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A Conformal Connection on Null Hypersurfaces of Indefinite Kenmotsu Manifolds

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Abstract In this paper, our main remark is that proper totally contact umbilical integral manifolds of screen integrable null hypersurfaces in indefinite Kenmotsu manifolds admit η-Weyl structures. Its geometry is closely related to the one of a normal sub-bundle over the indefinite Kenmotsu manifold.

Keywords Indefinite Kenmostu manifold · Null hypersurface · Conformal connection

Mathematics Subject Classification Primary 53C15 · Secondary 53C05, 53C25, 53C50

1 Introduction

The manifolds of indefinite signature play a special role in geometry and physics. They generate models of spacetime of general relativity. For instance, in the tangent space at a point of a manifold with Lorentzian signature, a real isotropic cone is invariantly defined, and from physical point of view, this cone is the light cone. Null hypersurfaces are also studied in the theory of electromagnetism. This is the reason that there are many papers [11–15], and books [4–6,10] and references therein, in which null hypersurfaces are investigated.

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Many objects of general relativity are invariant under conformal transformations of a metric, and a null hypersurface is an example of the objects that are invariant under conformal transformations of a metric (see [1] for more details). Hence it is appropriate to study null hypersurfaces not only on a Kenmostu manifold of indefinite signature but also on an integral manifold endowed with a conformal structure.

A Weyl structure on a smooth manifold is a torsion-free affine connection D preserving a conformal structure $[g]$ [7]. The Ricci tensor of a Weyl connection is usually non-symmetric. This paper is devoted to the geometry of integral manifolds of the class of almost contact metric manifolds of Kählerian type known as Kenmotsu manifolds [8] with specific attention to its null subspace.

As is well known, contrary to timelike and spacelike submanifolds, the geometry of null submanifolds [5] is different because of the fact that their normal vector bundle intersects with the tangent bundle. To deal with this anomaly, the null submanifolds were introduced and presented in a book by Duggal and Bejancu [5]. They introduced a non-degenerate screen distribution to construct a non-intersecting null transversal vector bundle of the tangent bundle. Several authors have studied those spaces. Concerning the null contact geometry, some specific discussions can be found in [4, 11–15] and references therein.

In this paper, we investigate conformal connection on integral manifolds of null hypersurfaces in indefinite Kenmotsu manifolds, tangent to the structure vector field. Note that, being null manifold is invariant under conformal change of the metric, along with many geometric objects.

The paper is organized as follows. In Sect. 2, we give basic definition on indefinite Kenmotsu manifolds and null hypersurfaces of semi-Riemannian manifolds. In Sect. 3, we prove that a proper totally contact umbilical leaf of a screen integrable null hypersuface admits η-conformal structures. A geometric configuration of such a leaf is also established. By Theorem 3.11, we prove that, under some conditions, the local triviality implies that a locally symmetric integral manifolds of screen integrable null hypersurface turns out to be locally semi-Riemannian.

2 Preliminaries

Let $\overline{M}$ be a $(2n+1)$-dimensional manifold endowed with an almost contact structure $(\overline{\phi}, \xi, \eta)$, i.e., $\overline{\phi}$ is a tensor field of type $(1, 1)$, ξ is a vector field, and η is a 1-form satisfying

$$\overline{\phi}^2 = -\mathbb{I} + \eta \otimes \xi, \quad \eta(\xi) = 1, \quad \eta \circ \overline{\phi} = 0 \text{ and } \overline{\phi}\xi = 0. \tag{2.1}$$

Then $(\overline{\phi}, \xi, \eta, \overline{g})$ is called an indefinite almost contact metric structure on $\overline{M}$ if $(\overline{\phi}, \xi, \eta)$ is an almost contact structure on $\overline{M}$ and $\overline{g}$ is a semi-Riemannian metric on $\overline{M}$ such that [3], for any vector field $\overline{X}$, $\overline{Y}$ on $\overline{M}$

$$\overline{g}(\overline{\phi}\,\overline{X}, \overline{\phi}\,\overline{Y}) = \overline{g}(\overline{X}, \overline{Y}) - \eta(\overline{X})\,\eta(\overline{Y}). \tag{2.2}$$

It follows that $\eta(\cdot) = \overline{g}(\xi, \cdot)$. If, moreover, $(\overline{\nabla}_{\overline{X}}\overline{\phi})\overline{Y} = \overline{g}(\overline{\phi}\,\overline{X}, \overline{Y})\xi - \eta(\overline{Y})\overline{\phi}\,\overline{X}$, where $\overline{\nabla}$ is the Levi-Civita connection for the semi-Riemannian metric $\overline{g}$, we call

$\overline{M}$ *an indefinite Kenmotsu manifold*. Without loss of generality, ξ is assumed to be spacelike, that is, $\overline{g}(\xi, \xi) = 1$. The Kenmotsu structure defined in [8] differs to the indefinite Kenmotsu one only by the positiveness of the metric involved and so, the results in [8] remain unchanged for the indefinite case. $\Gamma(\Xi)$ denotes the set of smooth sections of the vector bundle Ξ.

A plane section σ in $T_p\overline{M}$ is called a $\overline{\phi}$-section if it is spanned by $\overline{X}$ and $\overline{\phi}\,\overline{X}$, where $\overline{X}$ is a unit tangent vector field orthogonal to ξ. The sectional curvature of a $\overline{\phi}$-section σ is called a $\overline{\phi}$-sectional curvature. If an indefinite Kenmotsu manifold $\overline{M}$ has constant $\overline{\phi}$-sectional curvature c, then, by virtue of the Proposition 12 in [8], the curvature tensor $\overline{R}$ of $\overline{M}$ is given by the relation (5.6) in [8] with $c = H$.

A Kenmotsu manifold $\overline{M}$ of constant $\overline{\phi}$-sectional curvature c will be called *Kenmotsu space form* and denoted by $\overline{M}(c)$.

If $\overline{M}$ is an indefinite Kenmotsu space, then, $\overline{M}$ is an Einstein one and $c = -1$ [13] and the curvature tensor $\overline{R}$ of $\overline{M}(c)$ is given by

$$\overline{R}(\overline{X}, \overline{Y})\overline{Z} = \overline{g}(\overline{X}, \overline{Z})\overline{Y} - \overline{g}(\overline{Y}, \overline{Z})\overline{X}, \quad \overline{X},\ \overline{Y},\ \overline{Z} \in \Gamma(T\overline{M}). \tag{2.3}$$

Let $(\overline{M}, \overline{g})$ be a $(2n+1)$-dimensional semi-Riemannian manifold with index s, $0 < s < 2n+1$, endowed with an almost contact metric structure $(\overline{\phi}, \xi, \eta)$, and let (M, g) be a null hypersurface of $\overline{M}$ with $g = \overline{g}_{|M}$. It is well known that the normal bundle $TM^{\perp}$ of the null hypersurface M is a vector subbundle of TM of rank 1. A complementary vector bundle $S(TM)$ of $TM^{\perp}$ in TM is a rank $(2n-1)$ non-degenerate distribution over M, called a *screen distribution* on M, such that

$$TM = S(TM) \oplus_{orth} TM^{\perp}, \tag{2.4}$$

where $\oplus_{orth}$ denotes the orthogonal direct sum. Existence of $S(TM)$ is secured provided M is paracompact. A null hypersurface with a specific screen distribution is denoted by $(M, g, S(TM))$. We know [5] that for such a triplet, there exists a unique rank 1 vector subbundle $tr(TM)$ of $T\overline{M}$ over M, such that for any non-zero section E of $TM^{\perp}$ on a coordinate neighborhood $\mathcal{U} \subset M$, there exists a unique section N of $tr(TM)$ on $\mathcal{U}$ satisfying

$$\overline{g}(N, E) = 1, \ \text{ and } \ \overline{g}(N, N) = \overline{g}(N, W) = 0, \ \ \forall\, W \in \Gamma(S(TM)|_{\mathcal{U}}). \tag{2.5}$$

Then, $T\overline{M}$ is decomposed as follows:

$$T\overline{M} = TM \oplus tr(TM) = (TM^{\perp} \oplus tr(TM)) \oplus_{orth} S(TM). \tag{2.6}$$

We call $tr(TM)$ and N the transversal vector bundle and the null transversal vector field of M with respect to $S(TM)$, respectively. The local Gauss and Weingarten formulas are, for any $X, Y \in \Gamma(TM|_{\mathcal{U}})$,

$$\overline{\nabla}_X Y = \nabla_X Y + B(X, Y)N, \ \ \overline{\nabla}_X N = -A_N X + \tau(X)N, \tag{2.7}$$

$$\nabla_X PY = \nabla^*_X PY + C(X, PY)E, \ \ \nabla_X E = -A^*_E X - \tau(X)E, \tag{2.8}$$

where $\overline{\nabla}$ is the Levi-Civita connection of $\overline{M}$, P is the projection morphism of $\Gamma(TM)$ on $\Gamma(S(TM))$ with respect to the decomposition (2.4). Also, ∇ and ∇^* are the linear connections, B and C are the local second fundamental forms, A_N and A^*_E are the shape operators on TM and $S(TM)$, respectively, and τ is a 1-form on *TM*.

From the fact that $B(X, Y) = \overline{g}(\overline{\nabla}_X Y, E)$, we know that B is independent of the choice of a screen distribution and satisfies $B(\cdot, E) = 0$. Unfortunately, the induced connection ∇ on TM is not metric and satisfies

$$(\nabla_X g)(Y, Z) = B(X, Y)\theta(Z) + B(X, Z)\theta(Y), \tag{2.9}$$

where θ is a differential 1-form locally defined on M by $\theta(\cdot) := \overline{g}(N, \cdot)$. However, the connection ∇^* on $S(TM)$ is metric. The above two local second fundamental forms of M and $S(TM)$ are related to their shape operators by $B(X, PY) = g(A^*_E X, PY)$, $g(A^*_E X, N) = 0$, $C(X, PY) = g(A_N X, PY)$, $g(A_N X, N) = 0$.

Now consider $(\overline{M}^{2n+1}, \overline{\phi}, \xi, \eta, \overline{g})$ to be an indefinite Kenmotsu manifold and (M, g) a null hypersurface of $(\overline{M}, \overline{g})$ with $\xi \in \Gamma(TM)$. If E is a local section of $TM^{\perp}$, it is easy to check that $\overline{\phi}E \neq 0$ and $\overline{g}(\overline{\phi}E, E) = 0$, then $\overline{\phi}E$ is tangent to M. Thus $\overline{\phi}(TM^{\perp})$ is a distribution on M of rank 1 such that $\overline{\phi}(TM^{\perp}) \cap TM^{\perp} = \{0\}$. This enables us to choose a screen distribution $S(TM)$ such that it contains $\overline{\phi}(TM^{\perp})$ as a vector subbundle. If we consider a local section N of $tr(TM)$, we have $\overline{\phi}\, N \neq 0$. Since $\overline{g}(\overline{\phi}\, N, E) = -\overline{g}(N, \overline{\phi}\, E) = 0$, we deduce that $\overline{\phi}\, E \in \Gamma(S(TM))$ and $\overline{\phi}\, N$ is also tangent to M. At the same time, $\overline{\phi}\, N$ has no component with respect to E. Thus $\overline{\phi}\, N \in \Gamma(S(TM))$, that is, $\overline{\phi}(tr(TM))$ is also a vector subbundle of $S(TM)$ of rank 1. From (2.1), we have $\overline{g}(\overline{\phi}\, N, \overline{\phi}E) = 1$. Therefore, $\overline{\phi}(TM^{\perp}) \oplus \overline{\phi}(tr(TM))$ is a non-degenerate vector subbundle of $S(TM)$ of rank 2. If $\xi \in \Gamma(TM)$, we may choose $S(TM)$ so that ξ belongs to $S(TM)$. Using this, and since $\overline{g}(\overline{\phi}E, \xi) = \overline{g}(\overline{\phi}N, \xi) = 0$, there exists a non-degenerate distribution $\mathcal{D}_0$ of rank $2n - 4$ on M such that

$$S(TM) = \left\{\overline{\phi}(TM^{\perp}) \oplus \overline{\phi}(tr(TM))\right\} \oplus_{orth} \mathcal{D}_0 \oplus_{orth} < \xi >, \tag{2.10}$$

where $\langle\xi\rangle$ is the distribution spanned by ξ. $\mathcal{D}_0$ is invariant under $\overline{\phi}$, i.e., $\overline{\phi}(\mathcal{D}_0) = \mathcal{D}_0$. Moreover, from (2.4), we have the decomposition

$$TM = \left\{\overline{\phi}(TM^{\perp}) \oplus \overline{\phi}(tr(TM))\right\} \oplus_{orth} \mathcal{D}_0 \oplus_{orth} < \xi > \oplus_{orth} TM^{\perp}. \tag{2.11}$$

Now, we consider the distributions on M, $\mathcal{D} := TM^{\perp} \oplus_{orth} \overline{\phi}(TM^{\perp}) \oplus_{orth} \mathcal{D}_0$, $\mathcal{D}' := \overline{\phi}(tr(TM))$. Then, $\mathcal{D}$ is invariant under $\overline{\phi}$ and

$$TM = (\mathcal{D} \oplus \mathcal{D}') \oplus_{orth} \langle\xi\rangle. \tag{2.12}$$

Let us consider the local null vector fields $U := -\overline{\phi}N$, $V := -\overline{\phi}E$. Then, from (2.12), any $X \in \Gamma(TM)$ is written as $X = RX + QX + \eta(X)\xi$, $QX = u(X)U$, where R and Q are the projection morphisms of TM into $\mathcal{D}$ and $\mathcal{D}'$, respectively, and u is a differential 1-form locally defined on M by $u(\cdot) := g(V, \cdot)$. In addition, we have, $\nabla_X\xi = X - \eta(X)\xi$, $B(X, \xi) = 0$ and $C(X, \xi) = \theta(X)$.

3 Main Results

In this section, we deal with totally contact umbilicity of some foliations of null hypersurface of an indefinite Kenmotsu manifold $(\overline{M}, \overline{g})$.

First of all, a submanifold M is said to be a totally umbilical null hypersurface of a semi-Riemannian manifold $\overline{M}$ if its local second fundamental form B satisfies

$$B(X, Y) = \rho g(X, Y), \quad \forall X, \ Y \in \Gamma(TM), \tag{3.1}$$

where ρ is a smooth function on $\mathcal{U} \subset M$. If we assume that M is a totally umbilical null hypersurface of an indefinite Kenmotsu manifold $\overline{M}$ with $\xi \in \Gamma(TM)$, we have $0 = B(\xi, \xi) = \rho$. Hence M is totally geodesic. It follows that an indefinite Kenmotsu space form $\overline{M}(c)$ does not admit any non-totally geodesic totally umbilical null hypersurface. From this point of view, Bejancu [2] considered the concept of totally contact umbilical semi-invariant submanifolds.

The notion of totally contact umbilical submanifolds was first defined by Kon [9]. We follow Bejancu's definition of totally contact umbilical submanifolds and state the following definition for null hypersurfaces.

A null hypersurface M is said to be totally contact umbilical if its second fundamental form $h = B \otimes N$ satisfies ([11]),

$$h(X, Y) = \{g(X, Y) - \eta(X)\eta(Y)\} H + \eta(X)h(Y, \xi) + \eta(Y)h(X, \xi), \tag{3.2}$$

for any $X, Y \in \Gamma(TM)$, where H is a normal vector field to M. It is easy to check that a totally contact umbilical null hypersurface of an indefinite Kenmotsu manifold is η-*totally umbilical*. If the function λ is nowhere vanishing on M, then the latter is said to be proper totally contact umbilical. It is easy to check that this is an intrinsic notion that is independent, on $\mathcal{U}$, of the choice of a screen distribution, E, and hence N as in Sect. 2.

Let $\widehat{W}$ be an element of $TM^{\perp} \oplus tr(TM)$ which is a non-degenerate distribution of rank 2. Then there exist non-zero functions α and β such that

$$\widehat{W} = \alpha E + \beta N, \tag{3.3}$$

where α and β are defined as $\alpha = \overline{g}(\widehat{W}, N)$ and $\beta = \overline{g}(\widehat{W}, E)$. Note that $\overline{g}(\widehat{W}, \widehat{W}) = ||\widehat{W}||^2_{\overline{g}} = \alpha\beta \neq 0$.

The Lie derivative $L_{\widehat{W}}$ of $\overline{g}$ with respect to the vector field $\widehat{W}$ is given by, for any $X, Y \in \Gamma(TM)$,

$$\begin{aligned}(L_{\widehat{W}}\overline{g})(X, Y) = &- 2\alpha B(X, Y) - \beta \{C(X, Y) + C(Y, X)\} \\ &+ \beta \{\tau(X)\theta(Y) + \tau(Y)\theta(X)\} + X(\beta)\theta(Y) + Y(\beta)\theta(X).\end{aligned} \tag{3.4}$$

Let $\mathcal{A}_{\widehat{W}}$ be a tensor field of type (1, 1) locally defined by the combination of the shape operators A^*_E and A_N, that is,

$$\mathcal{A}_{\widehat{W}} X = \alpha A^*_E X + \beta A_N X, \quad \forall X \in \Gamma(TM). \tag{3.5}$$

Lemma 3.1 *Let* $(M, g, S(TM))$ *be a null hypersurface of an indefinite Kenmostu manifold* $(\overline{M}, \overline{g})$ *with* $\xi \in \Gamma(TM)$. *Then* $\mathcal{A}_{\widehat{W}}X = 0,\ \forall X \in \Gamma(TM)$ *if and only if* $A^*_E X = 0$ *and* $A_N X = 0,\ \forall X \in \Gamma(TM)$.

Proof Suppose that $\mathcal{A}_{\widehat{W}}X = 0,\ \forall X \in \Gamma(TM)$. Then, $\alpha A^*_E X + \beta A_N X = 0$. So, for any $Y \in \Gamma(TM)$, $\alpha g(A^*_E X, Y) + \beta g(A_N X, Y) = 0$, i.e., $\overline{g}(\widehat{W}, C(X, Y)E + B(X, Y)N) = 0$ which implies that $B(X, Y) = 0$ and $C(X, Y) = 0$, since $TM^\perp \oplus N(TM)$ is a non-degenerate distribution of rank 2. By Theorem 2.2 and Proposition 2.7 in [5] (pp. 88 and 89, respectively), A^*_E and A_N vanish identically on M. The converse is obvious. □

Let $(M, g, S(TM))$ be a screen integrable null hypersurface of an indefinite Kenmostu manifold $(\overline{M}, \overline{g})$ with $\xi \in \Gamma(TM)$. Then, (3.4) becomes, for any $X, Y \in \Gamma(TM)$,

$$(L_{\widehat{W}}\overline{g})(X, Y) = -2\alpha B(X, Y) - 2\beta C(X, Y) + \beta \{\tau(X)\theta(Y) + \tau(Y)\theta(X)\} + X(\beta)\theta(Y) + Y(\beta)\theta(X). \tag{3.6}$$

Let M' be a leaf of $S(TM)$. Then, using (2.7) and (2.8), we obtain

$$\overline{\nabla}_X Y = \nabla'_X Y + h'(X, Y), \tag{3.7}$$

for any $X, Y \in \Gamma(TM')$, where ∇' and h' are the Levi-Civita connection and the second fundamental form of M' in $\overline{M}$. Thus,

$$h'(X, Y) = C(X, Y)E + B(X, Y)N, \quad \forall X,\ Y \in \Gamma(TM'). \tag{3.8}$$

Note that, for any $X \in \Gamma(TM')$,

$$\nabla'_X \xi = X - \eta(X)\xi. \tag{3.9}$$

The relation (3.6) becomes

$$(L_{\widehat{W}}\overline{g})(X, Y) = -2\alpha B(X, Y) - 2\beta C(X, Y), \quad \forall X,\ Y \in \Gamma(TM'). \tag{3.10}$$

The action of the Levi-Civita connection $\overline{\nabla}$ (defined in (3.7)) on the normal bundle $TM^\perp \oplus N(TM)$ is given by

$$\overline{\nabla}_X \widehat{W} = -\mathcal{A}_{\widehat{W}}X + \nabla'^{\perp}_X \widehat{W}, \quad \forall X,\ Y \in \Gamma(TM'), \tag{3.11}$$

where $\nabla'^{\perp}_X \widehat{W} = \{X(\alpha) - \alpha\tau(X)\} E + \{X(\beta) + \beta\tau(X)\} N$.

From (3.7), we have

$$\overline{g}(h'(X, Y), \widehat{W}) = \overline{g}(\overline{\nabla}_X Y, \widehat{W}) = -\overline{g}(Y, \overline{\nabla}_X \widehat{W}) = g(\mathcal{A}_{\widehat{W}}X, Y). \tag{3.12}$$

Lemma 3.2 *Let $(M, g, S(TM))$ be a screen integrable null hypersurface of an indefinite Kenmotsu manifold $(\overline{M}, \overline{g})$ with $\xi \in \Gamma(TM)$. Let M' be a leaf of $S(TM)$ immersed in $\overline{M}$ as a non-degenerate submanifold. Then,*

$$(L_{\widehat{W}}\overline{g})(X, Y) = -2\overline{g}(h'(X, Y), \widehat{W}), \quad \forall X,\ Y \in \Gamma(TM').$$

If the leaf M' is totally contact umbilical, then the second fundamental form h' of M' satisfies

$$h'(X, Y) = \{g(X, Y) - \eta(X)\eta(Y)\}\, H' + \eta(X)h'(Y, \xi) + \eta(Y)h'(X, \xi), \tag{3.13}$$

where H' is the mean curvature vector of M'. As $TM^{\perp} \oplus tr(TM)$ is the normal bundle of M', there exist smooth function λ such that $H' = \lambda\widehat{W}$. But, for any $X \in \Gamma(TM')$, $h'(X, \xi) = C(X, \xi)E + B(X, \xi)N = 0$, the relation (3.13) becomes

$$h'(X, Y) = \left\{g'(X, Y) - \eta(X)\eta(Y)\right\} H'. \tag{3.14}$$

That is, M' is η-totally contact geodesic. Therefore, $C(X, Y) = \lambda\alpha\{g'(X, Y) - \eta(X)\eta(Y)\}$ and $B(X, Y) = \lambda\beta\{g'(X, Y) - \eta(X)\eta(Y)\}$, for any $X, Y \in \Gamma(TM')$ which lead to

$$A_N X = \lambda\alpha\{X - \eta(X)\xi\} \ \text{ and } \ A^*_E X = \lambda\beta\{X - \eta(X)\xi\}.$$

Consequently, the $(1, 1)$-tensor field $\mathcal{A}_{\widehat{W}}$ in (3.5) is deduced to

$$\mathcal{A}_{\widehat{W}} X = \lambda(\alpha^2 + \beta^2)\{X - \eta(X)\xi\}. \tag{3.15}$$

Note that totally contact umbilicity is the nearest situation from being totally geodesic. If the leaf M' is totally contact umbilical, then, the relation (3.15) may be rewritten for a given $\widehat{W}$ in $TM^{\perp} \oplus tr(TM)$ as

$$\overline{g}(\overline{\nabla}_X Y, \widehat{W}) = \omega(\widehat{W})\{g(X, Y) - \eta(X)\eta(Y)\}, \tag{3.16}$$

with ω a 1-form on $TM^{\perp} \oplus tr(TM)$ which coincides with the function $\lambda||\widehat{W}||^2_{\overline{g}}$ of normal vector $H' = \lambda\widehat{W}$ in (3.14). Therefore, the map

$$(X, Y) \longmapsto \overline{g}(\overline{\nabla}_X \widehat{W}, Y) = -\omega(\widehat{W})\{g'(X, Y) - \eta(X)\eta(Y)\}, \tag{3.17}$$

is a bilinear symmetric form on TM'.

Lemma 3.3 *Let $(M, g, S(TM))$ be a screen integrable null hypersurface of an indefinite Kenmotsu manifold $(\overline{M}, \overline{g})$ with $\xi \in \Gamma(TM)$. Let M' be a leaf of $S(TM)$ immersed in $\overline{M}$ as a non-degenerate submanifold. Then, M' is totally contact umbilical if and only if $TM^{\perp} \oplus tr(TM)$ is an η-conformal Killing distribution, that is, there exists a*

1-form ω on $TM^{\perp} \oplus tr(TM)$ such that, for any section $\widehat{W}$ of $TM^{\perp} \oplus tr(TM)$ and $X,\ Y \in \Gamma(TM')$,

$$(L_{\widehat{W}}\overline{g})(X, Y) = -2\omega(\widehat{W})\{g'(X, Y) - \eta(X)\eta(Y)\}.$$

Proof The proof follows from the relation $L_{\widehat{W}}\overline{g} = 2\alpha B - 2\beta C$. □

Denote by $\overline{R}$ and R' the curvature tensors of $\overline{\nabla}$ and ∇', respectively. Then, by using (3.7) and (3.11), we obtain,

$$\begin{aligned}\overline{R}(X, Y)Z =& R'(X, Y)Z + \mathcal{A}_{h'(X,Z)}Y - \mathcal{A}_{h'(Y,Z)}X \\ &+ (\nabla'^{\perp}_X h')(Y, Z) - (\nabla'^{\perp}_Y h')(X, Z),\end{aligned} \tag{3.18}$$

for any $X, Y, Z \in \Gamma(TM')$.

Comparing vector fields in M' and $TM^{\perp} \oplus tr(TM)$, we have

$$\overline{R}(X, Y)Z = R'(X, Y)Z + \mathcal{A}_{h'(X,Z)}Y - \mathcal{A}_{h'(Y,Z)}X, \tag{3.19}$$

$$(\nabla'^{\perp}_X h')(Y, Z) = (\nabla'^{\perp}_Y h')(X, Z). \tag{3.20}$$

Theorem 3.4 *Let $(M, g, S(TM))$ be a screen integrable null hypersurface of an indefinite Kenmotsu space form $(\overline{M}(c), \overline{g})$ with $\xi \in \Gamma(TM)$. Let M' be a leaf of $S(TM)$ immersed in $\overline{M}$ as a non-degenerate submanifold. If M' is a totally contact umbilical leaf, then,*

(i) *M' is η-Einstein.*
(ii) *The functions $\alpha\lambda$ and $\beta\lambda$ satisfy, respectively, the following partial differential equations*

$$X(\alpha\lambda) + \alpha\lambda\{\eta(X) - \tau(X)\} = 0, \tag{3.21}$$

$$X(\beta\lambda) + \beta\lambda\{\eta(X) + \tau(X)\} = 0, \tag{3.22}$$

for any $X \in \Gamma(TM')$. Moreover, if $\widehat{W}$ is a unit vector field in $TM^{\perp} \oplus tr(TM)$, the smooth function λ satisfies

$$X(\lambda) + \lambda\eta(X) = 0. \tag{3.23}$$

Proof Since M' is totally contact umbilical, then, by (3.14), $h'(X, Y) = \lambda\{g'(X, Y) - \eta(X)\eta(Y)\}\widehat{W}$, and using (2.3) and (3.15), the relation (3.19) becomes,

$$\begin{aligned}R'(X, Y)Z =& \{1 - \lambda^2(\alpha^2 + \beta^2)\}\{g'(X, Z)Y - g'(Y, Z)X\} \\ &+ \lambda^2(\alpha^2 + \beta^2)\{g'(X, Z)\eta(Y) - g'(Y, Z)\eta(X)\}\xi,\end{aligned}$$

for any $X, Y, X \in \Gamma(TM')$. Using this, we deduce that the Ricci tensor Ric' of M' is given by

$$
\begin{aligned}
Ric'(X, Y) = \{-2(n-1) + (2n-3)\lambda^2(\alpha^2 + \beta^2)\}g'(X, Y) \\
+ \lambda^2(\alpha^2 + \beta^2)\eta(X)\eta(Y),
\end{aligned}
$$

which prove the item (i). The proof of item (ii) follows from (3.20). If $\widehat{W}$ is a unit vector field in $TM^\perp \oplus tr(TM)$, then, $\alpha\beta = 1$ and combining the Eqs. (3.21) and (3.22), we deduce (3.23). □

As an example of a screen integrable null hypersurface of an indefinite Kenmotsu space form $(\overline{M}(c), \overline{g})$ with $\xi \in \Gamma(TM)$, containing a proper totally contact umbilical leaf which is η-Einstein leaf, have a proper totally contact umbilical SAC-lightlike hypersurface of an indefinite Kenmotsu space form. The concept of screen almost conformal (SAC), was firstly introduced by the author in [14], which means that the shape operators A_N and A^*_E of M and its screen distribution $S(TM)$, respectively, are related by $A_N = \varphi A^*_E + \theta \otimes \xi$, where φ is non-vanishing smooth function and α is a differential 1-form on $\mathcal{U}$ in M.

Next, we introduce the new concept of almost Weyl structures on leaves of screen integrable null hypersurfaces. In a (pseudo-) Riemannian setting, manifolds M with conformal structure $[g]$ and torsion-free connection D, such that the parallel translation induces conformal transformations, are called Weyl manifolds.

A conformal change of the metric leads to a metric which is no more compatible with the almost contact structure. This can be corrected by a convenient change of the structure vector field ξ and the 1-form η, which implies rather strong restrictions. Thus, in case there is an integral manifold of an integrable distribution of M, which has an indefinite Kenmotsu structure, we may consider a change of the form

$$
\widetilde{\phi} = \phi, \quad \widetilde{\xi} = e^{\rho}\xi, \quad \widetilde{\eta} = e^{-\rho}\eta, \quad \widetilde{g} = e^{-2\rho}g, \tag{3.24}
$$

where ρ is a smooth function in the considered integral manifold, to preserve the relations given by the Kenmotsu structure. To support this, we have the following.

Proposition 3.5 *Let $(M, g, S(TM))$ be a null hypersurface of an indefinite Kenmotsu manifold $(\overline{M}, \overline{g})$ with $\xi \in \Gamma(TM)$. Then, any integral manifold M_0 of an integrable distribution $\mathcal{D}_0 \oplus_{orth} \langle\xi\rangle$ is totally geodesic in both M and $\overline{M}$ and has an indefinite Kenmotsu structure.*

Proof Suppose that $\mathcal{D}_0 \oplus_{orth} \langle\xi\rangle$ is integrable. Let M_0 be a leaf of $\mathcal{D}_0 \oplus_{orth} \langle\xi\rangle$, then for any $p \in M_0$, we have $T_pM_0 = (\mathcal{D}_0 \oplus_{orth} \langle\xi\rangle)_p$ and $\dim M_0 = 2n-3$. If $X_0 = X'_0 + \eta(X_0)\xi \in \Gamma(TM_0)$, $\phi X_0 = \overline{\phi}RX'_0 = \overline{\phi}X'_0 = \overline{\phi}X_0$, $R : \Gamma(TM) \longrightarrow \Gamma(\mathcal{D})$ being the projection morphism and $\mathcal{D} = TM^\perp \oplus_{orth} \overline{\phi}(TM^\perp) \oplus_{orth} \mathcal{D}_0$. We put $\overset{\circ}{\phi} = \phi|_{\mathcal{D}_0 \oplus_{orth} \langle\xi\rangle}$ and $\overset{\circ}{\eta} = \eta|_{\mathcal{D}_0 \oplus_{orth} \langle\xi\rangle}$, so $\overset{\circ}{\phi}$ defines an $(1, 1)$-type tensor field on M_0 because $\overline{\phi}(\mathcal{D}_0 \oplus_{orth} \langle\xi\rangle) \subset \mathcal{D}_0$. Now we consider $(M_0, \phi_0, \xi, \eta_0, g)$ and check that, this is an indefinite Kenmotsu structure. We know that $\phi^2 X = -X + \eta(X)\xi + u(X)U$, $\forall X \in \Gamma(TM)$ and $u(X) = 0$, $\forall X \in \Gamma(\mathcal{D}_0 \oplus_{orth} \langle\xi\rangle)$, so we deduce $(\overset{\circ}{\phi})^2 X_0 = -X_0 + \eta(X_0)\xi$, for any $X_0 \in \Gamma(TM_0)$. Then $\overset{\circ}{\eta}(\xi) = 1$ and (ϕ_0, ξ, η_0) is an almost contact structure. Now, we prove the compatibility between the (ϕ_0, ξ, η_0)-structure and the metric g on M_0. By $\overset{\circ}{\phi} = \phi|_{\mathcal{D}_0 \oplus_{orth} \langle\xi\rangle}$, we have, for any $X_0, Y_0 \in \Gamma(TM_0)$, $g(\overset{\circ}{\phi} X_0, \overset{\circ}{\phi} Y_0) = g(X_0, Y_0) - \eta(X_0)\eta(Y_0)$. Let $\overset{\circ}{\nabla}$ be a linear connection on the bundle $\mathcal{D}_0 \oplus_{orth} \langle\xi\rangle$. For any $X \in \Gamma(TM)$, $Y_0 \in \Gamma(\mathcal{D}_0 \oplus_{orth} \langle\xi\rangle)$, we have $\nabla_X Y_0 = \overset{\circ}{\nabla}_X Y_0 + \overset{\circ}{h}(X, Y_0)$, where

$$\overset{\circ}{h}: \Gamma(TM) \times \Gamma(\mathcal{D}_0 \oplus_{orth} \langle\xi\rangle) \longrightarrow \Gamma(\{\overline{\phi}(TM^{\perp}) \oplus \overline{\phi}(N(TM))\} \oplus_{orth} TM^{\perp}),$$

is $\mathcal{F}(M)$-bilinear. Let $\mathcal{U} \subset M$ be a coordinate neighborhood as fixed in Sect. 2. Then, for any $X_0, Y_0 \in \Gamma(\mathcal{D}_0 \oplus_{orth} \langle\xi\rangle)$, $\nabla_{X_0} Y_0 = \overset{\circ}{\nabla}_{X_0} Y_0 + C(X_0, \phi Y_0)V + B(X_0, \phi Y_0)U + C(X_0, Y_0)E$, and the local expression of $\overset{\circ}{h}$ is $\overset{\circ}{h}(X_0, Y_0) = C(X_0, \phi Y_0)V + B(X_0, \phi Y_0)U + C(X_0, Y_0)E$. Since $\mathcal{D}_0 \oplus_{orth} \langle\xi\rangle$ is integrable, $\overset{\circ}{h}$ is symmetric, i.e., $C(X_0, \phi Y_0) = C(Y_0, \phi X_0)$, $B(X_0, \phi Y_0) = B(Y_0, \phi X_0)$ and $C(X_0, Y_0) = C(Y_0, X_0)$. The Levi-Civita connection $\overline{\nabla}$ on $\overline{M}$ and the induced one $\overset{\circ}{\nabla}$ are related as $\overline{\nabla}_{X_0} Y_0 = \overset{\circ}{\nabla}_{X_0} Y_0 + B(X_0, Y_0)N + \overset{\circ}{h}(X_0, Y_0)$. It is easy to check that $\overset{\circ}{\nabla}_{X_0} Y_0 \in \Gamma(\mathcal{D}_0 \oplus_{orth} \langle\xi\rangle)$, $\forall X_0, Y_0 \in \Gamma(\mathcal{D}_0 \oplus_{orth} \langle\xi\rangle)$, i.e., $\mathcal{D}_0 \oplus_{orth} \langle\xi\rangle$ defines a totally geodesic foliation. Hence M_0 is a totally geodesic leaf in both M and $\overline{M}$. Moreover, $\overset{\circ}{\nabla}$ is the Levi-Civita connection on M_0. In fact, since $\mathcal{D}_0 \oplus_{orth} \langle\xi\rangle \subset S(TM)$, we get, $(\overset{\circ}{\nabla}_{X_0} g)(Y_0, Z_0) = (\nabla_{X_0} g)(Y_0, Z_0) = 0$, and

$$(\overset{\circ}{\nabla}_{X_0} \overset{\circ}{\phi})Y_0 = g(\overset{\circ}{\phi} X_0, Y_0)\xi - \overset{\circ}{\eta}(Y_0) \overset{\circ}{\phi} X_0,$$

for any $X_0, Y_0, Z_0 \in \Gamma(\mathcal{D}_0 \oplus_{orth} \langle\xi\rangle)$. Thus, $(M_0, \phi_0, \xi, \eta_0, g)$ has an indefinite Kenmotsu structure. □

Definition 3.6 A connection D' on a leaf M' is said to be η-*conformal* if the covariant derivative of g is proportional to $g - \eta \otimes \eta$, that is, there exists a differential 1-form β such that the following

$$D'g' = -\gamma \otimes \{g' - \eta \otimes \eta\} \tag{3.25}$$

holds. Here $g' = g|_{M'}$. If in addition, D' is torsion-free, it is said to be *Weyl connection* [7] in the direction of the distribution $Ker(\eta)$. But on M', such a connection will be called η-*Weyl connection*.

For every metric $g^* \in [g']$, it is natural to consider the compatible almost contact structure of an integral manifold M^* of an integrable distribution of M with $\xi \in TM$ given by (3.24), so that, corresponding to $[g]$, we obtain a conformal class of almost contact metric structures denoted by $[\phi^*, \xi, \eta, g^*]$, where $\phi^{*2} = \overline{\phi}^2_{|_{M^*}} = -\mathbb{I} + \eta \otimes \xi$ and $g^* = g|_{|_{M^*}}$. From this viewpoint, given an η-conformal structure ∇ on $(M, [g])$ with $\xi \in TM$, its remarkable properties are actually the properties invariant for the conformal class $[\phi, \xi, \eta, g]$, where ϕ is defined by $\phi^2 = -\mathbb{I} + \eta \otimes \xi + u \otimes U$. This tells us that (ϕ, ξ, η, g) is not an almost contact metric structure. But, this structure is invariant under the change in (3.24).

Consider on M' conformal structure of the form (3.24). These metrics endow M' with a conformal structure denoting by $\mathcal{C}' = [\phi', \xi, \eta, g']$. Throughout this paper, M' endowed with this conformal structure is denoted as $(M', \mathcal{C}')$.

Definition 3.7 An integral manifold $(M', \mathcal{C}', D')$ is said to be almost Weyl if it is endowed with an η-Weyl connection D' satisfying the relation in (3.25).

Now, assume that the leaf M' is proper totally contact umbilical. The 1-form ω in (3.16) is related to the mean curvature vector H' of M' as $H' = \lambda \widehat{W}$. Hence, the 1-form ω is a section of $(TM^{\perp} \oplus tr(TM))^*$ and then there exists a section $\omega^{\sharp}$ of $T\overline{M}|_{M'}$ such that

$$\overline{g}(\omega^{\sharp}, \widehat{W}) = \omega(\widehat{W}), \ \forall \widehat{W} \in \Gamma(TM^{\perp} \oplus tr(TM)). \tag{3.26}$$

In fact, the section $\omega^{\sharp}$ is the metrical dual vector of ω. We also observe that two sections $\omega^{\sharp}$ differ by exactly one section of TM'. Let β be the differential 1-form on $\overline{M}$, locally defined by

$$\gamma(X) = 2\overline{g}(\omega^{\sharp}, X), \tag{3.27}$$

and we define D^{γ} as

$$\begin{aligned} D^{\gamma}_X Y = \overline{\nabla}_X Y + \frac{1}{2}\gamma(X)\{Y - \eta(Y)\xi\} + \frac{1}{2}\gamma(Y)\{X - \eta(X)\xi\} \\ - \{g'(X, Y) - \eta(X)\eta(Y)\}\omega^{\sharp}, \end{aligned} \tag{3.28}$$

for all $X, Y \in \Gamma(TM')$ and $\overline{\nabla}$ is the Levi-Civita connection on $(\overline{M}, \overline{g})$.

First of all, we have the following:

Lemma 3.8 *Let $(M, g, S(TM))$ be a screen integrable null hypersurface of an indefinite Kenmotsu manifold $(\overline{M}, \overline{g})$ with $\xi \in \Gamma(TM)$. Let M' be a totally contact umbilical leaf of $S(TM)$ immersed in $\overline{M}$ as a non-degenerate submanifold. Then, D^{γ} in (3.28) is a torsion-free connection on M' and for any $X, Y \in \Gamma(TM')$,*

$$D^{\gamma}_X Y \in \Gamma(TM').$$

Proof D^{β} is clearly a torsion-free connection on M. Using (3.16) and for any X, $Y \in \Gamma(TM')$, we have

$$\begin{aligned} \overline{g}(D^{\gamma}_X Y, \widehat{W}) &= \overline{g}(\overline{\nabla}_X Y, \widehat{W}) - \omega(\widehat{W})\{g'(X, Y) - \eta(X)\eta(Y)\} \\ &= \omega(\widehat{W})\{g'(X, Y) - \eta(X)\eta(Y)\} - \omega(\widehat{W})\{g'(X, Y) - \eta(X)\eta(Y)\} \\ &= 0, \end{aligned}$$

which completes the proof. □

Finally, we show that D^γ is η-conformal connection. Using Lemma 3.8 and (3.28), and let X, Y, and Z be tangent vector fields to M'. We have

$$\begin{aligned}(D^\gamma_X g')(Y, Z) &= X(g'(Y, Z) - g'(D^\gamma_X Y, Z) - g'(Y, D^\gamma_X Z)\\ &= (\overline{\nabla}_X \overline{g})(Y, Z) - \frac{1}{2}\gamma(X)\{g'(Y, Z) - \eta(Y)\eta(Z)\}\\ &\quad - \frac{1}{2}\gamma(Y)\{g'(X, Z) - \eta(X)\eta(Z)\} + \frac{1}{2}\gamma(Z)\{g'(X, Y) - \eta(X)\eta(Y)\}\\ &\quad - \frac{1}{2}\gamma(X)\{g'(Y, Z) - \eta(Y)\eta(Z)\} - \frac{1}{2}\gamma(Z)\{g'(X, Y) - \eta(X)\eta(Y)\}\\ &\quad + \frac{1}{2}\gamma(Y)\{g'(X, Z) - \eta(X)\eta(Z)\}.\end{aligned}$$

That is, $(D^\gamma_X g')(Y, Z) = -\gamma(X)\{g'(Y, Z) - \eta(Y)\eta(Z)\}$. This means that D^γ is η-Weyl connection on M'.

Now suppose that there exists an η-Weyl connection D' on (M', g'), i.e., D' is torsion-free and there exists a smooth 1-form γ such that $D'g' = -\gamma \otimes \{g' - \eta \otimes \eta\}$. Then, one has, for any $X, Y \in \Gamma(TM')$,

$$\begin{aligned}D'_X Y = \overline{\nabla}_X Y &+ \frac{1}{2}\gamma(X)\{Y - \eta(Y)\xi\} + \frac{1}{2}\gamma(Y)\{X - \eta(X)\xi\}\\ &- \{g'(X, Y) - \eta(X)\eta(Y)\}\omega^\sharp.\end{aligned} \tag{3.29}$$

From (3.7) and using the fact that the dual vector $\omega^\sharp$ can be written, in general, as

$$\omega^\sharp = P\omega^\sharp + \frac{1}{\alpha\beta}\omega(\widehat{W})\widehat{W}, \quad \alpha\beta \neq 0, \tag{3.30}$$

we have

$$\begin{aligned}D'_X Y = \nabla'_X Y + h'(X, Y) &+ \frac{1}{2}\gamma(X)\{Y - \eta(Y)\xi\} + \frac{1}{2}\gamma(Y)\{X - \eta(X)\xi\}\\ &- \{g'(X, Y) - \eta(X)\eta(Y)\}\left\{P\omega^\sharp + \frac{1}{\alpha\beta}\omega(\widehat{W})\widehat{W}\right\}.\end{aligned} \tag{3.31}$$

Comparing the elements of M' and $TM^\perp \oplus tr(TM)$, one obtains,

$$\begin{aligned}D'_X Y = \nabla'_X Y &+ \frac{1}{2}\gamma(X)\{Y - \eta(Y)\xi\} + \frac{1}{2}\gamma(Y)\{X - \eta(X)\xi\}\\ &- \{g'(X, Y) - \eta(X)\eta(Y)\}P\omega^\sharp,\end{aligned} \tag{3.32}$$

$$h'(X, Y) = \frac{1}{\alpha\beta}\omega(\widehat{W})\{g'(X, Y) - \eta(X)\eta(Y)\}\widehat{W}, \tag{3.33}$$

which implies that M' is totally contact umbilical. Therefore,

Theorem 3.9 *Let $(M, g, S(TM))$ be a screen integrable null hypersurface of an indefinite Kenmotsu manifold $(\overline{M}, \overline{g})$ with $\xi \in \Gamma(TM)$. For M', a leaf of $S(TM)$ immersed in $\overline{M}$ as a non-degenerate submanifold, to be proper totally contact umbilical, it is necessary and sufficient that it admits an η-Weyl connection.*

Let $(M, g, S(TM))$ be a screen integrable null hypersurface of an indefinite Kenmotsu manifold $(\overline{M}, \overline{g})$ with $\xi \in \gamma(TM)$.

Let M' be a totally contact umbilical leaf of $S(TM)$. Then, by Theorem 3.9, there is an η-Weyl connection D' in M' such that, for any $X, Y \in \Gamma(TM')$, we have,

$$D'_X Y = \overline{\nabla}_X Y + \widetilde{\gamma}_X Y, \tag{3.34}$$

where

$$\begin{aligned}\widetilde{\gamma}_X Y = &\frac{1}{2}\gamma(X)\{Y - \eta(Y)\xi\} + \frac{1}{2}\gamma(Y)\{X - \eta(X)\xi\}\\ &- \{g'(X, Y) - \eta(X)\eta(Y)\}\omega^{\sharp}.\end{aligned} \tag{3.35}$$

The Lie derivative of g' with respect to structure vector field ξ is given by

$$\begin{aligned}(L_\xi g')(X, Y) &= \xi(g'(X, Y)) - g'([\xi, X], Y) - g'([\xi, Y], X)\\ &= 2\{g'(X, Y) - \eta(X)\eta(Y)\}.\end{aligned} \tag{3.36}$$

By definition of D' in (3.25), we have, for any $X, Y \in \Gamma(TM')$,

$$-\gamma(\xi)\{g'(X, Y) - \eta(X)\eta(Y)\} = (L_\xi g')(X, Y) + g'(D'_X\xi, Y) + g'(X, D'_Y\xi),$$

which leads to the following identity, for any $X, Y \in \Gamma(TM')$,

$$g'(D'_X\xi, Y) + g'(D'_Y\xi, X) = -(\gamma(\xi) + 2)\{g'(X, Y) - \eta(X)\eta(Y)\}. \tag{3.37}$$

Using (3.37), one has, $\gamma(\xi) = -2$ and $D'_X\xi = 0$, for any $X \in \Gamma(TM')$.

Now, we want to understand better the position of the vector field $\omega^{\sharp}$ in $\overline{M}$. As is mentioned above, $\omega^{\sharp}$ is, in general, a vector field on $\overline{M}$, that is $\omega^{\sharp} = P\omega^{\sharp} + \frac{1}{\alpha\beta}\omega(\widehat{W})\widehat{W}$ with $\alpha\beta \neq 0$. Its location, on either in TM' or $T\overline{M}$, depends on the smooth function $\omega(\widehat{W})$. Therefore,

Lemma 3.10 *Let $(M, g, S(TM))$ be a screen null hypersurface of an indefinite Kenmotsu space form $(\overline{M}(c), \overline{g})$ with $\xi \in \Gamma(TM)$. Let M' be a totally contact umbilical leaf of $S(TM)$ immersed in $\overline{M}$ as a non-degenerate submanifold. Then, the following assertions are equivalent:*

(i) *The dual $\omega^{\sharp}$ of the differential form ω in (3.26) is a vector field on M'.*
(ii) *M' is totally geodesic in M.*
(iii) *$TM^{\perp} \oplus tr(TM)$ is a Killing distribution on M'.*

Moreover,

$$\overline{\nabla}_X\omega^\sharp + \frac{1}{2}\gamma(X)\omega^\sharp \in \Gamma(TM'), \quad \forall\, X \in \Gamma(TM'). \tag{3.38}$$

Proof Since M' is totally contact umbilical, then, the equivalences follow the following. By (3.30), the dual $\omega^\sharp$ of the differential form ω is a vector field on M', i.e., $\overline{g}(\omega^\sharp, \widehat{W}) = 0$ if and only if $\omega(\widehat{W}) = 0$. Now we prove (3.38). The curvature tensor $\overline{R}$ of the $\overline{g}$-compatible connection $\overline{\nabla}$ is given by, for any $X, Y, Z \in \Gamma(TM')$,

$$\overline{R}(X,Y)Z = \overline{\nabla}_X\overline{\nabla}_Y Z - \overline{\nabla}_Y\overline{\nabla}_X Z - \overline{\nabla}_{[X,Y]}Z. \tag{3.39}$$

Using the relation (3.34), one obtains,

$$\begin{aligned}\overline{\nabla}_X\overline{\nabla}_Y Z =& D'_X D'_Y Z - \frac{1}{2}\gamma(X)\{D'_Y Z - \eta(D'_Y Z)\xi\} - \frac{1}{2}\gamma(D'_Y Z)\{X - \eta(X)\xi\}\\ &+ \{g'(X, D'_Y Z) - \eta(X)\eta(D'_Y Z)\}\omega^\sharp - \frac{1}{2}X(\gamma(Y))\{Z - \eta(Z)\xi\}\\ &- \frac{1}{2}\gamma(Y)\{\overline{\nabla}_X Z - X(\eta(Z))\xi - \eta(Z)\overline{\nabla}_X\xi\} - \frac{1}{2}X(\gamma(Z))\{Y - \eta(Y)\xi\}\\ &- \frac{1}{2}\gamma(Z)\{\overline{\nabla}_X Y - X(\eta(Y))\xi - \eta(Y)\overline{\nabla}_X\xi\}\\ &+ \{X(g'(Y,Z)) - X(\eta(Y))\eta(Z) - \eta(Y)X(\eta(Z))\}\omega^\sharp\\ &+ \{g'(Y,Z) - \eta(Y)\eta(Z)\}\overline{\nabla}_X\omega^\sharp,\end{aligned} \tag{3.40}$$

and

$$\begin{aligned}\overline{\nabla}_{[X,Y]}Z =& D'_{[X,Y]}Z - \frac{1}{2}\gamma([X,Y])\{Z - \eta(Z)\xi\} - \frac{1}{2}\gamma(Z)\{[X,Y] - \eta([X,Y])\xi\}\\ &+ \{g'([X,Y],Z) - \eta([X,Y])\eta(Z)\}\omega^\sharp.\end{aligned} \tag{3.41}$$

Putting the pieces (3.40) and (3.41) together into (3.39), we get,

$$\begin{aligned}\overline{R}(X,Y)Z =& R^{D'}(X,Y)Z - \frac{1}{4}\{\gamma(Z) + 2\eta(Z)\}\{\gamma(X)(Y - \eta(Y)\xi)\\ &- \gamma(Y)(X - \eta(X)\xi)\} - d\gamma(X,Y)\{Z - \eta(Z)\xi\}\\ &+ \frac{1}{2}\{(D'_X g')(Y,Z) - (D'_Y g')(X,Z)\}\omega^\sharp - \frac{1}{2}(D'_Y\gamma)Z\{X - \eta(X)\xi\}\\ &- \frac{1}{2}(D'_X\gamma)Z\{Y - \eta(Y)\xi\} + \frac{1}{2}\gamma(Z)\{\eta(Y)X - \eta(X)Y\}\\ &+ \{g'(Y,Z) - \eta(Y)\eta(Z)\}\overline{\nabla}_X\omega^\sharp - \{g'(X,Z) - \eta(X)\eta(Z)\}\overline{\nabla}_Y\omega^\sharp.\end{aligned} \tag{3.42}$$

Since the vector fields X, Y, and Z are all taken in M', then, $\overline{g}$-doting the relation (3.42) with $\widehat{W}$ and using (2.3), one has,

$$\{g'(X, Z) - \eta(X)\eta(Z)\}\{\overline{g}(\omega^\sharp, Y)\overline{g}(\omega^\sharp, \widehat{W}) + \overline{g}(\overline{\nabla}_Y \omega^\sharp, \widehat{W})\}$$
$$= \{g'(Y, Z) - \eta(Y)\eta(Z)\}\{\overline{g}(\omega^\sharp, X)\overline{g}(\omega^\sharp, \widehat{W}) + \overline{g}(\overline{\nabla}_X \omega^\sharp, \widehat{W})\},$$

which implies

$$\overline{g}\left(\overline{\nabla}_Y \omega^\sharp + \frac{1}{2}\gamma(Y)\omega^\sharp, \widehat{W}\right)\widehat{P}X = \overline{g}\left(\overline{\nabla}_X \omega^\sharp + \frac{1}{2}\gamma(X)\omega^\sharp, \widehat{W}\right)\widehat{P}Y, \tag{3.43}$$

where $\widehat{P}$ is a projection defined by $\widehat{P} = P - \eta \otimes \xi$.

Now suppose that there exists a vector field X_0 on some neighborhood of M' such that $\overline{g}(\overline{\nabla}_{X_0}\omega^\sharp + \frac{1}{2}\gamma(X_0)\omega^\sharp, \widehat{W}) \neq 0$ at some point p in the neighborhood. Then, from (3.43) it follows that all vectors of the fiber $(TM' - \langle\xi\rangle))_p$ are collinear with $(\widehat{P}X_0)_p$. This contradicts $\dim(TM' - \langle\xi\rangle)_p > 1$, since $((TM') - \langle\xi\rangle)_p$ is a non-degenerate distribution of rank $2n - 2$, $n \geq 2$. Therefore, $\overline{g}(\overline{\nabla}_X \omega^\sharp + \frac{1}{2}\gamma(X)\omega^\sharp, \widehat{W}) = 0$. This completes the assertion (3.38). □

Let X and Y be vector fields on M' satisfying $[X, Y] = 0$ at p. Then, we have for any vector fields Z, W:

$$X(g(Z, W)) = -\gamma(X)\{g'(Z, W) - \eta(Z)\eta(W)\} + g'(D'_X Z, W) + g'(Z, D'_X W),$$

which implies

$$\begin{aligned} Y(X(g(Z, W))) = &- Y(\gamma(X))\{g'(Z, W) - \eta(Z)\eta(W)\} \\ &+ \gamma(X)\gamma(Y)\{g'(Z, W) - \eta(Z)\eta(W)\} \\ &- \gamma(X)g'(D'_Y Z, W) - \gamma(X)g'(Z, D'_Y W) \\ &+ \gamma(X)\eta(W)Y(\eta(Z)) + \gamma(X)\eta(Z)Y(\eta(W)) \\ &- \gamma(Y)\{g'(D'_X Z, W) - \eta(D'_X Z)\eta(W)\} \\ &+ g'(D'_Y D'_X Z, W) + g'(D'_X Z, D'_Y W) \\ &- \gamma(Y)\{g'(Z, D'_X W) - \eta(Z)\eta(D'_X W)\} \\ &+ g'(D'_Y Z, D'_X W) + g'(Z, D'_Y D'_X W), \end{aligned} \tag{3.44}$$

and obtain the formula for $X(Y(g(Z, W)))$ by exchanging X and Y in (3.44). Subtracting $Y(X(g(Z, W)))$ from $X(Y(g(Z, W)))$ yields

$$d\gamma(X, Y)\{g'(Z, W) - \eta(Z)\eta(W)\} = g'(R^{D'}(X, Y)Z, W) + g'(Z, R^{D'}(X, Y)W).$$

So,

$$(n - 1)d\gamma(X, Y) = \sum_{i=1}^{2n-1} g'(R^{D'}(X, Y)E_i, E_i), \tag{3.45}$$

where $\{E_i\}$ is an orthonormal base at p with respect to g'. By the aid of the first Bianchi identity (notice that the connection D' is torsion-free so that the Bianchi identity holds), we have

$$R^{D'}(X, Y)E_i = -R^{D'}(Y, E_i)X - R^{D'}(E_i, X)Y, \tag{3.46}$$

and therefore,

$$(n-1)d\gamma(X, Y) = Ric^{D'}(Y, X) - Ric^{D'}(X, Y), \tag{3.47}$$

where $Ric^{D'}(X, Y) = \sum_{i=1}^{2n-1} g'(R^{D'}(E_i, X)Y, E_i)$ which is in line with the convention (3.39).

Theorem 3.11 *Let $(M, g, S(TM))$ be a screen integrable null hypersurface of an indefinite Kenmotsu manifold $(\overline{M}, \overline{g})$ with $\xi \in \Gamma(TM)$. Let $(M', \mathcal{C}', D')$ be an almost Weyl leaf of $S(TM)$ immersed in $\overline{M}$ as a non-degenerate submanifold. Then, for $g' \in \mathcal{C}'$, the following assertions are equivalent:*

(i) $d\gamma_{g'} = 0$,
(ii) $Ric^{D'}$ *is symmetric,*
(iii) *each point of M' has a neighborhood on which D' is a torsion-free g'-compatible linear connection for a certain metric g' in $\mathcal{C}'$.*

The Theorem 3.11 shows that, under some conditions, the local triviality implies that a locally symmetric integral manifolds of integrable screen distribution $S(TM)$ of a null hypersurface $(M, g, S(TM))$ turns out to be locally semi-Riemannian.

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