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Metabelian Associative Algebras

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Abstract Metabelian algebras are introduced and it is shown that an algebra A is metabelian if and only if A is a nilpotent algebra having the index of nilpotency at most 3, i.e. $xyzt = 0$, for all $x, y, z, t \in A$. We prove that the Itô's theorem for groups remains valid for associative algebras. A structure theorem for metabelian algebras is given in terms of pure linear algebra tools and their classification from the view point of the extension problem is proven. Two border-line cases are worked out in detail: all metabelian algebras having the derived algebra of dimension 1 (resp. codimension 1) are explicitly described and classified. The algebras of the first family are parameterized by bilinear forms and classified by their homothetic relation. The algebras of the second family are parameterized by the set of all matrices $(X, Y, u) \in \mathrm{M}_n(k)^2 \times k^n$ satisfying $X^2 = Y^2 = 0$, $XY = YX$ and $Xu = Yu$.

Keywords Metabelian algebras · Congruence of bilinear forms · Classification of algebras

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1 Introduction

The concept of a metabelian group goes back to Fite [8] and since then they became a very important topic of study within group theory [14]. At the level of Lie, or more

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general Leibniz algebras, the corresponding concept of metabelian Lie/Leibniz algebra, as 2-step solvable algebra, is also well known [2,6,7]. In this paper, we introduce the associative algebra counterpart of a metabelian group by defining a *metabelian algebra* over a field k as an extension of an abelian algebra by an abelian algebra—the word 'abelian' is borrowed from Lie algebras, i.e. an algebra A having the trivial multiplication: $xy = 0$, for all $x, y \in A$. At first sight, such a restrictive definition seems to have limited chances of leading to an interesting theory of these associative algebras and, moreover, few of them can be expected to exist. We shall prove the contrary and, to begin with, we shall introduce the following argument: the classification of all metabelian associative algebras having the derived algebra of dimension 1 (hence only a small and apparently unattractive class of algebras) is equivalent to the classification of bilinear forms on a vector space up to the homothetic relation on bilinear forms—a relation that generalizes the classical isometric relation on bilinear forms [17]. On the other hand, the class of all $(n+1)$-dimensional metabelian associative algebras having the derived algebra of dimension n is parameterized by an interesting set of matrices, namely the matrices $(X, Y, u) \in \mathrm{M}_n(k)^2 \times k^n$ satisfying $X^2 = Y^2 = 0$, $XY = YX$ and $Xu = Yu$. We prove that an algebra A is metabelian if and only if A is a nilpotent algebra having the index of nilpotency at most 3. The classification of all nilpotent algebras of a given dimension is a classical problem in the theory of associative algebras: see [13, ChapterVI] where the classification of nilpotent algebras of dimension 3 and 4 is obtained. The classification of nilpotent algebras of a dimension higher than 4 is a difficult problem since the complexity of the computations increases very rapidly with the dimension [9]. Thus, one has higher chances to succeed in classifying metabelian algebras of a given dimension instead of nilpotent ones. The first bridge to solving the problem is given by Theorem 3.3, where a structure theorem is proposed. We show that any metabelian algebra A is isomorphic to an algebra of the form $P \star V$ associated to a system $(P, V, \triangleleft, \triangleright, \theta)$ consisting of two vector spaces P, V and three bilinear maps $\triangleleft : V \times P \to V$, $\triangleright : P \times V \to V$, $\theta : P \times P \to V$ satisfying the following compatibility conditions for any $p, q, r \in P$ and $x \in V$:

$$p \triangleright (x \triangleleft q) = (p \triangleright x) \triangleleft q, \quad (x \triangleleft p) \triangleleft q = p \triangleright (q \triangleright x) = 0, \quad p \triangleright \theta(q, r) = \theta(p, q) \triangleleft r.$$

We denoted $P \star V := P \times V$ with the multiplication given for any $p, q \in P$, $x, y \in V$ by

$$(p, x) \star (q, y) := (0, \theta(p, q) + p \triangleright y + x \triangleleft q)$$

Based on this, the classification of all metabelian algebras that are extensions of a given abelian algebra P_0 by an abelian algebra V_0 is obtained in Theorem 3.6, where the explicit description of the classifying object $\mathrm{Ext}(P_0, V_0)$ is given. $\mathrm{Ext}(P_0, V_0)$ classifies all metabelian algebras from the viewpoint of the extension problem [10], i.e. up to an isomorphism of algebras that stabilizes V_0 and co-stabilizes P_0. In Theorem 3.8, all metabelian algebras having the derived algebra of dimension 1 are explicitly described, classified and the automorphism groups of these algebras are determined. The algebras of this family are classified by bilinear forms on a vector space P up to the following equivalence relation: two bilinear forms θ and $\theta' \in \mathrm{Bil}\,(P \times P, k)$ are called homothetic [15] if there exists a pair $(u, \psi) \in k^* \times \mathrm{Aut}_k(P)$ such that

$u\,\theta(p,\, q) = \theta'\big(\psi(p),\, \psi(q)\big)$, for all $p, q \in P$. If $k = k^2$, this relation is equivalent to the classical classification of bilinear forms [17]. Examples are given in Corollary 3.10. Theorem 3.11 and Example 3.12 address the dual case: all $(n+1)$-dimensional metabelian algebras having the derived algebras of codimension 1 are explicitly described and classified by the set of all matrices $(X, Y, u) \in \mathrm{M}_n(k)^2 \times k^n$ satisfying $X^2 = Y^2 = 0$, $XY = YX$ and $Xu = Yu$.

2 Preliminaries

All vector spaces, algebras, linear or bilinear maps are over an arbitrary field k. For a vector space V, we denote by $\mathrm{Bil}\,(V \times V,\, k)$ all bilinear forms on V. Two bilinear forms θ and $\theta' \in \mathrm{Bil}\,(V \times V,\, k)$ are called *isometric*, and we denote this by $\theta \approx \theta'$, if there exists an automorphism $\varphi \in \mathrm{Aut}_k(V)$ such that $\theta(x,\, y) = \theta'(\varphi(x),\, \varphi(y))$, for all $x, y \in V$. If V is finite dimensional having $\{e_1, \ldots, e_n\}$ as a basis, we also denote by $\theta = (\theta(e_i,\, e_j)) \in \mathrm{M}_n(k)$ the matrix associated to a bilinear form $\theta \in \mathrm{Bil}\,(V \times V,\, k)$. Then $\theta \approx \theta'$ if and only if there exists an invertible matrix $C \in \mathfrak{gl}(n, k)$, such that $\theta = C^T\,\theta'\,C$—where C^T is the transposed of C. For future references to the classification problem of bilinear forms up to an isometry, see [11,16] and the references therein. Two bilinear forms θ and $\theta' \in \mathrm{Bil}\,(V \times V,\, k)$ are called *homothetic* [15], and we denote this by $\theta \equiv \theta'$, if there exists a pair $(u, \psi) \in k^* \times \mathrm{Aut}_k(P)$ such that $u\,\theta(p,\, q) = \theta'\big(\psi(p),\, \psi(q)\big)$, for all $p, q \in P$. Any isometric bilinear forms are homothetic and if $k = k^2 := \{x^2 \mid x \in k\}$, then any homothetic bilinear forms are isometric.

By an algebra we always mean an associative algebra, i.e. a pair (A, M_A) consisting of a vector space A and a bilinear map $M_A : A \times A \to A$, called the multiplication on A and denoted by $M_A\,(x, y) = xy$, such that $x(yz) = (xy)z$, for all $x, y, z \in A$. Any vector space V is an algebra with the trivial multiplication $xy = 0$, for all x, $y \in V$—such an algebra is called *abelian* and will be denoted by V_0. $\mathrm{Aut}_{\mathrm{Alg}}\,(A)$ will denote the group of algebra automorphisms of A. An algebra A is called *nilpotent* if there exists a positive integer n such that $x_1 x_2 \ldots x_n = 0$, for all $x_1, \ldots, x_n \in A$. If there is a non-zero product of $n-1$ elements of A, then n is called the *index of nilpotency* of A. For an algebra A, we denote by A' the *derived subalgebra* of A, i.e. the subspace of A generated by all xy, for any $x, y \in A$. Let P and V be two given algebras. An extension of P by V is triple $(A,\, i,\, \pi)$ consisting of an algebra A and two morphisms of algebras $i : V \to A$, $\pi : A \to P$ such that

$$0 \longrightarrow V \xrightarrow{\ i\ } A \xrightarrow{\ \pi\ } P \longrightarrow 0$$

is an exact sequence. Two extensions (A, i, π) and (A', i', π') of P by V are called *equivalent*, and we denote this by $(A, i, \pi) \approx (A', i', \pi)$ if there exists a morphism of algebras $\varphi : A \to A'$ that stabilizes V and co-stabilizes P, i.e. the following diagram

$$\begin{array}{ccccc} V & \xrightarrow{\ i\ } & A & \xrightarrow{\ \pi\ } & P \\ {\scriptstyle Id}\downarrow & & \downarrow{\scriptstyle \varphi} & & \downarrow{\scriptstyle Id} \\ V & \xrightarrow{\ i'\ } & A' & \xrightarrow{\ \pi'\ } & P \end{array}$$

is commutative. Any such morphism φ is an isomorphism and thus $\approx$ is an equivalence relation on the class of all extensions of P by V. We denote by $\mathrm{Ext}\,(P,\ V)$ the set of all equivalence classes of all extensions of P by V via $\approx$.

3 Metabelian Algebras

We introduce the concept of metabelian associative algebras having in mind one of the equivalent definitions of metabelian groups.

Definition 3.1 An algebra A is called *metabelian* if it is an extension of an abelian algebra by an abelian algebra, that is there exist two vector spaces V, P and an exact sequence of morphisms of algebras[1]

$$0 \longrightarrow V_0 \stackrel{i}{\longrightarrow} A \stackrel{\pi}{\longrightarrow} P_0 \longrightarrow 0 \tag{1}$$

Explicit examples will be provided at the end of the paper. In order to proceed with the characterization and structure theorem for metabelian algebras, some preparatory work is needed.

Definition 3.2 Let V be a vector space. A *discrete bimodule* over V is a bimodule over the abelian algebra V_0, i.e. a triple $(P, \triangleleft, \triangleright)$ consisting of a vector space P and two bilinear maps $\triangleleft : V \times P \to V$, $\triangleright : P \times V \to V$ satisfying the following compatibility conditions for any $p, q \in P$ and $x \in V$:

$$(x \triangleleft p) \triangleleft q = p \triangleright (q \triangleright x) = 0, \qquad p \triangleright (x \triangleleft q) = (p \triangleright x) \triangleleft q \tag{2}$$

For a discrete bimodule $(P, \triangleleft, \triangleright)$ over V, a bilinear map $\theta : P \times P \to V$ is called a *discrete* $(\triangleleft, \triangleright)$*-cocycle* if for any p, q and $r \in P$ we have

$$p \triangleright \theta(q,\ r) = \theta(p,\ q) \triangleleft r. \tag{3}$$

A system $(P,\ V,\ \triangleleft,\ \triangleright,\ \theta)$ consisting of a vector space V, a discrete bimodule $(P, \triangleleft, \triangleright)$ over V and a discrete $(\triangleleft, \triangleright)$-cocycle $\theta : P \times P \to V$ is called a *metabelian datum* of P by V. We denote by $\mathrm{DZ}^2\big((P, \triangleleft, \triangleright),\ V\big)$ the space of all discrete $(\triangleleft, \triangleright)$-cocycles and by $\mathrm{Met}\,(P,\ V)$ the set of all metabelian datums $(P,\ V,\ \triangleleft,\ \triangleright,\ \theta)$ of P by V.

Let $(P,\ V,\ \triangleleft,\ \triangleright,\ \theta) \in \mathrm{Met}\,(P,\ V)$ and $P \star V = P \star_{(\triangleleft,\, \triangleright,\, \theta)} V$ be the vector space $P \times V$ with the multiplication given for any $p, q \in P$, $x, y \in V$ by

$$(p, x) \star (q, y) := (0,\ \theta(p, q) + p \triangleright y + x \triangleleft q). \tag{4}$$

As a special case of [3, Proposition 1.2], we can easily see that $P \star V$ with the multiplication given by (4) is an associative algebra. In fact, we can easily show that the multiplication given by (4) is associative if and only if $(P,\ V,\ \triangleleft,\ \triangleright,\ \theta)$ is a

[1] We recall that for a vector space V, we denote $V_0 = V$ with the abelian algebra structure.

metabelian datum. $P \star V$ is a metabelian algebra since we have the following exact sequence of algebra maps

$$0 \longrightarrow V_0 \xrightarrow{i_V} P \star V \xrightarrow{\pi_P} P_0 \longrightarrow 0\,, \tag{5}$$

where $i_V(x) = (0, x)$ and $\pi_p(p, x) := p$, for all $x \in V$ and $p \in P$. The algebra $P \star V$ is called the *metabelian product* of P over V associated to $(P,\ V,\ \triangleleft,\ \triangleright,\ \theta) \in \mathrm{Met}\,(P,\ V)$. The next characterization and structure theorem for metabelian algebras shows that any metabelian algebra A is isomorphic to some $P \star V$.

Theorem 3.3 *For an associative algebra A the following statements are equivalent:*

(1) *A is metabelian;*
(2) *A is a nilpotent algebra having the index of nilpotency at most 3, i.e. $xyzt = 0$, for all $x, y, z, t \in A$;*
(3) *The derived algebra A' is an abelian subalgebra of A;*
(4) *There exists an isomorphism of algebras $A \cong P \star V$, for some vector spaces P and V and for some $(P,\ V,\ \triangleleft,\ \triangleright,\ \theta) \in \mathrm{Met}\,(P,\ V)$.*

Proof (3) is just an equivalent rephrasing of (2), i.e. (2) $\Leftrightarrow$ (3). The exactness of the sequence (5) proves that $P \star V$ is metabelian, that is (4) $\Rightarrow$ (1). Now, if A' is an abelian subalgebra of A, we have an exact sequence of algebra maps

$$0 \longrightarrow A' = A'_0 \xrightarrow{i} A \xrightarrow{\pi} (A/A')_0 \longrightarrow 0, \tag{6}$$

where i is the inclusion map and π the canonical projection. This proves (3) $\Rightarrow$ (1). We prove now that (1) $\Rightarrow$ (2). Assume that (1) is an exact sequence of algebra maps. We can assume that V is a subspace of A. The fact that i is an algebra map shows that $xy = 0$, for all $x, y \in V$. On the other hand, the fact that π is a morphism of algebras and the exactness of the sequence (1) translates to $\pi(ab) = 0$, i.e. $ab \in \mathrm{Ker}(\pi) = V$, for all $a, b \in A$. Thus, the product $xyzt = 0$, for all x, y, z and $t \in A$ since the product of any two elements of V is zero. If we show (1) $\Rightarrow$ (4), the proof is finished. Since k is a field, we can pick a k-linear section $s : P \to A$ of π, i.e. $\pi \circ s = \mathrm{Id}_P$. Using the section s, we define three bilinear maps $\triangleleft_s : V \times P \to V$, $\triangleright_s : P \times V \to V$ and $\theta_s : P \times P \to V$ by the following formulas:

$$x \triangleleft p := xs(p), \qquad p \triangleright x := s(p)x, \qquad \theta(p, q) := s(p)s(q)$$

for all $p, q \in P$ and $x \in V$. Then, by a straightforward computation (or as a special case of [3, Proposition 1.4]) we can show that $(P,\ V,\ \triangleleft_s,\ \triangleright_s,\ \theta_s) \in \mathrm{Met}\,(P,\ V)$ and the map

$$\varphi : P \star V \to A, \qquad \varphi(p, x) := s(p) + x \tag{7}$$

is an isomorphism of algebras with $\varphi^{-1}(y) = (\pi(y),\ y - s(\pi(y)))$, for all $y \in A$. □

One of the fundamental results in the factorization theory for groups is the famous Itô's theorem [12] which has been the key ingredient in proving many structural theorems for finite groups [4]: if G is a group such that $G = AB$, for two abelian subgroups A and B, then G is metabelian. As a special case of Theorem 3.3, we obtain the counterpart of Itô's theorem for associative algebras:

Corollary 3.4 (Itô's theorem for associative algebras) *Let A be an associative algebra such that $A = P_0 + V_0$, for two abelian subalgebras P_0 and V_0 of A. Then A is metabelian.*

Proof Using Theorem 3.3 we have to prove that $a_1\, a_2\, a_3\, a_4 = 0$, for all $a_i \in A$, $i = 1, \ldots, 4$. Indeed, since $a_i \in A = P_0 + V_0$ we can find $v_i \in V_0$ and $p_i \in P_0$ such that $a_i = v_i + p_i$, for all $i = 1, \ldots, 4$. Let $v_5, v_6 \in V_0$ and $p_5, p_6 \in P_0$ such that $p_2\, v_3 = v_5 + p_5$ and $v_2\, p_3 = v_6 + p_6$. Using intensively that P_0 and V_0 are both abelian, we obtain

$$\begin{aligned} a_1\, a_2\, a_3\, a_4 &= \Big(\underline{v_1\, v_2} + v_1\, p_2 + p_1\, v_2 + \underline{p_1\, p_2}\Big)\Big(\underline{v_3\, v_4} + v_3\, p_4 + p_3\, v_4 + \underline{p_3\, p_4}\Big) \\ &= (v_1\, p_2 + p_1\, v_2)\,(v_3\, p_4 + p_3\, v_4) \\ &= v_1\, p_2\, v_3\, p_4 + v_1\, \underline{p_2\, p_3}\, v_4 + p_1\, \underline{v_2\, v_3}\, p_4 + p_1\, v_2\, p_3\, v_4 \\ &= v_1\, p_2\, v_3\, p_4 + p_1\, v_2\, p_3\, v_4 \;=\; v_1\,(v_5 + p_5)\, p_4 + p_1\,(v_6 + p_6)\, v_4 \\ &= \underline{v_1\, v_5}\, p_4 + v_1\, \underline{p_5\, p_4} + p_1\, \underline{v_6\, v_4} + \underline{p_1\, p_6}\, v_4 \;=\; 0 \end{aligned}$$

as needed. □

Remark 3.5 There is more to be said related to the proof of Theorem 3.3: if A is a metabelian algebra, extension of P_0 by V_0, then the isomorphism $\varphi : P \star V \to A$ given by (7) stabilizes V and co-stabilizes P, i.e. the diagram

$$\begin{array}{ccccc} V_0 & \xrightarrow{i_V} & P \star V & \xrightarrow{\pi_P} & P_0 \\ \big\downarrow{\scriptstyle Id} & & \big\downarrow{\scriptstyle \varphi} & & \big\downarrow{\scriptstyle Id} \\ V_0 & \xrightarrow{i} & A & \xrightarrow{\pi} & P_0 \end{array}$$

is commutative. Hence, the extension (A, i, p) of P_0 by V_0 is equivalent to the extension $(P \star V, i_V, \pi_P)$ of P_0 by V_0 given by (5). Thus the classification of metabelian algebras is reduced to the classification of all metabelian products of $P \star V$ associated to all $(P,\ V,\ \triangleleft,\ \triangleright,\ \theta) \in \mathrm{Met}\,(P,\ V)$. This will be given below by indicating the explicit description of the classifying object $\mathrm{Ext}(P_0,\ V_0)$.

Let $(P, \triangleleft, \triangleright)$ be a fixed discrete bimodule over a vector space V. Two discrete $(\triangleleft, \triangleright)$-cocycles $\theta, \theta' : P \times P \to V$ are called *cohomologous* and we denote this by $\theta \equiv \theta'$ if there exists a linear map $r : P \to V$ such that

$$\theta(p,\ q) = \theta'(p,\ q) + p \triangleright r(q) + r(p) \triangleleft q \qquad (8)$$

for all $p, q \in P$. As a special case of [3, Lemma 1.5 and Definition 1.6], we can prove that $\equiv$ is an equivalence relation on $\mathrm{DZ}^2\big((P, \triangleleft, \triangleright),\ V\big)$ and denote by

$$\mathrm{DH}^2\big((P, \triangleleft, \triangleright),\ V\big) := \mathrm{DZ}^2\big((P, \triangleleft, \triangleright),\ V\big)/\equiv$$

the quotient set via $\equiv$, called the *discrete cohomological group*. In fact, [3, Lemma1.5], applied for the abelian case, shows that two extensions of P_0 by V_0 of the form $(P \star_{(\triangleleft, \triangleright, \theta)} V, i_V, \pi_P)$ and $(P \star_{(\triangleleft', \triangleright', \theta)} V, i_V, \pi_P)$ are equivalent if and only if $\triangleleft' = \triangleleft$, $\triangleright' = \triangleright$ and $\theta' \equiv \theta$. All these considerations lead to the following classification result that gives the decomposition of $\mathrm{Ext}(P_0, V_0)$ as the co-product of all discrete cohomological groups.

Theorem 3.6 *Let V and P be two vector spaces. Then there exists a bijection*

$$\mathrm{Ext}(P_0, V_0) \cong \sqcup_{(\triangleleft,\triangleright)} \mathrm{DH}^2\big((P, \triangleleft, \triangleright),\ V\big), \tag{9}$$

where $\sqcup_{(\triangleleft,\triangleright)}$ is the co-product in the category of sets over all possible discrete bimodule structures $(P, \triangleleft, \triangleright)$ over V. The explicit bijection sends an element $\overline{\theta} \in \mathrm{DH}^2\big((P, \triangleleft, \triangleright),\ V\big)$ to the metabelian product $P \star_{(\triangleleft, \triangleright, \theta)} V$, where $\overline{\theta}$ denotes the equivalence class of the discrete $(\triangleleft, \triangleright)$-cocycle θ via $\equiv$.

Remarks 3.7 (1) The formula (9) highlights an algorithm that breaks up the problem of computing $\mathrm{Ext}(P_0, V_0)$ into three steps: (1) first, all discrete bimodule structures $(P, \triangleleft, \triangleright)$ over V are described; (2) then, for a given structure $(P, \triangleleft, \triangleright)$, we compute the space $\mathrm{DZ}^2\big((P, \triangleleft, \triangleright),\ V\big)$ of all discrete $(\triangleleft, \triangleright)$-cocycles; (3) in the last step all quotient spaces $\mathrm{DH}^2\big((P, \triangleleft, \triangleright),\ V\big) = \mathrm{DZ}^2\big((P, \triangleleft, \triangleright),\ V\big)/\approx$ are described and their co-product is computed. Specific examples are given below.

(2) If we are interested in the classification of all metabelian algebras A of a given dimension, then the key object is the derived algebra A' since Theorem 3.3 shows that $A \cong P \star A'$, where $P := A/A'$, the quotient vector space viewed as an abelian algebra. Thus, there are two numbers involved in the problem of classifying finite-dimensional metabelian algebras: n, the dimension of the metabelian algebras that we are looking for, and $m := \dim_k(A') \le n$. Two border-line cases are immediately settled: if $m = n$, then $A' = A$, and thus we obtain, using (3) of Theorem 3.3, that A is the abelian algebra k_0^n. On the other hand, if $m = 0$, that is $A' = 0$, then A is also the abelian algebra k_0^n, by the definition of A'. The next two steps cover the cases when $m = 1$ (resp. $m = n - 1$).

First we shall describe and classify metabelian algebras having the derived algebra of dimension 1; the group of algebra automorphisms of such algebras is also determined.

Theorem 3.8 *Let P be a vector space. Then*

(1) $\mathrm{Ext}\,(P_0, k_0) \cong \mathrm{Bil}\,(P \times P, k)$ *and the equivalence classes of all metabelian algebras that are extensions of P_0 by k_0 are represented by the extensions of the form*

$$0 \longrightarrow k_0 \longrightarrow P_\theta := P \star_\theta k \longrightarrow P_0 \longrightarrow 0 \tag{10}$$

for any $\theta \in \mathrm{Bil}\,(P \times P, k)$; the multiplication on the algebra P_θ is given by: $(p, x) \star (q, y) = (0, \theta(p, q))$, for all $p, q \in P$, $x, y \in k$.
Any metabelian algebra having the derived algebra of dimension 1 is isomorphic to a P_θ, for some vector space P and $\theta \in \mathrm{Bil}\,(P \times P, k)$.

(2) *Two algebras P_θ and $P_{\theta'}$ are isomorphic if and only if the bilinear forms θ and θ are homothetic, i.e. there exists a pair $(u, \psi) \in k^* \times \mathrm{Aut}_k(P)$ such that for all p, $q \in P$*

$$u\,\theta(p, q) = \theta'\big(\psi(p), \psi(q)\big). \tag{11}$$

In particular, if $k = k^2$ then, $P_\theta \cong P_{\theta'}$ if and only if θ and θ' are isometric.

(3) *The group of algebra automorphisms $\mathrm{Aut}_{\mathrm{Alg}}\,(P_\theta)$ is isomorphic to*

$$\mathcal{G}(P, \theta) := \big\{(u, \lambda, \psi) \in k^* \times P^* \times \mathrm{Aut}_k(P) \mid u\,\theta(p, q) = \theta\big(\psi(p), \psi(q)\big), \ \forall\, p, q \in P\big\},$$

where $\mathcal{G}(P, \theta)$ is a group with respect to the following multiplication:

$$(u, \lambda, \psi) \cdot (u', \lambda', \psi') := (uu', \lambda \circ \psi' + u\lambda', \psi \circ \psi') \tag{12}$$

for all (u, λ, ψ) and $(u', \lambda', \psi') \in \mathcal{G}(P, \theta)$.

Proof (1) The proof is based on Theorems 3.3 and 3.6 following the three steps described in Remark 3.7 applied in the case that $V := k$. Using the first compatibility of (2) we can easily prove that P has only one structure of a discrete bimodule over k, namely the trivial one: $x \triangleleft p = p \triangleright x := 0$, for all $x \in k$ and $p \in P$. Hence, the set of all discrete $(\triangleleft, \triangleright)$-cocycles $\theta : P \times P \to k$ is precisely the space of all bilinear forms of P since (3) holds trivially. Moreover, the equivalence relation (8) is just the equality between the two bilinear maps. Thus, using the decomposition formula given by (9), we obtain that $\mathrm{Ext}\,(P_0, k_0) \cong \mathrm{Bil}\,(P \times P, k)$. The last statement follows from Theorem 3.3.

(2) If $\theta = 0$ (the trivial bilinear form), then $P_{\theta=0}$ is the abelian algebra; thus, if $P_{\theta=0} \cong P_{\theta'}$, then $\theta' = 0$ and there is nothing to prove. We will assume that $\theta \neq 0$. We prove a more general statement which shall also provide the proof of (3). More precisely, for two non-trivial bilinear forms θ and θ' on P we shall prove that there exists a bijection between the set of all isomorphisms of algebras $\varphi : P_\theta \to P_{\theta'}$ and the set of all triples $(u, \lambda, \psi) \in k^* \times \mathrm{Hom}_k(P, k) \times \mathrm{Aut}_k(P)$ satisfying the compatibility condition (11). Moreover, the bijection is given such that the isomorphism $\varphi = \varphi_{(u,\lambda,\psi)} : P_\theta \to P_{\theta'}$ corresponding to (u, λ, ψ) is given by

$$\varphi(p, x) = (\psi(p), \lambda(p) + u\,x) \tag{13}$$

for all $p \in P$ and $x \in k$. Indeed, any k-linear map $\varphi : P \times k \to P \times k$ is uniquely determined by a quadruple $(u, \lambda, \psi, p_0) \in k \times \mathrm{Hom}_k(P, k) \times \mathrm{End}_k(P) \times P$ such that

$$\varphi(p, x) = \varphi_{(u,\,\lambda,\,\psi,\,p_0)}\,(p, x) = (\psi(p) + x\,p_0,\ \lambda(p) + x\,u)$$

for all $p \in P$ and $x \in k$. Now, we can easily see that $\varphi_{(u,\,\lambda,\,\psi,\,p_0)} : P_\theta \to P_{\theta'}$ is an algebra map if and only if the following two compatibilities hold

$$\theta(p, q)\,p_0 = 0, \quad u\,\theta(p, q) = \theta'\big(\psi(p) + x\,p_0,\ \psi(q) + y\,p_0\big)$$

for all $p, q \in P$. Since $\theta \neq 0$, we obtain that $\varphi_{(u,\, \lambda,\, \psi,\, p_0)}$ is an algebra map if and only if $p_0 = 0$ and (11) holds. In what follows we denote by $\varphi_{(u,\, \lambda,\, \psi)}$ the algebra map corresponding to a quadruple $(u,\ \lambda,\ \psi,\ p_0)$ with $p_0 = 0$. It remains to be proven that such a morphism $\varphi = \varphi_{(u,\, \lambda,\, \psi)}$ is bijective if and only if ψ is bijective and $u \neq 0$. Assume first that φ is bijective: then its inverse φ^{-1} is an algebra map and thus has the form $\varphi^{-1}(q,\ y) = (\psi'(q),\ \lambda'(q) + yu')$, for some triple (u', λ', ψ'). If we write $\varphi^{-1} \circ \varphi(0,\ 1) = (0,\ 1)$, we obtain that $uu' = 1$ i.e. u is invertible. In the same way, $\varphi^{-1} \circ \varphi(p,\ 0) = (p,\ 0) = \varphi \circ \varphi^{-1}(p,\ 0)$ gives that ψ is bijective and $\psi' = \psi^{-1}$. Conversely, if $(u,\ \psi) \in k^* \times \mathrm{Aut}_k(P)$, then we can see that $\varphi_{(u,\, \lambda,\, \psi)}$ is bijective having the inverse given by $\varphi^{-1}_{(u,\, \lambda,\, \psi)} := \varphi_{(u^{-1},\, -\lambda\circ\psi^{-1},\, \psi^{-1})}$ as needed. For the last statement we remark that if $\theta \approx \theta'$, then (11) holds for $u = 1$. Conversely, if $k = k^2$, then we can write $u = v^2$, for some $v \in k^*$. Multiplying the equation (11) by v^{-2} and substituting ψ with $v^{-1}\psi$ we obtain that $\theta \approx \theta'$.

(3) Follows the proof of (2) once we observe that for two triples (u, λ, ψ) and $(u', \lambda', \psi') \in k^* \times P^* \times \mathrm{Aut}_k(P)$ we have that $\varphi_{(u,\, \lambda,\, \psi)} \circ \varphi_{(u',\, \lambda',\, \psi')} = \varphi_{(uu',\, \lambda\circ\psi' + u\lambda',\, \psi\circ\psi')}$. □

Theorem 3.8 reduces the classification of all $(n+1)$-dimensional metabelian algebras having the derived algebra of dimension 1 to the classification of bilinear forms on k^n up to the equivalence relation given by (11). If $k = k^2$ this is just the classical classification of bilinear forms solved in [11] for algebraically closed or real closed fields.

Example 3.9 Let $\{e_1, \ldots, e_n\}$ be the canonical basis of k^n. Applying Theorem 3.8 for $P = k^n$ we obtain that any $(n+1)$-dimensional metabelian algebra having the derived algebra of dimension 1 is isomorphic to an algebra denoted by $k^{n+1}_\theta := k^n \star_\theta k$, for some non-trivial bilinear form $\theta \in \mathrm{Bil}\,(k^n \times k^n,\ k)$. Explicitly, k^{n+1}_θ is the algebra having $\{E,\ F_1, \ldots, F_n\}$ as a basis and the multiplication defined for any $i, j = 1, \ldots, n$ by

$$F_i \star F_j := \theta(e_i, e_j)\, E$$

undefined multiplications on the elements of the basis are 0. Two such algebras k^{n+1}_θ and $k^{n+1}_{\theta'}$ are isomorphic if and only if there exists a pair $(u,\ C) \in k^* \times \mathfrak{gl}(n, k)$ such that $u\,\theta = C^T\,\theta'\,C$, where we write $\theta = (\theta(e_i,\ e_j))$ and $\theta' = (\theta'(e_i,\ e_j)) \in \mathrm{M}_n(k)$.

Assume that k is algebraically closed of characteristic $\neq 2$. Then, k^{n+1}_θ and $k^{n+1}_{\theta'}$ are isomorphic if and only if $\theta \approx \theta'$. If $n = 2$ the equivalence classes of all bilinear forms on k^2 are given by the following two families of matrices [11]:

$$\theta_{a,\,b} = \begin{pmatrix} 1 & a \\ b & 0 \end{pmatrix}, \qquad \theta = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

for all $a, b \in k$. On the other hand, for $n = 3$ the equivalence classes of all bilinear forms on k^3 are given by the following six families of matrices [11] for any $a, b \in k$:

$$\theta^1_{a,\,b} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & a \\ 0 & b & 0 \end{pmatrix}, \qquad \theta^2_{a,\,b} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & a \\ 0 & b & 0 \end{pmatrix}, \qquad \theta^3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}$$

$$\theta^4 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}, \quad \theta^5 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}, \quad \theta^6 = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

To conclude, we obtain the following classification results:

Corollary 3.10 *Let k be an algebraically closed field of characteristic $\neq 2$. Then*

(1) *The isomorphism classes of 3-dimensional metabelian algebras having the derived algebra of dimension 1 are the following two families of algebras defined for any $a, b \in k$:*

$$\begin{aligned} k^3_{a,b}: &\quad F_1 \star F_1 = E, \quad F_1 \star F_2 = a\,E, \quad F_2 \star F_1 = b\,E \\ k^3_{-1}: &\quad F_1 \star F_2 = -F_2 \star F_1 = E \end{aligned}$$

(2) *The isomorphism classes of 4-dimensional metabelian algebras having the derived algebra of dimension 1 are the following six families of algebras defined for any $a, b \in k$:*

$$\begin{aligned} k^{4,1}_{a,b}: &\quad F_1 \star F_1 = F_2 \star F_2 = E, \quad F_2 \star F_3 = a\,E, \quad F_3 \star F_2 = b\,E \\ k^{4,2}_{a,b}: &\quad F_2 \star F_2 = E, \quad F_2 \star F_3 = a\,E, \quad F_3 \star F_2 = b\,E \\ k^{4,3}: &\quad F_1 \star F_1 = F_2 \star F_3 = -F_3 \star F_2 = E \\ k^{4,4}: &\quad F_2 \star F_3 = -F_3 \star F_2 = E \\ k^{4,5}: &\quad F_1 \star F_2 = F_2 \star F_3 = -F_3 \star F_2 = E \\ k^{4,6}: &\quad F_1 \star F_1 = F_1 \star F_2 = F_2 \star F_2 = F_2 \star F_3 = F_3 \star F_2 = E. \end{aligned}$$

Now we shall describe the metabelian algebras having the derived algebra of codimension 1. For a vector space V, we denote by $\mathcal{T}(V) \subseteq \mathrm{End}_k(V)^2$ the set of all pairs of endomorphisms $(\lambda, \Lambda) \in \mathrm{End}_k(V) \times \mathrm{End}_k(V)$ satisfying the following compatibility condition

$$\lambda^2 = \Lambda^2 = 0, \qquad \Lambda \circ \lambda = \lambda \circ \Lambda. \tag{14}$$

Theorem 3.11 *Let V be a vector space. Then*

(1) *There exists a bijection*

$$\mathrm{Met}\,(k,\ V) \cong \{\,(\lambda,\ \Lambda,\ \zeta) \in \mathcal{T}(V) \times V \mid \zeta \in \mathrm{Ker}(\Lambda - \lambda)\}$$

given such that the metabelian datum $(k,\ V,\ \triangleleft,\ \triangleright, \theta)$ of k by V associated to a triple $(\lambda,\ \Lambda,\ \zeta)$ is given for any $x \in V$ and $p, q \in k$ by

$$x \triangleleft p := p\,\lambda(x) \quad p \triangleright x := p\,\Lambda(x), \quad \theta(p, q) := pq\,\zeta. \tag{15}$$

(2) *There exists a bijection*

$$\mathrm{Ext}\,(k_0,\ V_0)\ \cong\ \sqcup_{(\lambda,\Lambda)\in\mathcal{T}(V)} \big(\mathrm{Ker}(\Lambda - \lambda)/\mathrm{Im}(\Lambda + \lambda)\big). \tag{16}$$

that sends an element $\overline{\zeta} \in \mathrm{Ker}(\Lambda - \lambda)/\mathrm{Im}(\Lambda + \lambda)\big)$ *to the metabelian product* $k \star_{(\lambda,\Lambda,\zeta)} V = k \times V$*, with the multiplication given for any* $p, q \in k$ *and* $x, y \in V$ *by*

$$(p, x) \star (q, y) = (0, pq\,\zeta + p\,\Lambda(y) + q\,\lambda(x)) \tag{17}$$

(3) *Any metabelian algebra having the derived algebra of codimension* 1 *is isomorphic to an algebra* $k \star_{(\lambda,\Lambda,\zeta)} V$*, for some vector space* V*,* $(\lambda, \Lambda) \in \mathcal{T}(V)$ *and* $\zeta \in \mathrm{Ker}(\Lambda - \lambda)$.

Proof (1) We apply the definitions of the metabelian datum for $P := k$. First of all we show that there exists a bijection between the set of all discrete bimodule structures over V on k and $\mathcal{T}(V)$. Indeed, since $P = k$ then any bilinear map $\triangleleft : V \times k \to V$ (resp. $\triangleright : k \times V \to V$) is uniquely implemented by a linear map $\lambda : V \to V$ (resp. $\Lambda : V \to V$) via the formulas

$$x \triangleleft p := p\lambda(x) \qquad p \triangleright x := p\Lambda(x) \tag{18}$$

for all $x \in V$ and $p \in k$. We can easily see that the pair $(\triangleleft = \triangleleft_\lambda, \triangleright = \triangleright_\Lambda)$ satisfies the compatibility condition (2) if and only if (14) holds. Let now $(\lambda, \Lambda) \in \mathrm{End}_k(V)$ be a pair satisfying (14) and consider $(k, \triangleleft_\lambda, \triangleright_\Lambda)$ with the discrete bimodule structures over V implemented by (λ, Λ) via (18). Then we can see that $\mathrm{DZ}^2\big((k, \triangleleft_\lambda, \triangleright_\Lambda), V\big) \cong \mathrm{Ker}(\Lambda - \lambda)$, and the bijection sends any $\zeta \in \mathrm{Ker}(\Lambda - \lambda)$ to the associated discrete $(\triangleleft_\lambda, \triangleright_\Lambda)$-cocycle θ_ζ given by $\theta_\zeta(p, q) := pq\,\zeta$, for all $p, q \in P$. Thus, we have proved that there exists a one-to-one correspondence between Met (k, V) and the set of all triples $(\lambda, \Lambda, \zeta) \in \mathcal{T}(V) \times V$ such that $\zeta \in \mathrm{Ker}(\Lambda - \lambda)$, as needed.

(2) We fix a pair $(\lambda, \Lambda) \in \mathcal{T}(V)$; using (14), we observe first that $\mathrm{Im}(\Lambda + \lambda) \leq \mathrm{Ker}(\Lambda - \lambda)$. Hence, the quotient vector space from the right-hand side of (16) is defined. Let $\zeta \in \mathrm{Ker}(\Lambda - \lambda)$ and $\theta = \theta_\zeta$ be the associated discrete $(\triangleleft_\lambda, \triangleright_\Lambda)$-cocycle. Then, we can see that $\theta_\zeta \equiv \theta_{\zeta'}$ in the sense of (8) if and only if $\zeta - \zeta' \in \mathrm{Im}(\Lambda + \lambda)$. This shows that

$$\mathrm{DH}^2\big((k, \triangleleft_\lambda, \triangleright_\Lambda), V\big) \cong \mathrm{Ker}(\Lambda - \lambda)/\mathrm{Im}(\Lambda + \lambda)$$

and the conclusion follows from Theorems 3.6 and 3.3. □.

Example 3.12 By taking $V := k^n$ in Theorem 3.11 we obtain the explicit description of all $(n+1)$-dimensional metabelian algebras having the derived algebra of dimension n. Indeed, we denote by

$$\mathcal{T}(n) := \{\, (X, Y) \in \mathrm{M}_n(k) \mid X^2 = Y^2 = 0, \quad XY = YX\}.$$

For a pair of matrices $(X, Y) \in \mathcal{T}(n)$ we denote by $\mathcal{E}(X, Y) := \{u \in k^n \mid Xu = Yu\}$ the equalizer of X and Y. Then we have

$$\mathrm{Ext}\,(k_0, k_0^n) \cong \sqcup_{(X,Y)\in\mathcal{T}(n)} \big(\mathcal{E}(X, Y)/\equiv\big),$$

where $\equiv$ is the following equivalence relation on $\mathcal{E}(X, Y)$: $u \equiv u'$ if and only if there exists $r \in k^n$ such that $u - u' = (X + Y)r$. For $(X = (x_{ij}), Y = (x_{ij}), \overline{u} =$

$\overline{(u_1, \ldots, u_n)}) \in \mathcal{T}(n) \times \mathcal{E}(X, Y)/\equiv$, we denote by $k^{n+1}_{X,Y,\overline{u}}$ the associated metabelian product $k \star k^n$. Then $k^{n+1}_{X,Y,\overline{u}}$ is the algebra having $\{F, E_1, \ldots, E_n\}$ as a basis and the multiplication defined for any $i, j = 1, \ldots, n$ by

$$F \star F := \sum_{j=1}^{n} u_j E_j, \quad F \star E_i := \sum_{j=1}^{n} y_{ji} E_i, \quad E_i \star F := \sum_{j=1}^{n} x_{ji} E_i.$$

Any $(n+1)$-dimensional metabelian algebra having the derived algebra of codimension 1 is isomorphic to such an algebra $k^{n+1}_{X,Y,\overline{u}}$, for some $(X, Y, \overline{u}) \in \mathcal{T}(n) \times \mathcal{E}(X, Y)/\equiv$. Classifying these algebras for an arbitrary n is a very difficult task.

Final comments. Theorem 3.6 classifies metabelian algebras from the viewpoint of the extension problem [10]: for two given vector spaces P and V, $\mathrm{Ext}(P_0, V_0)$ classifies all metabelian algebras that are extensions of P_0 by V_0 up to an isomorphism of algebras that stabilizes V_0 and co-stabilizes P_0. Even if the explicit computation of the classifying object $\mathrm{Ext}(P_0, V_0)$ offers important information, it is not enough to classify up to an isomorphism all metabelian algebras of a given dimension. Based on Theorem 3.3, the next step is to ask when two arbitrary metabelian products $P \star V$ and $P' \star V'$ are isomorphic as algebras. Mutatis-mutandis, this is the associative algebra version of the isomorphism problem from metabelian groups, which is a very difficult question that seems to be connected to Hilbert's Tenth problem, i.e. the problem is algorithmically undecidable (cf. [5]). Theorem 3.8 gives the full answer in the particular case $V = V' := k$: the isomorphism of two algebras $P \star k$ and $P' \star k$ is equivalent to classifying the bilinear forms as stated there. Unfortunately, a similar result offering a necessary and sufficient criterion for two metabelian products $k \star V$ and $k' \star V'$ from Theorem 3.11 to be isomorphic could not be obtained: direct computations lead to a system of compatibilities that is very technical and impossible to apply in practice.

We end the paper with an open question. A classical and very difficult problem in the theory of groups is the following ([4, p. 18]): for two given abelian groups A and B describe and classify all groups G which can be written as a product $G = AB$. Having this question in mind as well as Corollary 3.4 we might ask as follows:

Question *Let V_0 and P_0 be two abelian algebras. Describe and classify all algebras A containing V_0 and P_0 as subalgebras such that $A = V_0 + P_0$.*

A particular case of the question, corresponding to the additional condition $P_0 \cap V_0 = \{0\}$, is the following: *for two given abelian algebras V_0 and P_0, describe and classify all bicrossed products $P_0 \bowtie V_0$ of algebras*—for details and the construction of the bicrossed product associated to a matched pair of associative algebras we refer to [1].

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