(\zeta)g'(\zeta) = \gamma\zeta^{-1}$ has solutions on $\mathbb{C} \setminus \{0\}$, where γ is a given nonzero constant.*

Proof Without loss of generality, we may assume that $\gamma = 1$.

Suppose first that g is transcendental. Then by Lemma 2.4, $\frac{1}{2}(g^2)' - \zeta^{-1}$ has infinitely many zeros on $\mathbb{C}$, hence $g(\zeta)g'(\zeta) = \zeta^{-1}$ has infinitely many zeros on $\mathbb{C} \setminus \{0\}$.

Next we suppose that g is a polynomial. Since $g'(\zeta) \neq 0$ on $\mathbb{C} \setminus \{0\}$, we have $g(\zeta) = a\zeta^n + b$, where $a \neq 0$. Then $g(\zeta)g'(\zeta) - \zeta^{-1} = n\zeta^{-1}(a\zeta^{2n} + b\zeta^n + 1)$ must have zero on $\mathbb{C} \setminus \{0\}$. □

Finally, we suppose that g is non-polynomial rational function.

Case 1. If $g'(\zeta) \neq 0$ on $\mathbb{C}$, then by Lemma 2.3, $g(\zeta) = B + A/(z + a)^n$, where $A \neq 0$, B are two constants. Then

$$g(\zeta)g'(\zeta) - \zeta^{-1} = \frac{-An[A + B(\zeta + a)^n]\zeta - (\zeta + a)^{2n+1}}{\zeta(\zeta + a)^{2n+1}}.$$

If $a \neq 0$, we see that $g(\zeta)g'(\zeta) - \zeta^{-1}$ must have zeros on $\mathbb{C} \setminus \{0\}$. If $a = 0$, then

$$g(\zeta)g'(\zeta) - \zeta^{-1} = \frac{-An(A + B\zeta^n) - \zeta^{2n}}{\zeta^{2n+1}}$$

also has zeros on $\mathbb{C} \setminus \{0\}$.

Case 2. If $g'(\zeta) \neq 0$ on $\mathbb{C} \setminus \{0\}$ and $g'(0) = 0$, then we can suppose that

$$g'(\zeta) = \frac{\mu\zeta^l}{\prod_{i=1}^n (\zeta - \zeta_i)^{p_i+1}}, \tag{2.14}$$

where $\zeta_i \neq 0 (i = 1, 2, \ldots, n)$ are all distinct poles of g, μ is a nonzero constant and $l \in \mathbb{Z}$ is a positive integer. If $g(\zeta)g'(\zeta) \neq \zeta^{-1}$ on $\mathbb{C} \setminus \{0\}$, then we can suppose that

$$g(\zeta)g'(\zeta) = \zeta^{-1} + \frac{\lambda \zeta^s}{\prod_{i=1}^n (\zeta - \zeta_i)^{2p_i+1}},$$

where λ is a nonzero constant. We see that $s = -1$, otherwise $\zeta = 0$ would be a pole of gg', hence of g, which contradicts that $g'(0) = 0$. Then we have

$$g(\zeta)g'(\zeta) = \frac{\prod_{i=1}^n (\zeta - \zeta_i)^{2p_i+1} + \lambda}{\zeta \prod_{i=1}^n (\zeta - \zeta_i)^{2p_i+1}},$$

hence

$$g(\zeta) = \frac{Q(\zeta)}{\mu \zeta^{l+1} \prod_{i=1}^n (\zeta - \zeta_i)^{p_i}}, \text{ where } Q(\zeta) = \prod_{i=1}^n (\zeta - \zeta_i)^{2p_i+1} + \lambda.$$

Case 2.1. If $g(0) = 0$, then $\zeta = 0$ is a zero of $Q(\zeta)$ with multiplicity $2(l+1)$ and

$$g(\zeta) = \frac{\zeta^{l+1} P(\zeta)}{\mu \prod_{i=1}^n (\zeta - \zeta_i)^{p_i}},$$

where $P(\zeta)$ is a monic polynomial and

$$\deg P = \deg Q - 2(l+1) = \sum_{i=1}^n (2p_i + 1) - 2(l+1) \geq 0.$$

Then we have

$$g'(\zeta) = \frac{\zeta^l P_1(\zeta)}{\mu \prod_{i=1}^n (\zeta - \zeta_i)^{p_i+1}}, \tag{2.15}$$

where $P_1(\zeta) = [(l+1)P(\zeta) + \zeta P'(\zeta)] \prod_{i=1}^n (\zeta - \zeta_i) - \zeta P(\zeta) \sum_{i=1}^n p_i \prod_{j \neq i} (\zeta - \zeta_j)$. We see that the polynomial $P_1(\zeta)$ is not a constant, since the first coefficient of $P_1(\zeta)$ is

$$l + 1 + \deg P - \sum_{i=1}^n p_i = \sum_{i=1}^n (p_i + 1) - (l+1) \geq \frac{n}{2} > 0.$$

Hence comparing with (2.15) and (2.14), it is a contradiction.

Case 2.2. If $g(0) \neq 0$, then $\zeta = 0$ is a zero of $Q(\zeta)$ with multiplicity $l+1$ and

$$g(\zeta) = \frac{P(\zeta)}{\mu \prod_{i=1}^n (\zeta - \zeta_i)^{p_i}},$$

where $P(\zeta)$ is a monic polynomial and

$$\deg P = \deg Q - (l+1) = \sum_{i=1}^{n}(2p_i+1) - (l+1) \geq 0.$$

Then we have

$$g'(\zeta) = \frac{P_2(\zeta)}{\mu \prod_{i=1}^{n}(\zeta-\zeta_i)^{p_i+1}}, \tag{2.16}$$

where $P_2(\zeta) = P'(\zeta)\prod_{i=1}^{n}(\zeta-\zeta_i) - P(\zeta)\sum_{i=1}^{n} p_i \prod_{j\neq i}(\zeta-\zeta_j)$. We see that the leading term of $P_2(\zeta)$ is

$$\left(\deg P - \sum_{i=1}^{n} p_i\right)\zeta^{\deg P+n-1} = \left[\sum_{i=1}^{n}(p_i+1)-(l+1)\right]\zeta^{\sum_{i=1}^{n}(2p_i+2)-(l+2)}.$$

If $\sum_{i=1}^{n}(p_i+1)-(l+1) \neq 0$, then $\sum_{i=1}^{n}(2p_i+2)-(l+2) \neq l$. Hence comparing with (2.16) and (2.14), it is a contradiction.

If $\sum_{i=1}^{n}(p_i+1)-(l+1) = 0$, then $\sum_{i=1}^{n}(2p_i+2)-(l+2) = l$. Hence comparing with (2.16) and (2.14), it is also a contradiction.

The lemma is proved.

3 Proof of Theorem 1.1

In this section, we first prove the following theorem.

Theorem 3.1 *Let $\{f_n\}$ be a sequence of meromorphic functions on D whose zeros have multiplicity at least k, where k is a positive integer. Let $\{a_n\}$ and $\{h_n\}$ be two sequences of meromorphic functions on D such that $a_n(z) \xrightarrow{\chi} a(z)$ and $h_n(z) \xrightarrow{\chi} h(z)$ on D, where $a(z) \neq 0, \infty$, $h(z) \neq 0, \infty$ on D, and let $l \in \mathbb{Z}$ be an integer such that $2l < k$. Then the family $\{f_n\}$ is normal on D provided that $f_n(z)f_n^{(k)}(z) = a_n(z) \Leftrightarrow f_n^{(k)}(z) = z^{-l}h_n(z)$ for every $f_n \in \{f_n\}$.*

Proof Suppose that $\{f_n\}$ is not normal at some point $z_0 \in D$. Then by Lemma 2.1, there exist points $z_n \to z_0$, a subsequence of $\{f_n\}$ (we still denote $\{f_n\}$) and positive numbers $\rho_n \to 0$, such that

$$g_n(\zeta) = \rho_n^{-\frac{k}{2}} f_n(z_n+\rho_n\zeta) \xrightarrow{\chi} g(\zeta)$$

on D, where g a nonconstant meromorphic function with bounded spherical derivative (and hence of order at most two), all of whose zeros are of multiplicity at least k. We denote $a_0 = a(z_0)(\neq 0, \infty)$. □

Case 1. $l \leq 0$, or $l > 0$ with $z_0 \neq 0$.

We claim that (i) $gg^{(k)} \not\equiv a_0$, and (ii) $g^{(k)} \not\equiv 0$.

In fact, if $gg^{(k)} \equiv a_0$, then g is a nonconstant entire function (and hence of exponential type) and $g \neq 0$. Hence $g(\zeta) = \mathrm{e}^{c\zeta+d}$, where $c(\neq 0), d \in \mathbb{C}$. But then $g(\zeta)g^{(k)}(\zeta) = c^k \mathrm{e}^{2c\zeta+2d} \not\equiv a_0$, a contradiction. Similarly, if $g^{(k)} \equiv 0$, then g is a nonconstant polynomial of degree less than k. This contradicts that all zeros of g have multiplicity at least k.

We further claim that (iii) $gg^{(k)} \neq a_0$, and (iv) $g^{(k)} \neq 0$.

To prove (iii), suppose that $g(\zeta_0)g^{(k)}(\zeta_0) = a_0$ for some $\zeta_0 \in \mathbb{C}$. Then g is holomorphic on some close neighborhood U of ζ_0, and hence $g_n(\zeta)g_n^{(k)}(\zeta) - a_n(z_n + \rho_n\zeta) \to g(\zeta)g^{(k)}(\zeta) - a_0$ on U uniformly. Since $gg^{(k)} \not\equiv a_0$, by Hurwitz's theorem, there exist points $\zeta_n \to \zeta_0$ such that (for n sufficiently large)

$$a_n(z_n + \rho_n\zeta_n) = g_n(\zeta_n)g^{(k)}(\zeta_n) = f_n(z_n + \rho_n\zeta_n)f_n^{(k)}(z_n + \rho_n\zeta_n).$$

Hence by the condition, $f_n^{(k)}(z_n + \rho_n\zeta_n) = (z_n + \rho_n\zeta_n)^{-l}h_n(z_n + \rho_n\zeta_n)$, so that $g_n^{(k)}(\zeta_n) = \rho_n^{\frac{k}{2}} f_n^{(k)}(z_n + \rho_n\zeta_n) = \rho_n^{\frac{k}{2}}(z_n + \rho_n\zeta_n)^{-l}h_n(z_n + \rho_n\zeta_n)$. Thus $g^{(k)}(\zeta_0) = \lim_{n\to\infty} g^{(k)}(\zeta_n) = 0$, which contradicts that $g(\zeta_0)g^{(k)}(\zeta_0) = a_0 \neq 0$. This proves (iii).

Next we prove (iv). Suppose that $g^{(k)}(\zeta_0) = 0$ for some $\zeta_0 \in \mathbb{C}$. Then g is holomorphic on some close neighborhood U of ζ_0, and hence $g_n^{(k)}(\zeta) - \rho_n^{\frac{k}{2}}(z_n + \rho_n\zeta_n)^{-l}h_n(z_n + \rho_n\zeta) \to g^{(k)}(\zeta)$ on U uniformly. Since $g^{(k)}(\zeta) \not\equiv 0$, by Hurwitz's theorem, there exist points $\zeta_n \to \zeta_0$ such that (for n sufficiently large)

$$g_n^{(k)}(\zeta_n) - \rho_n^{\frac{k}{2}}(z_n + \rho_n\zeta_n)^{-l}h_n(z_n + \rho_n\zeta_n) = 0.$$

It follows that $f_n^{(k)}(z_n + \rho_n\zeta_n) = (z_n + \rho_n\zeta_n)^{-l}h_n(z_n + \rho_n\zeta_n)$, and hence by the condition, we have

$$a_n(z_n + \rho_n\zeta_n) = f_n(z_n + \rho_n\zeta_n)f_n^{(k)}(z_n + \rho_n\zeta_n) = g_n(\zeta_n)g_n^{(k)}(\zeta_n).$$

This leads to a contradiction that

$$a_0 = a(z_0) = \lim_{n\to\infty} g_n(\zeta_n)g_n^{(k)}(\zeta_n) = g(\zeta_0)g^{(k)}(\zeta_0) = 0.$$

(iv) is also proved.

However, by Lemma 2.2, there is no nonconstant meromorphic function g on $\mathbb{C}$ with the properties (iii) and (iv) such that all zeros have multiplicity at least k.

Case 2. $l \geq 1$ and $z_0 = 0$. Then we have $k > 2$ for the condition $2l < k$. In this part, we consider two cases.

Case 2.1. Suppose that $\frac{z_n}{\rho_n} \to \infty$. Let

$$G_n(\zeta) = z_n^{-\frac{k}{2}} f_n(z_n + z_n\zeta).$$

Then we see that

$$G_n(\zeta)G_n^{(k)}(\zeta) = a_n(z_n + z_n\zeta) \Longleftrightarrow G_n^{(k)}(\zeta) = z_n^{\frac{k}{2}-l}(1+\zeta)^{-l}h_n(z_n + z_n\zeta).$$

By Case 1, we see that $\{G_n\}$ is normal on $\Delta(0, 1)$. Say $G_n \xrightarrow{\chi} G$ on $\Delta(0, 1)$. We claim that $G(0) = 0$ and hence $G \not\equiv \infty$. Suppose $G(0) \neq 0$, then by $\frac{z_n}{\rho_n} \to \infty$, we have

$$g_n(\zeta) = \rho_n^{-\frac{k}{2}} f_n(z_n + \rho_n\zeta) = \left(\frac{z_n}{\rho_n}\right)^{\frac{k}{2}} G_n\left(\frac{\rho_n}{z_n}\zeta\right) \xrightarrow{\chi} \infty$$

on $\mathbb{C}$. This is a contradiction. Hence $G(0) = 0$, so that $G_n^{(k)} \to G^{(k)}$ in some neighborhood of 0. It follows that

$$g_n^{(k)}(\zeta) = \left(\frac{\rho_n}{z_n}\right)^{\frac{k}{2}} G_n^{(k)}\left(\frac{\rho_n}{z_n}\zeta\right) \xrightarrow{\chi} 0$$

on $\mathbb{C}$. Thus $g^{(k)} \equiv 0$, which contradicts that all zeros of g have multiplicity at least k and g is nonconstant.

Case 2.2. So we may assume that $\frac{z_n}{\rho_n} \to c$, a finite complex number. Then we have

$$H_n(\zeta) = \rho_n^{-\frac{k}{2}} f_n(\rho_n\zeta) = g_n\left(\zeta - \frac{z_n}{\rho_n}\right) \xrightarrow{\chi} g(\zeta - c) := H(\zeta)$$

on $\mathbb{C}$, and all zeros of $H(\zeta)$ have multiplicity at least k. And since g is nonconstant, we see that H is also nonconstant. We see from the condition that

$$H_n(\zeta)H_n^{(k)}(\zeta) = a_n(\rho_n\zeta) \Longleftrightarrow H_n^{(k)}(\zeta) = \rho_n^{\frac{k}{2}-l}\zeta^{-l}h_n(\rho_n\zeta). \tag{3.1}$$

We claim that (i) $HH^{(k)} \not\equiv a_0$ and (ii) $H^{(k)} \not\equiv 0$.

If $HH^{(k)} \equiv a_0$, then H is a zero-free entire function of finite order and H is not a polynomial. Thus $H(\zeta) = \mathrm{e}^{Q(\zeta)}$, where Q is a nonconstant polynomial, then $H^{(k)}(\zeta) = P(\zeta)\mathrm{e}^{Q(\zeta)}$, where P is a polynomial. It follows that $H(\zeta)H^{(k)}(\zeta) = P(\zeta)\mathrm{e}^{2Q(\zeta)} \not\equiv a_0$, which is a contradiction. So $HH^{(k)} \not\equiv a(0)$. If $H^{(k)} \equiv 0$, H would be a polynomial of degree less than k. Since H is nonconstant, H has at least one zero. The multiplicity of the zero cannot be larger than the degree of the polynomial H. This contradicts that all zeros of H have multiplicity at least k.

We further claim that (iii) $HH^{(k)} \neq a_0$ on $\mathbb{C} \setminus \{0\}$, and (iv) $H^{(k)} \neq 0$ on $\mathbb{C} \setminus \{0\}$.

Suppose that $H(\zeta_0)H^{(k)}(\zeta_0) = a_0$ at some point $\zeta_0 \neq 0$. Then $H(\zeta_0) \neq \infty$, and hence H is holomorphic on some close neighborhood U of ζ_0. Thus

$$H_n(\zeta)H_n^{(k)}(\zeta) - a_n(\rho_n\zeta) \to H(\zeta)H^{(k)}(\zeta) - a_0$$

on U uniformly. Since $H(\zeta)H^{(k)}(\zeta) \not\equiv a_0$, by Hurwitz's theorem, there exist points ζ_n, $\zeta_n \to \zeta_0$, such that (for n sufficiently large)

$$H_n(\zeta_n)H_n^{(k)}(\zeta_n) - a_n(\rho_n\zeta_n) = 0.$$

By (3.1), we have $H_n^{(k)}(\zeta_n) = \rho_n^{\frac{k}{2}-l}\zeta_n^{-l}h_n(\rho_n\zeta_n)$ and hence

$$H^{(k)}(\zeta_0) = \lim_{n\to\infty} H_n^{(k)}(\zeta_n) = \lim_{n\to\infty} \rho_n^{\frac{k}{2}-l}\zeta_n^{-l}h_n(\rho_n\zeta_n) = 0,$$

which contradicts that $H(\zeta_0)H^{(k)}(\zeta_0) = a_0 \neq 0$. The claim (iii) is proved.

Next we suppose that $H^{(k)}(\zeta_0) = 0$ at some point $\zeta_0 \neq 0$. Then $H(\zeta_0) \neq \infty$, so that H is holomorphic on some close neighborhood U of ζ_0, and hence

$$H_n^{(k)}(\zeta) - \rho_n^{\frac{k}{2}-l}\zeta^{-l}h_n(\rho_n\zeta) \to H^{(k)}(\zeta)$$

on U uniformly. Since $H^{(k)}(\zeta) \not\equiv 0$, by Hurwitz's theorem, there exist points ζ_n, $\zeta_n \to \zeta_0$, such that (for n sufficiently large)

$$H^{(k)}(\zeta_n) - \rho_n^{\frac{k}{2}-l}\zeta_n^{-l}h_n(\rho_n\zeta_n) = 0.$$

Then by (3.1), we have $H_n(\zeta_n)H_n^{(k)}(\zeta_n) = a_n(\rho_n\zeta_n)$, and hence

$$H(\zeta_0)H^{(k)}(\zeta_0) = \lim_{n\to\infty} H_n(\zeta_n)H_n^{(k)}(\zeta_n) = \lim_{n\to\infty} a_n(\rho_n\zeta_n) = a_0.$$

This contradicts the claim (iii). The claim (iv) is also proved.

Thus, by Lemma 2.5 with $m = 0$, $H(\zeta) = C\zeta^{\frac{k}{2}}$. This contradicts that all zeros of H have multiplicity at least k.

Hence $\mathcal{F}$ is normal on D. The proof is completed. □

Proof of Theorem 1.1. By the proof of Theorem 3.1, we have showed that $\mathcal{F}$ is normal on $D \setminus a^{-1}(0) \bigcup a^{-1}(\infty)$, where $a^{-1}(0)$ stands for the set of zeros of a and $a^{-1}(\infty)$ stands for the set of poles of a. Next, we prove that $\mathcal{F}$ is also normal at every zero or pole of a in D.

Suppose that $\mathcal{F}$ is not normal at $z_0 \in D$, where z_0 is a zero or a pole of a. Without loss of generality, we may say $z_0 = 0$ and assume that $a(z) = z^m h(z)$ and $b(z) = z^{-l}b_1(z)$, where $m, l \in \mathbb{Z}$, $h(z)$, and $b_1(z)$ are holomorphic and zero free on $\Delta(0, \delta) \subset D$. We assume that $h(0) = 1$. We note by the condition that $-k \leq m \leq k-1$, $m \neq 0$ and $l < \frac{k-m}{2}$ if $l > 0$. In particular, $0 \leq \frac{m+k}{2} < k$.

Then by Lemma 2.1, there exist points $z_n \to 0$, functions $f_n \in \mathcal{F}$, and positive numbers $\rho_n \to 0$ such that

$$g_n(\zeta) = \rho_n^{-\frac{m+k}{2}} f_n(z_n + \rho_n\zeta) \xrightarrow{\chi} g(\zeta)$$

on $\mathbb{C}$, where g is a nonconstant meromorphic function of finite order, and all zeros of g have multiplicity at least k.

Case 1. Suppose that $\frac{z_n}{\rho_n} \to \infty$. Let

$$G_n(\zeta) = z_n^{-\frac{m+k}{2}} f_n(z_n + z_n\zeta).$$

Then by the condition $f(z)f^{(k)}(z) = a(z) \Leftrightarrow f^{(k)}(z) = b(z)$, we have

$$G_n(\zeta)G_n^{(k)}(\zeta) = (1+\zeta)^m h(z_n + z_n\zeta) \Longleftrightarrow G_n^{(k)}(\zeta) = z_n^{\frac{k-m}{2}-l}(1+\zeta)^{-l} b_1(z_n + z_n\zeta).$$

Since $z_n \to 0$ and $h(0),\ b_1(0) \neq 0, \infty$, by Theorem 3.1, we see that $\{G_n\}$ is normal on $\Delta(0,1)$. Say $G_n \xrightarrow{\chi} G$ on $\Delta(0,1)$. We claim that $G(0) = 0$ and hence $G \not\equiv \infty$. Suppose $G(0) \neq 0$, then by $\frac{z_n}{\rho_n} \to \infty$, we have

$$g_n(\zeta) = \rho_n^{-\frac{m+k}{2}} f_n(z_n + \rho_n\zeta) = \left(\frac{z_n}{\rho_n}\right)^{\frac{m+k}{2}} G_n\left(\frac{\rho_n}{z_n}\zeta\right) \xrightarrow{\chi} \begin{cases} \infty, & m+k > 0 \\ G(0), & m+k = 0 \end{cases}$$

on $\mathbb{C}$. This is a contradiction. Hence $G(0) = 0$, so that $G_n^{(k)} \to G^{(k)}$ in some neighborhood of 0. It follows that

$$g_n^{(k)}(\zeta) = \left(\frac{\rho_n}{z_n}\right)^{\frac{k-m}{2}} G_n^{(k)}\left(\frac{\rho_n}{z_n}\zeta\right) \xrightarrow{\chi} 0$$

on $\mathbb{C}$. Thus $g^{(k)} \equiv 0$, which contradicts that all zeros of g have multiplicity at least k and g is nonconstant.

Case 2. So we may assume that $\frac{z_n}{\rho_n} \to c$, a finite complex number. Then we have

$$H_n(\zeta) = \rho_n^{-\frac{m+k}{2}} f_n(\rho_n\zeta) = g_n\left(\zeta - \frac{z_n}{\rho_n}\right) \xrightarrow{\chi} g(\zeta - c) := H(\zeta)$$

on $\mathbb{C}$, and all zeros of $H(\zeta)$ have multiplicity at least k. And since g is nonconstant, we see that H is also nonconstant. We see from the condition that

$$H_n(\zeta)H_n^{(k)}(\zeta) = \zeta^m h(\rho_n\zeta) \Longleftrightarrow H_n^{(k)}(\zeta) = \rho_n^{\frac{k-m}{2}-l}\zeta^{-l} b_1(\rho_n\zeta). \tag{3.2}$$

We claim that (i) $H(\zeta)H^{(k)}(\zeta) \not\equiv \zeta^m$ and (ii) $H^{(k)}(\zeta) \not\equiv 0$.

In fact, if $H(\zeta)H^{(k)}(\zeta) \equiv \zeta^m$, then $\zeta = 0$ is the only possible zero or pole of H. If H is a transcendental function, then $H(\zeta) = \zeta^\alpha \mathrm{e}^{Q(\zeta)}$ for some $\alpha \in \mathbb{Z}$ and polynomial Q. Thus $H^{(k)}(\zeta) = P(\zeta)\mathrm{e}^{Q(\zeta)}$, where $P(\zeta)(\not\equiv 0)$ is a rational function. It follows that $HH^{(k)}$ is also a transcendental function, which is a contradiction. If H is a rational function and $\zeta = 0$ is a pole of H, then $\zeta = 0$ is the pole of $HH^{(k)}$ with multiplicity at least $k+2$, which contradicts $H(\zeta)H^{(k)}(\zeta) \equiv \zeta^m$, $-k \leq m \leq k-1$.

If H is a rational function and $\zeta = 0$ is not a pole of H, then H is a polynomial. If $\deg H \geq k$, then $\deg(HH^{(k)}) \geq k$. Otherwise, $HH^{(k)} \equiv 0$. Both cases contradict that $H(\zeta)H^{(k)}(\zeta) \equiv \zeta^m$. So $H(\zeta)H^{(k)}(\zeta) \not\equiv \zeta^m$.

If $H^{(k)} \equiv 0$, H would be a polynomial of degree less than k. Since H is nonconstant, H has at least one zero. The multiplicity of the zero cannot be larger than the degree of the polynomial H. This contradicts that all zeros of H have multiplicity at least k.

We further claim that (iii) $H(\zeta)H^{(k)}(\zeta) \neq \zeta^m$ on $\mathbb{C} \setminus \{0\}$, and (iv) $H^{(k)}(\zeta) \neq 0$ on $\mathbb{C} \setminus \{0\}$.

Suppose that $H(\zeta_0)H^{(k)}(\zeta_0) = \zeta_0^m$, $\zeta_0 \neq 0$. Then $H(\zeta_0) \neq \infty$. H is holomorphic on some close neighborhood U of ζ_0, and hence

$$H_n(\zeta)H_n^{(k)}(\zeta) - \zeta^m h(\rho_n\zeta) \to H(\zeta)H^{(k)}(\zeta) - \zeta^m$$

on U uniformly. Since $H(\zeta)H^{(k)}(\zeta) \not\equiv \zeta^m$, by Hurwitz's theorem, there exist points ζ_n, $\zeta_n \to \zeta_0$, such that (for n sufficiently large)

$$H_n(\zeta_n)H_n^{(k)}(\zeta_n) - \zeta_n^m h(\rho_n\zeta_n) = 0.$$

By (3.2), we have

$$H_n^{(k)}(\rho_n\zeta_n) = \rho_n^{\frac{k-m}{2}-l}\zeta_n^{-l}b_1(\rho_n\zeta_n).$$

By the condition $\frac{k-m}{2} - l > 0$ and $\zeta_n \to \zeta_0 \neq 0$, we have

$$H^{(k)}(\zeta_0) = \lim_{n\to\infty} H_n^{(k)}(\zeta_n) = \lim_{n\to\infty} \rho_n^{\frac{k-m}{2}-l}\zeta_n^{-l}b_1(\rho_n\zeta_n) = 0,$$

which contradicts that $H(\zeta_0)H^{(k)}(\zeta_0) = \zeta_0^m \neq 0$. Then (iii) is proved.

Next we suppose that $H^{(k)}(\zeta_0) = 0$, $\zeta_0 \neq 0$. Thus $H(\zeta_0) \neq \infty$. H is holomorphic on some close neighborhood U of ζ_0, and hence

$$H_n^{(k)}(\zeta) - \rho_n^{\frac{k-m}{2}-l}\zeta^{-l}b_1(\rho\zeta) \to H^{(k)}(\zeta)$$

on U uniformly. Since $H^{(k)}(\zeta) \not\equiv 0$, by Hurwitz's theorem, there exist points ζ_n, $\zeta_n \to \zeta_0$, such that (for n sufficiently large)

$$H^{(k)}(\zeta_n) - \rho_n^{\frac{k-m}{2}-l}\zeta_n^{-l}b_1(\rho\zeta_n) = 0.$$

Then by (3.2) we have $H_n(\zeta_n)H_n^{(k)}(\zeta_n) = \zeta_n^m h(\rho_n\zeta_n)$, thus

$$H(\zeta_0)H^{(k)}(\zeta_0) = \lim_{n\to\infty} H_n(\zeta_n)H_n^{(k)}(\zeta_n) = \lim_{n\to\infty} \zeta_n^m h(\rho_n\zeta_n) = \zeta_0^m.$$

This contradicts to claim (iii). So (iv) is proved.

If $k \geq 2$, then by Lemma 2.5 and claims (iii) and (iv), we get $m \geq k$ or $m \leq -(k+2)$, which are ruled out by the assumption.

If $k = 1$, then $m = -1$. By Lemma 2.6, there is no meromorphic function satisfying claims (iii) and (iv).

The proof of Theorem 1.1 is completed. □

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References

1. Hayman, W.K.: Meromorphic Functions. Clarendon, Oxford, UK (1964)
2. Schiff, J.L.: Normal Families. Springer, New York (1993)
3. Yang, L.: Value Distribution Theory. Spring, Berlin (1993)
4. Schwick, W.: Exceptional functions and normality. Bull. Lond. Math. Soc. **29**, 425–432 (1997)
5. Liu, D., Deng, B.M., Yang, D.G.: Normal families of meromorphic functions with sharing functions. Bull. Malays. Math. Sci. Soc. (to appear)
6. Meng, D.W., Hu, P.C.: Normality criteria of meromorphic functions sharing a holomorphic function. Bull. Malays. Math. Sci. Soc. (to appear)
7. Qi, J.M., Jie, D., Yang, L.Z.: Normality criteria for families of meromorphic function concerning shared values. Bull. Malays. Math. Sci. Soc. **35**(2), 449–457 (2012)
8. Xia, J.Y., Xu, Y.: Normality criterion concerning sharing functions II. Bull. Malays. Math. Sci. Soc. **33**(3), 479–486 (2010)
9. Zeng, S.P., Lahiri, I.: Montel's criterion and shared set. Bull. Malays. Math. Sci. Soc. (to appear)
10. Fang, M.L., Zalcman, L.: Normal families and shared values of meromorphic functions II. Comput. Methods Funct. **1**, 289–299 (2001)
11. Chen, H.H., Gu, Y.X.: Improvement of Marty's criterion and its applications. Sci. China Ser. A **36**, 674–681 (1993)
12. Pang, X.C., Zalcman, L.: Normal families and shared values. Bull. Lond. Math. Soc. **32**, 325–331 (2000)
13. Wang, Y.F., Fang, M.L.: Picard values and normal families of meromorphic functions with multiple zeros. Acta Mathematica Sinica **14**, 17–26 (1998)
14. Bergweiler, W., Pang, X.C.: On the derivative of meromorphic functions with multiple zeros. J. Math. Anal. Appl. **278**, 285–292 (2003)