Bull. Malays. Math. Sci. Soc. (2016) 39:1–16
DOI 10.1007/s40840-015-0159-4

BULLETIN of the MALAYSIAN MATHEMATICAL SCIENCES SOCIETY
http://math.usm.my/bulletin

Approximation by Complex Bivariate Balázs-Szabados Operators

Esma Yildiz Özkan[1]

Received: 27 May 2013 / Revised: 23 August 2013 / Published online: 30 July 2015

Abstract This study deals with approximation properties by the complex bivariate Balázs-Szabados operators of tensor-product kind. The upper and lower estimates and a Voronovskaja-type theorem of these operators are given. The exact degree of approximation for these operators is obtained.

Keywords Complex approximation · Rate of convergence · Voronovskaja-type theorem · Exact degree of approximation

Mathematics Subject Classification 30E10 · 41A25

1 Introduction

The applications on the real and complex approximations properties of the operators have been recently an active subject in the area of the approximation theory (see [7,10–14]).

Balázs [3] defined the Bernstein type rational functions. He gave an estimate for the order of its convergence and proved an asymptotic approximation theorem and a convergence theorem concerning the derivative of these operators.

In [4] , Balázs and Szabados obtained the best possible estimate under the more restrictive conditions, in which both the weight and the order of convergence would be

Communicated by Lee See Keong.

✉ Esma Yildiz Özkan
esmayildiz@gazi.edu.tr

1 Department of Mathematics, Faculty of Arts and Sciences, Gazi University, 06500 Ankara, Turkey

better than [3]. They applied their results to the approximate certain improper integrals by quadrature sums of positive coefficients based on finite number of equidistant nodes.

Atakut and Ispir [2] defined the bivariate real Bernstein type rational functions of the Bernstein type rational functions given by Balázs [3] and proved the approximation theorems for these functions. In [8], Gupta and Ispir studied on the Bezier variant of generalized Kantorovitch type Balázs operators.

The rational complex Balázs-Szabados operators was defined by Gal [6] as follows:

$$R_n(f;z) = \frac{1}{(1+a_n z)^n} \sum_{j=0}^{n} f\left(\frac{j}{b_n}\right) \binom{n}{j} (a_n z)^j,$$

where $D_R = \{z \in: |z| < R\}$ with $R > 0$, $f : D_R \cup [R, \infty) \to$ is a function, $a_n = n^{\beta-1}$, $b_n = n^{\beta}$ for $0 < \beta \leq 2/3$, $n \in \mathbb{N}$, $z \in \mathbb{C}$ with $z \neq -1/a_n$. He obtained the uniform convergence of $R_n(f;z)$ to $f(z)$ on compact disk and proved the upper estimate in approximation of these operators. Also, he obtained the Voronovskaja-type formula and the exact degree of its approximation.

Gal extended some complex univariate operators to the case of several complex variables. To illustrate, he studied the approximation properties of the complex Bernstein polynomials, both tensor-product and non-tensor-product types, the complex Favard-Szász-Mirakjan operators of tensor-product kind without exponential growth conditions on f and the complex Baskakov operators of tensor-product kind ([6], see pp. 155–179).

We consider the following real bivariate Balázs-Szabados operators of the univariate Bernstein type rational functions given by Balázs and Szabados[4]

$$R_{n,m}(f)(x_1, x_2) = \sum_{k=0}^{n} \sum_{j=0}^{m} f\left(\frac{k}{b_n}, \frac{j}{b_m}\right) p_{n,k}(x_1) p_{m,j}(x_2),$$

where $f : [0, \infty) \times [0, \infty) \to \mathbb{R}$ is a function, $a_n = n^{\beta-1}$, $b_n = n^{\beta}$, $a_m = m^{\beta-1}$, $b_m = m^{\beta}$ for $n, m \in \mathbb{N}$, $0 < \beta \leq 2/3$, and $x_{1,}\ x_2 \in \mathbb{R}$ with $x_1 \neq -1/a_n, x_2 \neq -1/a_m$,

$$p_{n,k}(x_1) = \frac{\binom{n}{k}(a_n x_1)^k}{(1+a_n x_1)^n} \quad \text{and} \quad p_{m,j}(x_2) = \frac{\binom{m}{j}(a_m x_2)^j}{(1+a_m x_2)^m}.$$

The real bivariate Balázs-Szabados operators are well defined, linear, and positive. For $\beta = 2/3$, we get the real bivariate Balázs-Szabados operators given by Atakut and Ispir [2].

We define the complex bivariate Balázs-Szabados operators as follows:

$$R_{n,m}(f)(z_1, z_2) = \sum_{k=0}^{n} \sum_{j=0}^{m} f\left(\frac{k}{b_n}, \frac{j}{b_m}\right) p_{n,k}(z_1) p_{m,j}(z_2), \tag{1.1}$$

where $f : (D_{R_1} \cup [R_1, \infty)) \times (D_{R_2} \cup [R_2, \infty)) \to \mathbb{C}$ is a function , $a_n = n^{\beta-1}$, $b_n = n^{\beta}$, $a_m = m^{\beta-1}$, $b_m = m^{\beta}$, for $n, m \in \mathbb{N}$, $0 < \beta \leq 2/3$, and $z_{1,}\ z_2 \in \mathbb{C}$, with $z_1 \neq -1/a_n, z_2 \neq -1/a_m$,

$$p_{n,k}\left(z_1\right)=\frac{\binom{n}{k}\left(a_n z_1\right)^k}{\left(1+a_n z_1\right)^n} \quad \text{and} \quad p_{m,j}\left(z_2\right)=\frac{\binom{m}{j}\left(a_m z_2\right)^j}{\left(1+a_m z_2\right)^m}.$$

The complex bivariate Balázs-Szabados operators are well defined, linear, and these operators are analytic for all $n \geq n_0$, $m \geq m_0$, $|z_1| \leq r_1 < n_0^{1-\beta}$ and $|z_2| \leq r_2 < m_0^{1-\beta}$ since $z_1 \neq -1/a_n$ and $z_2 \neq -1/a_m$.

Throughout the paper, we denote with $\|f\|_{r_1,r_2} = \max\left\{\left|f\left(z_1,z_2\right)\right| : \left(z_1,z_2\right) \in \bar{D}_{r_1} \times \bar{D}_{r_2}\right\}$ the uniform norm of f in the space of continuous functions on $\bar{D}_{r_1} \times \bar{D}_{r_2}$ and with $\|f\|_{B([0,\infty)\times[0,\infty))} = \sup\left\{\left|f\left(z_1,z_2\right)\right| : \left(z_1,z_2\right) \in [0,\infty)\times[0,\infty)\right\}$ the uniform norm of f in the space of bounded functions on $[0,\infty)\times[0,\infty)$, where $D_r = \{z \in \mathbb{C} : |z| < r\}$ for $r > 0$.

2 Auxiliary Results

We need following lemmas and theorems in order to prove the main results for the operators (1.1).

Lemma 1 *For all $n, m \in \mathbb{N}$, we have*

$$\begin{aligned}
R_{n,m}\left(e_{0,0}\right)\left(x_1,x_2\right)&=1,\\
R_{n,m}\left(e_{1,0}\right)\left(x_1,x_2\right)&=\frac{x_1}{1+a_n x_1},\\
R_{n,m}\left(e_{0,1}\right)\left(x_1,x_2\right)&=\frac{x_2}{1+a_m x_2},\\
R_{n,m}\left(e_{2,0}\right)\left(x_1,x_2\right)&=\frac{x_1^2}{\left(1+a_n x_1\right)^2}+\frac{x_1}{b_n\left(1+a_n x_1\right)^2},\\
R_{n,m}\left(e_{0,2}\right)\left(x_1,x_2\right)&=\frac{x_2^2}{\left(1+a_m x_2\right)^2}+\frac{x_2}{b_m\left(1+a_m x_2\right)^2},
\end{aligned}$$

where $\left(e_{i,j}\right)\left(x_1,x_2\right)=e_1^i\left(x_1\right)e_2^j\left(x_2\right)$ with $e_1^i\left(x_1\right)=x_1^i$ and $e_2^j\left(x_2\right)=x_2^j$ for $i,j=0,1,2$.

Proof Using Barbosu technique in [5] and Lemma 2.1 in [3], the lemma can be easily proved, so we will omit the proof of the lemma. □

Lemma 2 *Let $f : [0,\infty)\times[0,\infty) \to \mathbb{R}$ be a continuous function. If*

$$\begin{aligned}
\lim_{n,m\to\infty}\left\|R_{n,m}\left(e_{0,0}\right)-e_{0,0}\right\|_{r_1,r_2}&=0,\\
\lim_{n,m\to\infty}\left\|R_{n,m}\left(e_{1,0}\right)-e_{1,0}\right\|_{r_1,r_2}&=0,\\
\lim_{n,m\to\infty}\left\|R_{n,m}\left(e_{0,1}\right)-e_{0,1}\right\|_{r_1,r_2}&=0,\\
\lim_{n,m\to\infty}\left\|R_{n,m}\left(e_{2,0}+e_{0,2}\right)-\left(e_{2,0}+e_{0,2}\right)\right\|_{r_1,r_2}&=0,
\end{aligned}$$

then $\left\{R_{n,m}(f)\right\}$ *converges uniformly to* f *on* $[0, r_1] \times [0, r_2]$ *for* $r_1, r_2 > 0$.

Proof From Lemma 2, taking into account Volkov's theorem in [15] (also see in [1], p.245),the lemma can be easily proved, so we will omit the proof of the lemma. □

Lemma 3 *Let* $n_0, m_0 \geq 2$, $0 < \beta \leq 2/3$, $1/2 < r_1 < R_1 \leq n_0^{1-\beta}/2$ *and* $1/2 < r_2 < R_2 \leq m_0^{1-\beta}/2$. *If* $f : \left(D_{R_1} \cup [R_1, \infty)\right) \times \left(D_{R_2} \cup [R_2, \infty)\right) \to \mathbb{C}$ *is a uniformly continuous bounded on* $[0, \infty) \times [0, \infty)$ *and analytic in* $D_{R_1} \times D_{R_2}$, *then we have the form*

$$R_{n,m}(f)(z_1, z_2) = \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} c_{k,j} R_{n,m}(e_{k,j})(z_1, z_2)$$

for all $(z_1, z_2) \in D_{r_1} \times D_{r_2}$, *where* $(e_{k,j})(z_1, z_2) = e_1^k(z_1) e_2^j(z_2)$ *with* $e_1^k(z_1) = z_1^k$ *and* $e_2^j(z_2) = z_2^j$ *for* $k, j \in \mathbb{N}$.

Proof For any $s, r \in \mathbb{N}$, we define

$$f_{s,r}(z_1, z_2) = \sum_{k=0}^{s} \sum_{j=0}^{r} c_{k,j} e_{k,j}(z_1, z_2) \text{ if } |z_1| \leq r_1, |z_2| \leq r_2$$

and

$$f_{s,r}(z_1, z_2) = f(z_1, z_2) \text{ if } (z_1, z_2) \in (r_1, \infty) \times (r_2, \infty).$$

From the hypothesis on f, it is clear that each $f_{s,r}$ is bounded on $[0, \infty) \times [0, \infty)$, which implies that

$$\left|R_{n,m}(f_{s,r})(z_1, z_2)\right| \leq \sum_{k=0}^{n} \sum_{j=0}^{m} \left|p_{n,k}(z_1)\right| \left|p_{m,j}(z_2)\right| M_{f_{s,r}} < \infty,$$

where $M_{f_{s,r}}$ is a constant depending on $f_{s,r}$, so all $R_{n,m}(f_{s,r})$ are well defined for all $n, m \in \mathbb{N}$, $n \geq n_0$, $m \geq m_0$, $r_1 < n_0^{1-\beta}/2$, $r_2 < m_0^{1-\beta}/2$ and $(z_1, z_2) \in D_{r_1} \times D_{r_2}$.

Defining

$$f_{s,r,k,j}(z_1, z_2) = c_{k,j} e_{k,j}(z_1, z_2) \text{ if } |z_1| \leq r_1, \ |z_2| \leq r_2$$

and

$$f_{s,r,k,j}(z_1, z_2) = \frac{f(z_1, z_2)}{(s+1)(r+1)} \text{ if } (z_1, z_2) \in (r_1, \infty) \times (r_2, \infty).$$

It is clear that each $f_{s,r,k,j}$ is bounded on $[0,\infty)\times[0,\infty)$ and

$$f_{s,r}(z_1,z_2)=\sum_{k=0}^{s}\sum_{j=0}^{r} f_{s,r,k,j}(z_1,z_2).$$

From the linearity of $R_{n,m}$, we have

$$R_{n,m}\left(f_{s,r}\right)(z_1,z_2)=\sum_{k=0}^{s}\sum_{j=0}^{r} c_{k,j}R_{n,m}\left(e_{k,j}\right)(z_1,z_2).$$

It suffices to prove that

$$\lim_{s,r\to\infty} R_{n,m}\left(f_{s,r}\right)(z_1,z_2)=R_{n,m}(f)(z_1,z_2)$$

for any fixed $n, m\in\mathbb{N}$, $n\geq n_0$, $m\geq m_0$, $|z_1|\leq r_1$ and $|z_2|\leq r_2$. Since

$$\left\|f_{s,r}-f\right\|_{B([0,\infty)\times[0,\infty))}\leq\left\|f_{s,r}-f\right\|_{r_1,r_2},$$

we can write

$$\begin{aligned}&\left|R_{n,m}\left(f_{s,r}\right)\left(z_1,z_2\right)-R_{n,m}(f)\left(z_1,z_2\right)\right|\\&\quad\leq M_{r_1,r_2,m,n}\left\|f_{s,r}-f\right\|_{B([0,\infty)\times[0,\infty))}\\&\quad\leq M_{r_1,r_2,m,n}\left\|f_{s,r}-f\right\|_{r_1,r_2}\end{aligned}\tag{2.1}$$

for $|z_1|\leq r_1$ and $|z_2|\leq r_2$.

In (2.1), taking limit as $s,r\to\infty$ and using $\lim_{s,r\to\infty}\left\|f_{s,r}-f\right\|_{r_1,r_2}=0$, we get the result. □

Next lemma and theorem will help to get the convergence result of the operators (1.1).

Lemma 4 *Let* $n_0, m_0\geq 2$, $0<\beta\leq 2/3$, $1/2<r_1<R_1\leq n_0^{1-\beta}/2$ *and* $1/2<r_2<R_2\leq m_0^{1-\beta}/2$. *For all* $n\geq n_0$, $m\geq m_0$, $|z_1|\leq r_1$ *and* $|z_2|\leq r_2$,*the following inequality holds*

$$\left|R_{n,m}\left(e_{k,j}\right)\left(z_1,z_2\right)\right|\leq k!j!\,(2r_1)^k\,(2r_2)^j.$$

Proof Using Lemma 1.10.2 in [6] (see p. 141), the lemma is easily proved, so we will omit the proof of the theorem. □

3 Main Results

Let us denote with $A_C(f)$ the space of the all complex valued functions, which are uniformly continuous on $\left(D_{R_1} \cup [R_1, \infty)\right) \times \left(D_{R_2} \cup [R_2, \infty)\right)$, bounded on $[0, \infty) \times [0, \infty)$ and analytic in $D_{R_1} \times D_{R_2}$, that is $f(z_1, z_2) = \sum_{k=0}^{\infty} f_k(z_2) z_1^k$ for all $(z_1, z_2) \in D_{R_1} \times D_{R_2}$ with $f_k(z_2) = \sum_{j=0}^{\infty} c_{k,j} z_2^j$, and for which there exist $M > 0$, $0 < A_1 < 1/2r_1$ and $0 < A_2 < 1/2r_2$ with $\left|c_{k,j}\right| \leq M A_1^k A_2^j / k! j!$ (which implies $|f(z_1, z_2)| \leq M e^{A_1|z_1| + A_2|z_2|}$ for all $(z_1, z_2) \in D_{R_1} \times D_{R_2}$).

Now, we can give the following convergence result.

Theorem 1 *Let $n_0, m_0 \geq 2$, $0 < \beta \leq 2/3$, $1/2 < r_1 < R_1 \leq n_0^{1-\beta}/2$ and $1/2 < r_2 < R_2 \leq m_0^{1-\beta}/2$. If $f \in A_C(f)$, then the sequence of operators $\left\{R_{n,m}(f)(z_1, z_2)\right\}$ is uniformly convergent to f on $\bar{D}_{r_1} \times \bar{D}_{r_2}$ for all $n \geq n_0$ and $m \geq m_0$.*

Proof From Lemma 3 and Lemma 4, we can write

$$\begin{aligned} \left|R_{n,m}(f)\left(z_1, z_2\right)\right| &\leq \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \left|c_{k,j}\right| \left|R_{n,m}\left(e_{k,j}\right)\left(z_1, z_2\right)\right| \\ &\leq M \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \left(2r_1 A_1\right)^k \left(2r_2 A_2\right)^j < \infty, \end{aligned} \tag{3.1}$$

where the series $\sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \left(2r_1 A_1\right)^k \left(2r_2 A_2\right)^j$ is convergent since $0 < A_1 < 1/2r_1$ and $0 < A_2 < 1/2r_2$.

On the other hand, from Lemma 2, we know that

$$\lim_{n,m \to \infty} R_{n,m}(f)\left(x_1, x_2\right) = f\left(x_1, x_2\right) \tag{3.2}$$

for all $\left(x_1, x_2\right) \in [0, r_1] \times [0, r_2]$.

Using (3.1) and (3.2) and taking into account the Vitali's theorem [9] (see p.112, Theorem 3.2.10), it follows that $\left\{R_{n,m}(f)\left(z_1, z_2\right)\right\}$ uniformly converges to f on $\bar{D}_{r_1} \times \bar{D}_{r_2}$ for all $n \geq n_0$ and $m \geq m_0$.

We have the following upper estimate.

Theorem 2 *Let $n_0, m_0 \geq 2$, $0 < \beta \leq 2/3$, $1/2 < r_1 < R_1 \leq n_0^{1-\beta}/2$ and $1/2 < r_2 < R_2 \leq m_0^{1-\beta}/2$. If $f \in A_C(f)$, then for all $n \geq \max\left\{n_0, 1/r_1^{1/\beta}\right\}$, $m \geq \max\left\{m_0, 1/r_2^{1/\beta}\right\}$, $|z_1| \leq r_1$ and $|z_2| \leq r_2$ the following inequality holds*

$$\left|R_{n,m}(f)\left(z_1, z_2\right) - f\left(z_1, z_2\right)\right| \leq \left(a_n + \frac{1}{b_n}\right) C^1(f) + \left(a_m + \frac{1}{b_m}\right) C^2(f),$$

where

$$C^1(f) = \max\left\{2r_1 M e^{2r_1 A_1 + r_2 A_2}, \frac{4M}{r_1} e^{r_2 A_2} \sum_{k=1}^{\infty} k\left(2r_1 A_1\right)^{k-1}\right\},$$

$$C^2(f) = \max\left\{2r_2 M e^{2r_2 A_2} \sum_{k=0}^{\infty}\left(2r_1 A_1\right)^k, \frac{4M}{r_2} \sum_{k=0}^{\infty}\left(2r_1 A_1\right)^k \sum_{j=1}^{\infty} j\left(2r_2 A_2\right)^{j-1}\right\},$$

and also the series $\sum_{k=0}^{\infty}\left(2r_1 A_1\right)^k$, $\sum_{k=1}^{\infty} k\left(2r_1 A_1\right)^{k-1}$ *and* $\sum_{j=1}^{\infty} j\left(2r_2 A_2\right)^{j-1}$ *are convergent.*

Proof Using Lemma 3, we can write

$$\left|R_{n,m}(f)(z_1, z_2) - f(z_1, z_2)\right| \leq \sum_{k=0}^{\infty}\sum_{j=0}^{\infty} \left|c_{k,j}\right| \left|R_{n,m}\left(e_{k,j}\right)(z_1, z_2) - e_{k,j}(z_1, z_2)\right|. \tag{3.3}$$

Taking into account Lemma 1.10.2 and the estimate given in the proof of Theorem 1.10.5 in [6] (see p. 141 pp 144-145), for all $|z_1| \leq r_1$ and $|z_2| \leq r_2$, we obtain

$$\begin{aligned}
&\left|R_{n,m}\left(e_{k,j}\right)(z_1, z_2) - e_{k,j}(z_1, z_2)\right| \\
&\quad = \left|R_n\left(e_1^k\right)(z_1) . R_m\left(e_2^j\right)(z_2) - z_1^k z_2^j\right| \\
&\quad \leq \left|R_n\left(e_1^k\right)(z_1)\right| \left|R_m\left(e_2^j\right)(z_2) - z_2^j\right| + \left|z_2^j\right| \left|R_n\left(e_1^k\right)(z_1) - z_1^k\right| \\
&\quad \leq (k!)\,(2r_1)^k \left\{(2a_m r_2)\,(2r_2)^j + \frac{8\,(2r_2)^{j-1}}{b_m} j\,(j!)\right\} \\
&\qquad + r_2^j \left\{2a_n r_1\,(2r_1)^k + \frac{8\,(2r_1)^{k-1}}{b_n} k\,(k!)\right\} \\
&= (2a_n r_1)\,(2r_1)^k\, r_2^j + (2a_m r_2)\,(k!)\,(2r_1)^k\,(2r_2)^j \\
&\qquad + \frac{8k\,(k!)\,(2r_1)^{k-1}\, r_2^j}{b_n} + \frac{8\,(k!)\,(2r_1)^k\, j\,(j!)\,(2r_2)^{j-1}}{b_m}.
\end{aligned} \tag{3.4}$$

Applying (3.4) in(3.3), we get

$$\begin{aligned}
\left|R_{n,m}(f)(z_1, z_2) - f(z_1, z_2)\right| &\leq 2a_n r_1 M \sum_{k=0}^{\infty}\sum_{j=0}^{\infty} \frac{\left(2r_1 A_1\right)^k (r_2 A_2)^j}{k!\,j!} \\
&\quad + 2a_m r_2 M \sum_{k=0}^{\infty}\sum_{j=0}^{\infty} \frac{\left(2r_1 A_1\right)^k (2r_2 A_2)^j}{j!}
\end{aligned}$$

$$+\frac{8MA_1}{b_n}\sum_{k=1}^{\infty}\sum_{j=0}^{\infty}\frac{k\left(2r_1A_1\right)^{k-1}(r_2A_2)^j}{j!}$$
$$+\frac{8MA_2}{b_m}\sum_{k=0}^{\infty}\sum_{j=1}^{\infty}\left(2r_1A_1\right)^k j\,(2r_2A_2)^{j-1}$$

which complete the prove. □

In what follows a Voronovskaja-type result for the operators (1.1) is presented. It will be the product of the parametric extensions generated by Voronovskaja's formula in univariate case in Theorem 1.10.6 in [6] (see p. 145).

Theorem 3 *Let* $n_0, m_0 \geq 2$, $0 < \beta \leq 2/3$, $1/2 < r_1 < R_1 \leq n_0^{1-\beta}/2$ *and* $1/2 < r_2 < R_2 \leq m_0^{1-\beta}/2$. *If* $f \in A_C(f)$, *then for all* $n \geq \max\left\{n_0, 1/r_1^{1/\beta}\right\}$, $m \geq \max\left\{m_0, 1/r_2^{1/\beta}\right\}$, $|z_1| \leq r_1$ *and* $|z_2| \leq r_2$ *the following inequality holds*

$$\left|z_2T_m(f)\left(z_1,z_2\right)\circ z_1T_n(f)\left(z_1,z_2\right)\right| \leq C^3(f)\left[\left(a_n+\frac{1}{b_n}\right)^2+\left(a_m+\frac{1}{b_m}\right)^2\right],$$

where $C^3(f) = 1/2\max\left\{C_{r_1,r_2}^{r_1^*}(f), C_{r_1,r_2}^{r_2^*}(f)\right\}$ *with*

$$C_{r_1,r_2}^{r_1^*}(f) = MC_{r_1,r_1^*}e^{A_2r_2}\sum_{k=2}^{\infty}(k-1)k(k+1)(k+2)(2A_1r_1)^k$$
$$\times\max\left\{1, e^{-A_2r_2}\sum_{j=0}^{\infty}(2A_2r_2)^j, \frac{a_mr_2^2}{1-a_mr_2}, \frac{a_m^2b_mr_2^4+r_2}{2b_m(1-a_mr_2)^2}\right\}$$
$$C_{r_1,r_2}^{r_2^*}(f) = MC_{r_2,r_2^*}e^{A_1r_1}\sum_{j=2}^{\infty}(j-1)j(j+1)(j+2)(2A_2r_2)^j$$
$$\times\max\left\{1, e^{-A_1r_1}\sum_{k=0}^{\infty}(2A_1r_1)^k, \frac{a_nr_1^2}{1-a_nr_1}, \frac{a_n^2b_nr_1^4+r_1}{2b_n(1-a_nr_1)^2}\right\},$$

for $1/2 < r_1 < r_1^* < n_0^{1-\beta}/2$ *and* $1/2 < r_2 < r_2^* \leq m_0^{1-\beta}/2$.

Proof For $f(z_1,z_2)$, we define the parametric extensions of the Voronovskaja's formula by

$$z_1T_n(f)(z_1,z_2) := R_n(f(.,z_2))(z_1) - f(z_1,z_2) + \frac{a_nz_1^2}{1+a_nz_1}\frac{\partial f}{\partial z_1}(z_1,z_2)$$
$$-\frac{a_n^2b_nz_1^4+z_1}{2b_n(1+a_nz_1)^2}\frac{\partial^2 f}{\partial z_1^2}(z_1,z_2)$$

and

$$z_2T_m(f)(z_1,z_2) := R_m(f(z_1,.))(z_2) - f(z_1,z_2) + \frac{a_m z_2^2}{1+a_m z_2}\frac{\partial f}{\partial z_2}(z_1,z_2)$$
$$-\frac{a_m^2 b_m z_2^4 + z_2}{2b_m(1+a_m z_2)^2}\frac{\partial^2 f}{\partial z_2^2}(z_1,z_2).$$

Their product (composition) gives

$$\begin{aligned} &z_2T_m(f)(z_1,z_2)\circ z_1T_n(f)(z_1,z_2)\\ &= R_m\Bigg[R_n\big(f(.,.)\big)(z_1) - f(z_1,.) + \frac{a_n z_1^2}{1+a_n z_1}\frac{\partial f}{\partial z_1}(z_1,.)\\ &\quad -\frac{a_n^2 b_n z_1^4 + z_1}{2b_n(1+a_n z_1)^2}\frac{\partial^2 f}{\partial z_1^2}(z_1,.)\Bigg](z_2) - \Bigg[R_n\big(f(.,z_2)\big)(z_1) - f(z_1,z_2)\\ &\quad +\frac{a_n z_1^2}{1+a_n z_1}\frac{\partial f}{\partial z_1}(z_1,z_2) - \frac{a_n^2 b_n z_1^4 + z_1}{2b_n(1+a_n z_1)^2}\frac{\partial^2 f}{\partial z_1^2}(z_1,z_2)\Bigg] + \frac{a_m z_2^2}{1+a_m z_2}\frac{\partial}{\partial z_2}\\ &\quad\times\Bigg[R_n\big(f(.,z_2)\big)(z_1) - f(z_1,z_2) + \frac{a_n z_1^2}{1+a_n z_1}\frac{\partial f}{\partial z_1}(z_1,z_2)\\ &\quad -\frac{a_n^2 b_n z_1^4 + z_1}{2b_n(1+a_n z_1)^2}\frac{\partial^2 f}{\partial z_1^2}(z_1,z_2)\Bigg] - \frac{a_m^2 b_m z_2^4 + z_2}{2b_m(1+a_m z_2)^2}\frac{\partial^2}{\partial z_2^2}\\ &\quad\times\Bigg[R_n\big(f(.,z_2)\big)(z_1) - f(z_1,z_2) + \frac{a_n z_1^2}{1+a_n z_1}\frac{\partial f}{\partial z_1}(z_1,z_2)\\ &\quad -\frac{a_n^2 b_n z_1^4 + z_1}{2b_n(1+a_n z_1)^2}\frac{\partial^2 f}{\partial z_1^2}(z_1,z_2)\Bigg] := E_1 - E_2 + E_3 - E_4. \end{aligned}$$

After simple calculation, it is obtained the commutativity property

$$z_2T_m(f)(z_1,z_2)\circ z_1T_n(f)(z_1,z_2) = z_1T_n(f)(z_1,z_2)\circ z_2T_m(f)(z_1,z_2). \quad (3.5)$$

From the analyticity of f, since the all partial derivatives of f are analytic in $D_{R_1}\times D_{R_2}$, using Lemma 3, we can write

$$\begin{aligned} &R_n\big(f(.,z_2)\big)(z_1) - f(z_1,z_2) + \frac{a_n z_1^2}{1+a_n z_1}\frac{\partial f}{\partial z_1}(z_1,z_2) - \frac{a_n^2 b_n z_1^4 + z_1}{2b_n(1+a_n z_1)^2}\frac{\partial^2 f}{\partial z_1^2}(z_1,z_2)\\ &= \sum_{k=2}^{\infty} f_k(z_2)\left[R_n\left(e_1^k\right)(z_1) - e_1^k(z_1) + \frac{\left(a_n z_1^2\right)k z_1^{k-1}}{1+a_n z_1} - \frac{\left(a_n^2 b_n z_1^4 + z_1\right)k(k-1)z_1^{k-2}}{2b_n(1+a_n z_1)^2}\right]. \end{aligned} \quad (3.6)$$

Applying now R_m to (3.6) with respect to z_2, we obtain

$$\begin{aligned} E_1 &= \sum_{k=2}^{\infty} R_m\left(f_k\right)\left(z_2\right)\left[R_n\left(e_1^k\right)\left(z_1\right)-e_1^k\left(z_1\right)\right. \\ &\quad \left. +\frac{\left(a_n z_1^2\right) k z_1^{k-1}}{1+a_n z_1}-\frac{\left(a_n^2 b_n z_1^4+z_1\right) k(k-1) z_1^{k-2}}{2 b_n\left(1+a_n z_1\right)^2}\right] \\ &= \sum_{k=2}^{\infty} \sum_{j=0}^{\infty} c_{k, j} R_m\left(e_2^j\right)\left(z_2\right)\left[R_n\left(e_1^k\right)\left(z_1\right)-e_1^k\left(z_1\right)\right. \\ &\quad \left. +\frac{\left(a_n z_1^2\right) k z_1^{k-1}}{1+a_n z_1}-\frac{\left(a_n^2 b_n z_1^4+z_1\right) k(k-1) z_1^{k-2}}{2 b_n\left(1+a_n z_1\right)^2}\right]. \end{aligned} \tag{3.7}$$

In (3.7), passing now to absolute value for $|z_1| \leq r_1$ and $|z_2| \leq r_2$ and taking into account the Lemma 1.10.2 and the estimate given in the proof of Theorem 1.10.6 in [6] (see p. 141 and pp. 145–146), it follows

$$\begin{aligned} |E_1| &\leq \left(a_n+\frac{1}{b_n}\right)^2 C_{r_1, r_1^*} \\ &\quad \times \sum_{k=2}^{\infty} \sum_{j=0}^{\infty}\left|c_{k, j}\right| j!\left(2 r_2\right)^j(k-1) k(k+1)(k+2) k!\left(2 r_1\right)^k \\ &\leq \left(a_n+\frac{1}{b_n}\right)^2 M C_{r_1, r_1^*} e^{A_2 r_2} \\ &\quad \times \sum_{k=2}^{\infty}(k-1) k(k+1)(k+2)\left(2 A_1 r_1\right)^k \sum_{j=0}^{\infty}\left(2 A_2 r_2\right)^j, \end{aligned} \tag{3.8}$$

where $r_1 < r_1^* < R < n_0^{1-\beta}/2$ with $r_1^* < 1/2A_1$ and C_{r_1,r_1^*} is a constant depending on r_1 and r_1^*.

Similarly, using Lemma 3, we have

$$\begin{aligned} |E_2| &\leq \sum_{k=2}^{\infty}\left|f_k\left(z_2\right)\right|\left|R_n\left(e_1^k\right)\left(z_1\right)-e_1^k\left(z_1\right)+\frac{\left(a_n z_1^2\right) k z_1^{k-1}}{1+a_n z_1}\right. \\ &\quad \left.-\frac{\left(a_n^2 b_n z_1^4+z_1\right) k(k-1) z_1^{k-2}}{2 b_n\left(1+a_n z_1\right)^2}\right| \\ &\leq \left(a_n+\frac{1}{b_n}\right)^2 C_{r_1, r_1^*} \sum_{k=2}^{\infty} \sum_{j=0}^{\infty}\left|c_{k, j}\right| r_2^j(k-1) k(k+1)(k+2)(k!)\left(2 r_1\right)^k \\ &\leq \left(a_n+\frac{1}{b_n}\right)^2 M C_{r_1, r_1^*} e^{A_2 r_2} \sum_{k=2}^{\infty}(k-1) k(k+1)(k+2)\left(2 A_1 r_1\right)^k. \end{aligned} \tag{3.9}$$

Using

$$R_n\left(\frac{\partial f}{\partial z_2}(., z_2)\right)(z_1) = \sum_{k=0}^{\infty} \frac{\partial f_k}{\partial z_2}(z_2)\, R_n\left(e_1^k\right)(z_1)$$
$$= \sum_{k=0}^{\infty}\sum_{j=1}^{\infty} c_{k,j}\, j z_2^{j-1} R_n\left(e_1^k\right)(z_1),$$

we can write

$$E_3 = \frac{a_m z_2^2}{1+a_m z_2}\left[R_n\left(\frac{\partial f}{\partial z_2}(., z_2)\right)(z_1) - \frac{\partial f}{\partial z_2}(z_1, z_2)\right.$$
$$\left. + \frac{a_n z_1^2}{1+a_n z_1}\frac{\partial^2 f}{\partial z_1 \partial z_2}(z_1, z_2) - \frac{a_n^2 b_n z_1^4 + z_1}{2b_n (1+a_n z_1)^2}\frac{\partial^3 f}{\partial z_1^2 \partial z_2}(z_1, z_2)\right]$$
$$= \frac{a_m z_2^2}{1+a_m z_2}\sum_{k=2}^{\infty}\sum_{j=1}^{\infty} c_{k,j}\, j z_2^{j-1}\left[R_n\left(e_1^k\right)(z_1) - e_1^k(z_1)\right.$$
$$\left. + \frac{\left(a_n z_1^2\right) k z_1^{k-1}}{1+a_n z_1} - \frac{\left(a_n^2 b_n z_1^4 + z_1\right) k (k-1) z_1^{k-2}}{2b_n (1+a_n z_1)^2}\right],$$

which implies

$$|E_3| \le \left(a_n + \frac{1}{b_n}\right)^2 \frac{a_m r_2^2}{1-a_m r_2} C_{r_1, r_1^*}$$
$$\times \sum_{k=2}^{\infty}\sum_{j=1}^{\infty}\left|c_{k,j}\right| j r_2^{j-1}(k-1)k(k+1)(k+2)(k!)(2r_1)^k,$$
$$\le \left(a_n + \frac{1}{b_n}\right)^2 \frac{a_m r_2^2}{1-a_m r_2}$$
$$\times M C_{r_1, r_1^*} e^{A_2 r_2}\sum_{k=2}^{\infty}(k-1)k(k+1)(k+2)(2A_1 r_1)^k. \tag{3.10}$$

And also, using

$$R_n\left(\frac{\partial^2 f}{\partial z_2^2}(., z_2)\right)(z_1) = \sum_{k=0}^{\infty}\frac{\partial^2 f_k}{\partial z_2^2}(z_2)\, R_n\left(e_1^k\right)(z_1)$$
$$= \sum_{k=0}^{\infty}\sum_{j=2}^{\infty} c_{k,j}\, j (j-1) z_2^{j-2} R_n\left(e_1^k\right)(z_1),$$

we can write

$$E_4 = \frac{a_m^2 b_m z_2^4 + z_2}{2b_m (1 + a_m z_2)^2} \left[R_n \left(\frac{\partial^2 f}{\partial z_2^2} (., z_2) \right) (z_1) - \frac{\partial^2 f}{\partial z_2^2} (z_1, z_2) \right.$$
$$\left. + \frac{a_n z_1^2}{1 + a_n z_1} \frac{\partial^3 f}{\partial z_1 \partial z_2^2} (z_1, z_2) - \frac{a_n^2 b_n z_1^4 + z_1}{2b_n (1 + a_n z_1)^2} \frac{\partial^4 f}{\partial z_1^2 \partial z_2^2} (z_1, z_2) \right]$$
$$= \frac{a_m^2 b_m z_2^4 + z_2}{2b_m (1 + a_m z_2)^2} \sum_{k=2}^{\infty} \sum_{j=2}^{\infty} c_{k,j} j (j-1) z_2^{j-2} \left[R_n \left(e_1^k \right) (z_1) - e_1^k (z_1) \right.$$
$$\left. + \frac{\left(a_n z_1^2 \right) k z_1^{k-1}}{1 + a_n z_1} - \frac{\left(a_n^2 b_n z_1^4 + z_1 \right) k (k-1) z_1^{k-2}}{2b_n (1 + a_n z_1)^2} \right],$$

which implies

$$|E_4| \leq \left(a_n + \frac{1}{b_n} \right)^2 \frac{a_m^2 b_m r_2^4 + r_2}{2b_m (1 - a_m r_2)^2} C_{r_1, r_1^*} \sum_{j=2}^{\infty} j (j-1) r_2^{j-2}$$
$$\times \sum_{k=2}^{\infty} \left| c_{k,j} \right| (k-1) k (k+1) (k+2) (k!) (2r_1)^k ,$$
$$\leq \left(a_n + \frac{1}{b_n} \right)^2 \frac{a_m^2 b_m r_2^4 + r_2}{2b_m (1 - a_m r_2)^2}$$
$$\times M C_{r_1, r_1^*} e^{A_2 r_2} \sum_{k=2}^{\infty} (k-1) k (k+1) (k+2) (2A_1 r_1)^k . \tag{3.11}$$

Using (3.7)–(3.11), we get

$$|z_2 T_m (f) (z_1, z_2) \circ z_1 T_n (f) (z_1, z_2)| \leq |E_1| + |E_2| + |E_3| + |E_4|$$
$$\leq C_{r_1, r_2}^{r_1^*} (f) \left(a_n + \frac{1}{b_n} \right)^2 .$$

If we estimate $|z_1 T_n (f) (z_1, z_2) \circ z_2 T_m (f) (z_1, z_2)|$, then by reason of symmetry we get a similar order of approximation, simply interchanging above the places of n with m and r_1 with r_2.

In conclusion, using the commutativity property given in (3.5), we reach the result. □

Let us denote with $A_C^{(2)} (f)$ the space of the all complex valued functions that they and their first and second partial derivatives are uniformly continuous on $\left(D_{R_1} \cup [R_1, \infty) \right) \times \left(D_{R_2} \cup [R_2, \infty) \right)$, bounded on $[0, \infty) \times [0, \infty)$ and analytic in $D_{R_1} \times D_{R_2}$ and there exist $M > 0$, $0 < A_1 < 1/2r_1$, $0 < A_2 < 1/2r_2$ with $\left| c_{k,j} \right| \leq M A_1^k A_2^j / k! j!$(which implies $|f (z_1, z_2)| \leq M e^{A_1 |z_1| + A_2 |z_2|}$ for all $(z_1, z_2) \in D_{R_1} \times D_{R_2}$).

Theorem 2 and Theorem 3 will be used to find the exact degree in approximation of $R_{n,n}(f)$. In this sense we have the following lower estimate.

Theorem 4 *Let $n_0 \geq 2$, $0 < \beta < 1/2$, $1/2 < r_1 < R_1 \leq n_0^{1-\beta}/2$, and $1/2 < r_2 < R_2 \leq n_0^{1-\beta}/2$. If $f \in A_C^{(2)}(f)$ and f is not a solution of the complex partial differential equation*

$$K(f)\left(z_1, z_2\right) = z_1 \frac{\partial^2 f}{\partial z_1^2}\left(z_1, z_2\right) + z_2 \frac{\partial^2 f}{\partial z_2^2}\left(z_1, z_2\right) = 0, \tag{3.12}$$

then for all $n \geq n_0$ we have

$$\left\| R_{n,n}(f) - f \right\|_{r_1, r_2} \geq \frac{1}{36\left(1 + a_n b_n\right)} \left(a_n + \frac{1}{b_n}\right) \|K(f)\|_{r_1, r_2}.$$

Proof We can write

$$\begin{aligned} &R_{n,n}(f)\left(z_1, z_2\right) - f\left(z_1, z_2\right) \\ &= 2\left(a_n + \frac{1}{b_n}\right)\left\{K_n(f)(z_1, z_2) + 2\left(a_n + \frac{1}{b_n}\right)\left[\frac{D_n(f)(z_1, z_2)}{4\left(a_n + \frac{1}{b_n}\right)^2}\right]\right. \\ &\quad \left. + E_n(f)(z_1, z_2) + F_n(f)(z_1, z_2)\right\}, \end{aligned} \tag{3.13}$$

where

$$D_n(f)(z_1, z_2) = z_2 T_n(f)(z_1, z_2) \circ z_1 T_n(f)(z_1, z_2),$$

$$E_n(f)(z_1, z_2) = \sum_{h=1}^{4} E_n^h(f)(z_1, z_2)$$

with

$$E_n^1(f)(z_1, z_2) = \frac{-a_n b_n z_1^2}{2\left(1 + a_n b_n\right)\left(1 + a_n z_1\right)}\left[R_n\left(\frac{\partial f}{\partial z_1}(z_1, .)\right)(z_2) - \frac{\partial f}{\partial z_1}(z_1, z_2)\right],$$

$$E_n^2(f)(z_1, z_2) = \frac{-a_n b_n z_2^2}{2\left(1 + a_n b_n\right)\left(1 + a_n z_2\right)}\left[R_n\left(\frac{\partial f}{\partial z_2}(., z_2)\right)(z_2) - \frac{\partial f}{\partial z_2}(z_1, z_2)\right],$$

$$E_n^3(f)(z_1, z_2) = \frac{a_n^2 b_n z_1^4 + z_1}{4\left(1 + a_n b_n\right)\left(1 + a_n z_1\right)^2}\left[R_n\left(\frac{\partial^2 f}{\partial z_1^2}(z_1, .)\right)(z_2) - \frac{\partial^2 f}{\partial z_1^2}(z_1, z_2)\right],$$

$$E_n^4(f)(z_1, z_2) = \frac{a_n^2 b_n z_2^4 + z_2}{4\left(1 + a_n b_n\right)\left(1 + a_n z_2\right)^2}\left[R_n\left(\frac{\partial^2 f}{\partial z_2^2}(., z_2)\right)(z_1) - \frac{\partial^2 f}{\partial z_2^2}(z_1, z_2)\right],$$

$$F_n(f)(z_1, z_2) = \frac{z_1 T_n(f)(z_1, z_2) + z_2 T_n(f)(z_1, z_2)}{2\left(a_n + \frac{1}{b_n}\right)}$$

$$
\begin{aligned}
&-\frac{a_n b_n z_1^2}{2\left(1+a_n z_1\right)} \frac{\partial f}{\partial z_1}\left(z_1, z_2\right)-\frac{a_n b_n z_2^2}{2\left(1+a_n z_2\right)} \frac{\partial f}{\partial z_2}\left(z_1, z_2\right) \\
&-\frac{a_n^2 b_n z_1^2 z_2^2}{2\left(1+a_n b_n\right)\left(1+a_n z_1\right)\left(1+a_n z_2\right)} \frac{\partial^2 f}{\partial z_1 \partial z_2}\left(z_1, z_2\right) \\
&+\frac{\left(a_n^2 b_n z_1^4+z_1\right)\left(a_n z_2^2\right)}{4\left(1+a_n b_n\right)\left(1+a_n z_1\right)^2\left(1+a_n z_2\right)} \frac{\partial^3 f}{\partial z_1^2 \partial z_2}\left(z_1, z_2\right) \\
&+\frac{\left(a_n^2 b_n z_2^4+z_2\right)\left(a_n z_1^2\right)}{4\left(1+a_n b_n\right)\left(1+a_n z_1\right)\left(1+a_n z_2\right)^2} \frac{\partial^3 f}{\partial z_1 \partial z_2^2}\left(z_1, z_2\right) \\
&-\frac{\left(a_n^2 b_n z_1^4+z_1\right)\left(a_n^2 b_n z_2^4+z_2\right)}{8 b_n\left(1+a_n b_n\right)\left(1+a_n z_1\right)^2\left(1+a_n z_2\right)^2} \frac{\partial^4 f}{\partial z_1^2 \partial z_2^2}\left(z_1, z_2\right),
\end{aligned}
$$

and

$$
K_n(f)\left(z_1, z_2\right)=\frac{1}{1+a_n b_n}\left\{\frac{a_n^2 b_n z_1^4+z_1}{4\left(1+a_n z_1\right)^2} \frac{\partial^2 f}{\partial z_1^2}\left(z_1, z_2\right)+\frac{a_n^2 b_n z_2^4+z_2}{4\left(1+a_n z_2\right)^2} \frac{\partial^2 f}{\partial z_2^2}\left(z_1, z_2\right)\right\}.
$$

Under the conditions of theorem, since $\lim\limits_{n \to \infty} a_n = 0$, $\lim\limits_{n \to \infty} 1/b_n = 0$ and $\lim\limits_{n \to \infty} a_n b_n = 0$ for $0 < \beta < 1/2$, considering Theorem 1.10.5 and Theorem 1.10.6 in [6] (see p. 145), it is clear that

$$
\lim_{n \to \infty} E_n(f)\left(z_1, z_2\right)=0 \text{ and } \lim_{n \to \infty} F_n(f)\left(z_1, z_2\right)=0. \tag{3.14}
$$

From Theorem 3, we obtain

$$
\lim_{n \to \infty}\left\|2\left(a_n+\frac{1}{b_n}\right)\left[\frac{D_n(f)}{4\left(a_n+\frac{1}{b_n}\right)^2}\right]+E_n(f)+F_n(f)\right\|_{r_1, r_2}=0.
$$

Using $\lim\limits_{n \to \infty} a_n b_n = 0$ for $0 < \beta < 1/2$ and $1/1 + a_n |z_1| \geq 2/3$, we get

$$
\left\|K_n(f)\right\|_{r_1, r_2} \geq \frac{1}{18\left(1+a_n b_n\right)}\left|z_1\right|\left|\frac{\partial^2 f}{\partial z_1^2}\left(z_1, z_2\right)\right|. \tag{3.15}
$$

Similarly, it follows

$$
\left\|K_n(f)\right\|_{r_1, r_2} \geq \frac{1}{18\left(1+a_n b_n\right)}\left|z_2\right|\left|\frac{\partial^2 f}{\partial z_2^2}\left(z_1, z_2\right)\right|. \tag{3.16}
$$

From (3.15) and (3.16), we can write

$$
\left\|K_n(f)\right\|_{r_1, r_2} \geq \frac{1}{36\left(1+a_n b_n\right)}\|K(f)\|_{r_1, r_2}. \tag{3.17}
$$

In (3.13), taking into account the inequalities

$$\|H+G\|_{r_1,r_2} \geq \left|\|H\|_{r_1,r_2} - \|G\|_{r_1,r_2}\right| \geq \|H\|_{r_1,r_2} - \|G\|_{r_1,r_2},$$

and (3.17), it follows

$$\begin{aligned}
&\left\| R_{n,n}(f) - f \right\|_{r_1,r_2} \\
&\geq 2\left(a_n + \frac{1}{b_n}\right)\Bigg\{\|K_n(f)\|_{r_1,r_2} \\
&\quad - \left\| 2\left(a_n + \frac{1}{b_n}\right)\left[\frac{D_n(f)}{4\left(a_n + \frac{1}{b_n}\right)^2}\right] + E_n(f) + F_n(f) \right\|_{r_1,r_2}\Bigg\} \\
&\geq \left(a_n + \frac{1}{b_n}\right)\|K_n(f)\|_{r_1,r_2} \\
&\geq \left(a_n + \frac{1}{b_n}\right)\frac{1}{36(1+a_n b_n)}\|K(f)\|_{r_1,r_2}
\end{aligned}$$

for all $n \geq n_0$ with n_0 depending only f, r_1 and $r_{2.}$ We used here that by hypothesis we have $\|K(f)\|_{r_1,r_2} > 0$. □

Combining Theorem 3 with Theorem 4, we immediately obtain the following result giving the exact degree of the operators (1.1).

Corollary 1 *Suppose that the hypothesis in the statement of Theorem 6 holds. If Taylor series of* f *contains at least one term of the form* $c_{k+1,0}z_1^k$ *with* $c_{k+1,0} \neq 0$ *and* $k = 1, 2, \ldots$ *or of the form* $c_{0,j+1}z_2^j$ *with* $c_{0,j+1} \neq 0$ *and* $j = 1, 2, \ldots$*, then for all* $n \geq n_0$ *we have*

$$\left\| R_{n,n}(f) - f \right\|_{r_1,r_2} \sim \left(a_n + \frac{1}{b_n}\right).$$

Proof It suffices to prove that under the hypothesis on f, it cannot be a solution of the complex partial differential equation

$$z_1 \frac{\partial^2 f}{\partial z_1^2}(z_1,z_2) + z_2 \frac{\partial^2 f}{\partial z_2^2}(z_1,z_2) = 0, \ |z_1| < R_1, |z_2| < R_{2.}$$

Indeed, suppose the contrary. Since simple calculation gives

$$\begin{aligned}
z_1 \frac{\partial^2 f}{\partial z_1^2}(z_1,z_2) + z_2 \frac{\partial^2 f}{\partial z_2^2}(z_1,z_2) &= \sum_{k=1}^{\infty} c_{k+1,0}k(k+1)z_1^k + \sum_{k=1}^{\infty} c_{k+1,1}k(k+1)z_1^k z_2 \\
&\quad + 2\sum_{j=2}^{\infty} c_{2,j}z_1 z_2^j + \sum_{k=2}^{\infty}\sum_{j=2}^{\infty} c_{k+1,j}k(k+1)z_1^k z_2^j,
\end{aligned}$$

$$+\sum_{j=1}^{\infty} c_{0,j+1} j\,(j+1)\, z_2^j + \sum_{j=1}^{\infty} c_{1,j+1} j\,(j+1)\, z_1 z_2^j$$
$$+2\sum_{k=2}^{\infty} c_{k,2} z_1^k z_2 + \sum_{k=2}^{\infty}\sum_{j=2}^{\infty} c_{k,j+1} j\,(j+1)\, z_1^k z_2^j,$$

by equaling with zero and by the identification of coefficients, from the terms under the first and fifth sign $\sum$, we immediately get that $c_{k+1,0} = c_{0,j+1} = 0$, for all $k = 1, 2, \ldots$ and $j = 1, 2, \ldots$, which contradicts the hypothesis on f. Therefore the hypothesis and the lower estimate in Theorem 4 satisfy, which complete the proof. □

Acknowledgments The author is grateful to the editor and the reviewers for making valuable suggestions, leading to a better presentation of the work.

References

1. Altomare, F., Campiti, M.: Korovkin-Type Approximation Theory and Its Applications. Walter de Gruyter, Berlin (1994)
2. Atakut, Ç., İspir, N.: On Bernstein type rational functions of two variables. Math. Slovacka **54**(3), 291–301 (2004)
3. Balázs, K.: Approximation by Bernstein type rational function. Acta Math. Acad. Sci. Hungar **26**, 123–134 (1975)
4. Balázs, K., Szabados, J.: Approximation by Bernstein type rational function II. Acta Math. Acad. Sci. Hungar **40**(3–4), 331–337 (1982)
5. Barbosu, D.: Some generalized bivariate Bernstein operators. Math. Notes (Miskolc) **1**, 3–10 (2000)
6. Gal, S.G.: Approximation by Complex Bernstein and Convolution Type Operators. World Scientific Publishing Co. Pte.Ltd., New Jersey (2009)
7. Gal, S.G., Gupta, V., Mahmudov, N.I.: Approximation by a complex q-Durmeyer type operator. Ann. Univ. Ferrara **58**, 65–87 (2012)
8. Gupta, V., Ispir, N.: On the Bezier variant of generalized Kantorovitch type Balazs operators. Appl. Math. Lett. **18**(9), 1053–1061 (2005)
9. Kohr, G., Mocanu, P.T.: Special Chapters of Complex Analysis. University Press, Cluj-Napuca (2005). (in Romania)
10. Mahmudov, N.I.: Approximation by Bernstein-Durmeyer-type Polynomials in compact disks. Appl. Math. Lett. **24**, 1231–1238 (2011)
11. Mahmudov, N.I.: Approximation by genuine q-Bernstein-Durmeyer operators in compact disks. Hacet J. Math. Stat. **40**(1), 77–89 (2011)
12. Mursaleen, M., Khan, A.: Statistical approximation properties of modified q-Stancu-Beta operators. Bull. Malays. Math. Sci. Soc. (2) **36**(3), 683–690 (2013)
13. Ispir, N., Yıldız Özkan, E.: Approximation properties of complex q-Balazs-Szabados operators in compact disks. J. İnequal. Appl. **2013**, 361 (2013). doi:10.1186/1029-242X-2013-361
14. Agarwal, R.P., Gupta, V.: On q-analogue of a complex summation-integral type operators in compact disks. J. İnequal. Appl. **2012**, 111 (2012)
15. Volkov, V.I.: On the convergence of sequences of linear positive operators in the space of continuous functions of two variables. (Russian) Dokl. Akad. Nauk. SSSR (N.S.) 115 (1957)