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Finite Nilpotent Groups Whose Cyclic Subgroups are TI-Subgroups

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Abstract A subgroup H of a group G is called a TI-subgroup if $H^g \cap H = 1$ or H for all $g \in G$; and H is called quasi TI if $\mathcal{C}_G(x) \leq \mathcal{N}_G(H)$ for all non-trivial elements $x \in H$. A group G is called (quasi CTI-group) CTI-group if every cyclic subgroup of G is a (quasi TI-subgroup) TI-subgroup. It is clear that TI subgroups are quasi TI. We first show that finite nilpotent quasi CTI-groups are CTI. In this paper, we classify all finite nilpotent CTI-groups.

Keywords TI-group · CTI-groups · p-Group

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1 Introduction and Results

A subgroup H of a finite group G is called a TI-subgroup, if $H \cap H^g = 1$ or H for all $g \in G$. A group G is called a TI-group if all of whose subgroups are TI-subgroups. A group G is called an ATI-group if all of whose abelian subgroups are TI-subgroups.

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Similarly, a group G is called a CTI-group if any cyclic subgroup of G is a TI-subgroup or equivalently $\langle x\rangle^g \cap \langle x\rangle = 1$ or $\langle x\rangle$ for all elements $x, g \in G$.

A subgroup H of a finite group G is called a quasi TI-subgroup or a QTI-subgroup if $\mathcal{C}_G(x) \leq \mathcal{N}_G(H)$ for any $1 \neq x \in H$. A group G is called a QTI-group if all of whose subgroups are QTI-subgroups. Similarly, a group G is called a AQTI-group (a quasi CTI-group, respectively) if all of whose abelian subgroups (cyclic subgroups, respectively) are QTI-subgroups.

In [1], Walls classified finite groups all of whose subgroups are TI-subgroups. In [2] and [3], Guo, Li and Flavell classified finite groups whose abelian subgroups are TI-subgroups. Also in [4], Qian and Tang classified finite groups all of whose abelian subgroups are QTI-subgroups.

A subgroup H of G is called n-embedded in G if for every $1 \neq K \leqslant H$ we have $\mathcal{N}_G(K) \leqslant \mathcal{N}_G(H)$, and a group G is called n-group if every subgroup H of G is n-embedded in G. If G is nilpotent, then any quasi CTI-subgroup of G is n-embedded in G and quasi CTI-group is n-group. The n-groups have been classified by Kazarin in [5,6], but those Russian papers are not accessible. In [7] one can find English translation of [5] but without any proofs. So we give here a proof of this particular case independently with full details.

Here, we show that a finite nilpotent group is CTI if and only if it is quasi CTI. The structure of non-nilpotent finite CTI-groups are classified in [8]. In this paper, we complete the classification of finite CTI-groups by classifying nilpotent ones.

Our main results are the following.

Theorem 1.1 *Every finite nilpotent group is quasi CTI if and only if it is CTI.*

Theorem 1.2 *Let G be a finite nilpotent group. Then G is a CTI-group if and only if one of the following holds*:

(1) *G is Dedekindian, i.e., all subgroups of G are normal in G;*
(2) *G is a p-group of exponent p for some prime p;*
(3) *G is a p-group for some prime p such that $G' = \Omega_1(G^p)$ is of order p and $\Phi(G) = G^p$ is a central cyclic subgroup of G; and*
(4) *G is a 2-group such that $G = A\langle x\rangle$, where A is an abelian subgroup of G and x is an involution in $G \setminus A$ such that $a^x = a^{-1}$.*

Our notations are as follows. Let G be a group, $x \in G$ and $H \leq G$. Then, $\mathcal{C}_G(x)$ denotes the centralizer of x in G and $\mathcal{N}_G(H)$ denotes the normalizer of H in G. The derived subgroup, the center, and the Frattini subgroup of G are denoted by G', $Z(G)$, and $\Phi(G)$, respectively; and if G is a p-group, G^p denotes $\langle x^p \mid x \in G\rangle$ and $\Omega_1(G)$ denotes $\langle x \in G \mid x^p = 1\rangle$. The cyclic group of order n is denoted by $\mathbb{Z}_n$ and Q_8 denotes the quaternion group of order 8. The exponent of a finite group G is denoted by $\exp(G)$, and the order of an element $x \in G$ is denoted by $|x|$. For two elements x and y in a group, $[x, y]$ denotes the commutator $x^{-1}y^{-1}xy$.

2 Quasi CTI-Groups

Clearly a TI-subgroup is a QTI-subgroup. So TI-, ATI-, and CTI-groups are QTI-, AQTI-, and quasi CTI-groups, respectively. In [4] Qian and Tang proved that:

Theorem 2.1 ([4], Theorem 2.1). *For a finite p-group G, the following statements are equivalent:*

(i) *G is a TI-group;*
(ii) *G is an ATI-group; and*
(iii) *G is an $AQTI$-group.*

In Theorem 1.1, we show that a finite p-group is CTI if and only if it is quasi CTI. We need the following proposition in the proof of Theorem 1.1.

Proposition 2.2 *Let G be a finite quasi CTI-group and H is a non-normal cyclic subgroup of G.*

(i) *$H \cap Z(G) = 1$ and in particular no non-trivial subgroup of H is normal in G.*
(ii) *If $x \in H$ is of prime order, then $\mathcal{C}_G(x) = \mathcal{N}_G(H)$.*
(iii) *If $|Z(G)|$ divides by two distinct primes, then G is Dedekindian.*

Proof (i) If $1 \neq x \in H \cap Z(G)$, then $G = \mathcal{C}_G(x) \leqslant \mathcal{N}_G(H)$, a contradiction.
(ii) Since $\langle x \rangle \trianglelefteq \mathcal{N}_G(H)$, $x \in Z(\mathcal{N}_G(H))$ and so $\mathcal{C}_G(x) = \mathcal{N}_G(H)$.
(iii) Let $h \in G$ be of prime order. By assumption, $Z(G)$ contains an element x of order coprime to $|h|$. By (i) $\langle xh \rangle \trianglelefteq G$. Since $\langle h \rangle \leqslant \langle xh \rangle$, $\langle h \rangle \trianglelefteq G$. Therefore, any subgroup of prime order of G is normal and so G is Dedekindian by (i). □

We can now prove Theorem 1.1. Let us restate the statement of Theorem 1.1.

Theorem 1.1 *Every finite nilpotent group is quasi CTI if and only if it is CTI.*

Proof Suppose that G is a finite nilpotent non-Dedekindian quasi CTI-group. By Proposition 2.2 (iii), G must be a non-abelian p-group for some prime p. Suppose H is a cyclic subgroup of G and $x \in H$ is of order p. If $H \cap H^g \neq 1$ for some element $g \in G$, then $\langle x \rangle = \langle x \rangle^g$. Hence $g \in \mathcal{C}_G(x) = \mathcal{N}_G(H)$ by Proposition 2.2 (ii). Therefore $H = H^g$. This completes the proof. □

3 CTI-p-Groups with Center of Exponent > p

The following lemma shows that to classify finite nilpotent CTI-groups it is enough to classify the finite CTI-p-groups. Recall that a group is called Dedekindian (or Hamiltonian) if all of its subgroups are normal.

Lemma 3.1 ([8], Corollary 1.3). *Let G be a CTI-group with non-trivial center.*

(i) *Assume that the order of $1 \neq g \in G$ is coprime to the order of an element of $Z(G)$. Then $\langle g \rangle \trianglelefteq G$.*
(ii) *If two distinct primes p and q divide the order of $Z(G)$, then G is a Dedekindian group.*

The preceding lemma implies that a finite non-Dedekindian nilpotent CTI-group is necessarily a non-Dedekindian p-group.

It is easy to see that a finite non-Dedekindian CTI-p-group contains a non-normal (non-central) subgroup of order p. This justifies the hypothesis of the existence of a non-central element of order p in the following lemma:

Lemma 3.2 *Let G be a non-Dedekindian finite CTI-p-group such that the exponent of $Z(G)$ is at least p^2 and x is a non-central element of order p of G.*

(i) *For any $g \in G \backslash \mathcal{C}_G(x)$, $[x, g]$ is an element of $Z(G)^p$ of order p.*
(ii) *$\mathcal{C}_G(x)$ is a maximal subgroup of G.*
(iii) *$Z(G) \cong \mathbb{Z}_{p^e} \times E$ where $e \geq 2$ and E is an elementary abelian subgroup.*
(iv) *If for any non-central element y of order p, $\mathcal{C}_G(x) = \mathcal{C}_G(y)$, then $\mathcal{C}_G(x)$ is abelian.*

Proof (i) Let $y \in Z(G)$ be of order at least p^2. Since $(yx)^p$ is central, $\langle yx \rangle \trianglelefteq G$. Hence for any $g \in G$, $yx^g \in \langle yx \rangle$, so $yx^g = y^i x^i$ for some i where $(i, p) = 1$, then $[x, g] = x^{i-1} y^{1-i}$. Therefore $[x, g]^p = [x^p, g] = 1$. Accordingly $y^{p(i-1)} = 1$, thus p divides $i - 1$ and so $x^g = y^{i-1}x$. This implies that $[x, g] \in Z(G)^p$ and $\langle y, x \rangle \trianglelefteq G$. Let $Z \leqslant \langle y \rangle$ be of prime order. Then

$$\langle x^g, x \rangle = \langle Z, x \rangle = \Omega_1(\langle y, x \rangle) \trianglelefteq G.$$

(ii) By (i), $\langle x^g, x \rangle$ contains any conjugate of x, so $|G : \mathcal{N}_G(\langle x \rangle)| = p$, now (ii) follows from the equality $\mathcal{C}_G(x) = \mathcal{N}_G(\langle x \rangle)$.
(iii) Let $\langle y_1 \rangle$ be a central subgroup of order at least p^2 such that $\langle y \rangle \cap \langle y_1 \rangle = 1$. By (i), we have $\langle Z, x \rangle = \langle x^g, x \rangle = \langle Z_1, x \rangle$, where $Z_1 \leqslant \langle y_1 \rangle$ is of order p. Then we have $Z = Z_1$, which is a contradiction.
(iv) The hypothesis implies that $\mathcal{C}_G(x)$ is Dedekindian, as it is a CTI-group and any subgroup of prime order is normal in $\mathcal{C}_G(x)$. Since $\mathcal{C}_G(x)$ contains $Z(G)$, so $\exp(Z(\mathcal{C}_G(x)))$ is at least p^2. Hence $\mathcal{C}_G(x)$ is an abelian group. □

Theorem 3.3 *Let G be a finite non-abelian p-group such that* $\exp(Z(G)) > p$. *If G is a CTI-group then every cyclic subgroup of order at least p^2 is normal in G.*

Proof Suppose that G is a non-abelian CTI-p-group having a cyclic subgroup $\langle a \rangle$ which is not normal in G and $|a| = p^2$. Thus $a^p \notin Z(G)$ and $\langle a \rangle \cap Z(G) = 1$.

Step 1 We show that $\mathcal{C}_G(a)$ is abelian and $\Omega_1(G) \leqslant \mathcal{C}_G(a)$.

Since $a \in Z(\mathcal{C}_G(a))$ and $\langle a \rangle \cap Z(G) = 1$, it follows from Lemma 3.2 (iii) that $\mathcal{C}_G(a)$ is abelian. Now for any non-central element y of order p, we have $a^p \in \mathcal{C}_G(y)$, since $\mathcal{C}_G(y)$ is a maximal subgroup of G by Lemma 3.2 (ii). Hence $y \in \mathcal{C}_G(a^p)$. Since G is CTI, $\langle a \rangle \trianglelefteq \mathcal{C}_G(a^p)$ and so $[a, y] \in \langle a \rangle$ because $y \in C_G(a^p)$. Also $[a, y] \in Z(G)$ by Lemma 3.2 (i). Hence $[a, y] = 1$ (otherwise $\langle a \rangle \trianglelefteq G$) and so $\mathcal{C}_G(a)$ contains any element of order p of G.

Step 2 $M = \mathcal{C}_G(a^p)$ is abelian.

We show that for any non-central element y of order p, $\mathcal{C}_G(y) = M$. Suppose, for a contradiction, that $y \in G \backslash Z(G)$ is of order p such that $\mathcal{C}_G(y) \neq M$. By Step 1, $C = \mathcal{C}_G(a)$ is abelian and $y \in C$. Thus $C \leqslant \mathcal{C}_G(y)$, so $C \neq M$ and by Normalizer–Centralizer theorem C is maximal in M. Now we have $C = M \cap \mathcal{C}_G(y)$. Let $g \in M \backslash C$, then $|g| \geq p^2$ as by the previous part $\Omega_1(G) \leq C$. Since $M = \langle g \rangle C$, we have $\langle g \rangle \trianglelefteq M$ (because $g^p \in Z(M)$). Therefore

$$1 \neq [a, g] \in \langle g \rangle \cap \langle a \rangle = \langle a^p \rangle,$$

and so $\langle g\rangle \ntrianglelefteq G$, because $\langle x\rangle \ntrianglelefteq G$. Now since $g \notin \mathcal{C}_G(y)$ and $\langle g\rangle \trianglelefteq M$, then $1 \neq [g, y] \in \langle g\rangle$ is a central element of prime order, a contradiction. Hence, $M = \mathcal{C}_G(y)$ for any non-central element y of prime order and it follows from Lemma 3.2 (iv) that M is abelian.

Let $x \in G\backslash M$. Then $|x| \geq p^2$. Since $G = M\langle x\rangle$ and M is abelian, $\langle x^p\rangle \leqslant M \cap \langle x\rangle \leqslant Z(G)$. Therefore $\langle x\rangle \trianglelefteq G$. Thus $[a, x] \in M \cap \langle x\rangle \leq Z(G)$. Hence $[a^p, x] = [a, x^p] = 1$ and so $a^p \in Z(G)$, a contradiction. □

In Theorem 4.3 (see Sect. 4), we shall prove that all finite CTI-p-groups G satisfy the property mentioned in the conclusion of Theorem 3.3, i.e., every cyclic subgroup of order at least p^2 is normal in G. In the proof of Theorem 4.3 we use Theorem 3.3.

Theorem 3.4 *Let G be a finite non-Dedekindian p-group such that* $\exp(Z(G)) > p$. *If G is a CTI-group then the followings hold:*

(i) *G' is of order p and $G' = \langle x\rangle$ for all $x \in G$ such that $|x| \geq p^2$;*
(ii) *G^p is a cyclic central subgroup of G and $G' = \Omega_1(G^p)$. In particular, $\Phi(G) = G^p$ is a cyclic central subgroup of G.*

Proof (i) Let H be a non-normal subgroup of G and $h \in H$ such that $h^g \notin H$ for some $g \in G$. Let $p^e = \exp(Z(G))$. Since $\langle h\rangle \ntrianglelefteq G$, Theorem 3.3 implies that $|h| = p$ and it follows from Lemma 3.2 (iii) that $[g, h] \in Z$, where $Z = Z(G)^{p^{e-1}}$. Hence $\langle hZ\rangle \trianglelefteq G/Z$. Therefore G/Z is Dedekindian. If $p > 2$ or $p = 2$ and $\exp(Z(G)) > 4$, then G/Z is abelian and so $G' = Z$ is of order p.

Now let $p = 2$ and $\exp(Z(G)) = 4$ and assume that $G/Z \cong Q_8 \times E$ for some elementary abelian 2-group E. It follows that there exists a normal subgroup K of G such that $K/Z \cong Q_8$. By [9, 3.2.10], $Z \lneqq Z(K)$. Thus $K/Z(K) \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ which implies that $|K'| = 2$. Hence $K' \leqslant Z(G)$ since $K \trianglelefteq G$. It follows that $K' \cap Z = 1$. Now we have $K \cong Q_8 \times Z$ and so G has a subgroup H isomorphic to $\mathbb{Z}_4 \times Q_8$ which is not CTI, since $\Omega_1(H) \leqslant Z(H)$, a contradiction. Therefore, in this case again G/Z is abelian and $|G'| = 2$.

Since G is not Dedekindian, it follows from Lemma 3.2 (i) that $G' = \Omega_1(Z(G)^p)$. Let x be an element of G of order greater than p. By Theorem 3.3, $\langle x\rangle \trianglelefteq G$. If $x \notin Z(G)$, then $[x, g] \neq 1$ for some $g \in G$. Since $|G'| = p$, $G' = \langle [x, g]\rangle = \langle x^{\frac{|x|}{p}}\rangle$ as $\langle x\rangle \trianglelefteq G$. Now assume that $x \in Z(G)$. Then it follows from Lemma 3.2 (iii) that $Z(G)^p \cap \langle x\rangle \neq 1$ and so $G' = \langle x^{\frac{|x|}{p}}\rangle$.

(ii) Note that $G^p = \langle a^p \ : \ |a| \geq p^2\rangle$. Since $G^p \leq Z(G)$, G^p is abelian and so

$$\Omega_1(G^p) = \langle a^{\frac{|a|}{p}} \ : \ |a| \geq p^2\rangle.$$

Now by part (i), $G' = \langle a^{\frac{|a|}{p}}\rangle$ for all $a \in G$ such that $|a| \geq p^2$. Therefore $G' = \Omega_1(G^p)$. □

4 CTI-p-Groups with Center of Exponent = p

In this section, we classify CTI-p-groups whose centers are elementary abelian.

The following lemma holds in any CTI-group.

Lemma 4.1 *Let G be a finite CTI-group and g be any element of G. Then $\langle g^x, g\rangle$ is abelian for all $x \in \mathcal{N}_G(\mathcal{N}_G(\langle g\rangle))$.*

Proof Since $x \in \mathcal{N}_G(\mathcal{N}_G(\langle g\rangle))$, $\langle g\rangle^x \leq \mathcal{N}_G(\langle g\rangle)$ and $\langle g\rangle^{x^{-1}} \leq \mathcal{N}_G(\langle g\rangle)$. It follows that $\langle g\rangle^x$ and $\langle g\rangle$ are normal subgroups of $\langle g, g^x\rangle$. Thus $\langle g, g^x\rangle' \leq \langle g\rangle^x \cap \langle g\rangle$. Suppose, if possible, that $\langle g, g^x\rangle' \neq 1$. Since G is a CTI-group, $\langle g\rangle^x = \langle g\rangle$ and so $[g^x, g] = 1$. This completes the proof. □

The following Lemma is a consequence of Theorem 3.3.

Lemma 4.2 *Let G be a finite non-abelian CTI-p-group and let H be any normal non-abelian subgroup of G such that $Z(H)$ is of exponent at least p^2. Then any cyclic subgroup of order at least p^2 of H is normal in G.*

Proof By Theorem 3.4, $H' \leq \langle x\rangle$ for all $x \in H$ of order at least p^2. Now Theorem 3.4 implies that $|H'| = p$ and since H is normal in G, $H' \leq Z(G)$. It follows that $\langle x\rangle$ is normal in G. □

As we promised in Sect. 2, we now prove that the conclusion of Theorem 3.4 holds for all finite CTI-p-groups.

Theorem 4.3 *Let G be a finite CTI-p-group and g be any element of order p^2 of G. Then $\langle g\rangle$ is normal in G.*

Proof Suppose, for a contradiction, that $\langle g\rangle$ is not normal in G. Thus $\mathcal{N}_G(\langle g\rangle) \neq G$ and so $\mathcal{N}_G(\mathcal{N}_G(\langle g\rangle)) \setminus \mathcal{N}_G(\langle g\rangle) \neq \varnothing$ since G is a nilpotent group. Let $x \in \mathcal{N}_G(\mathcal{N}_G(\langle g\rangle)) \setminus \mathcal{N}_G(\langle g\rangle)$. Let $H = \langle g, x\rangle$. Note that $\langle g\rangle$ is not also normal in H.

(i) It follows from Lemma 4.1 that the normal closure $\langle g\rangle^H$ is abelian.
(ii) The centralizer $\mathcal{C}_H(g)$ is normal in H, since $H' \leq \langle g\rangle^H \leq \mathcal{C}_H(g)$. By Lemma 4.2, the centralizer $\mathcal{C}_H(g)$ is abelian, otherwise $\langle g\rangle$ is normal.
(iii) Let k be the largest positive integer such that $|[g,_k x]| = p^2$ and $|[g,_{k+1} x]| \leq p$. Note that such k exists; for G is nilpotent and so $[g,_c x] = 1$, where c is the nilpotency class of H and $|[g, x]| = p^2$, otherwise $[g, x]^p = 1$, implies then $(g^{-1}g^x)^p = g^{-p}(g^p)^x = 1$ and so $g^p = (g^p)^x$. It follows that $\langle g\rangle \cap \langle g\rangle^x \neq 1$ and so $\langle g\rangle = \langle g\rangle^x$, a contradiction (note that $[g, x]^{p^2} = [g^{p^2}, x] = 1$). Since $[g,_{k+1} x]^p = 1$, $[g,_k x]^p = ([g,_k x]^p)^x$ and so $[g,_k x]^x = [g,_k x]^i$ for some integer i such that $1 \leq i \leq p^2$ and $\gcd(i, p) = 1$. Therefore, $\langle [g,_k x]\rangle \trianglelefteq H$ and so $\mathcal{C}_G([g,_k x])$ is a maximal subgroup of H.

Suppose, for a contradiction, that $\mathcal{C}_H([g,_k x])$ is not abelian. Since $\langle g\rangle^H \leq \mathcal{C}_H([g,_k x])$, $H' \leq \mathcal{C}_H([g,_k x])$ and so $\mathcal{C}_H([g,_k x])$ is a normal non-abelian subgroup of H such that the exponent of its center is greater than p (as $[g,_k x] \in Z(\mathcal{C}_H([g,_k x]))$). Now Lemma 4.2 implies that $\langle g\rangle$ is normal in H, a contradiction. Hence $\mathcal{C}_H([g,_k x])$ is abelian. By (ii), $\mathcal{C}_H(g)$ is abelian and so $\mathcal{C}_H(g) = \mathcal{C}_H([g,_k x])$, because $[g, [g,_k x]] = 1$. Hence $\mathcal{C}_H(g)$ is a maximal subgroup of H. Therefore, $x^p \in \mathcal{C}_H(g)$ and so $[g, x^p] = 1$. Thus

$$1 = [g,_{k-1} x, x^p] = [g,_k x]^{x^{p-1}+\cdots+x+1} = [g,_k x]^{i^{p-1}+\cdots+i+1}.$$

If $i = 1$, then it follows that $[g,_k x]^p = 1$, a contradiction. Thus $i > 1$, and so

$$i^{p-1} + \cdots + 1 = \frac{i^p - 1}{i - 1}.$$

Since $[g,_k x]^p = ([g,_k x]^p)^x$, $i = pk + 1$ for some integer. Therefore, $\frac{i^p-1}{i-1} = p(1 + p\ell)$ for some integer ℓ and so $[g,_k x]^p = 1$, a contradiction. This last contradiction completes the proof. □

Lemma 4.4 *Let G be a finite CTI-p-group and x and y be elements of G of orders at least p^2. Then either $\langle x\rangle \cap \langle y\rangle \neq 1$ or $\mathcal{C}_G(x) = \mathcal{C}_G(y)$ is abelian.*

Proof Suppose that $\langle x\rangle \cap \langle y\rangle = 1$ and let $H = \mathcal{C}_G(x)$. We show that H is abelian and $H = \mathcal{C}_G(y)$. By Theorem 4.3, $\langle x\rangle$ and $\langle y\rangle$ are normal subgroups of G and so $y \in H$. Suppose, for a contradiction, that H is not abelian. Since $x \in Z(H)$, it follows from Theorem 3.4 (ii) that $[g, y] \in \langle x^p\rangle$ for all $g \in H$. Since $\langle y\rangle \lhd G$, $[g, y] \in \langle x^p\rangle \cap \langle y\rangle$ and so $y \in Z(H)$. This means that $\langle x\rangle \times \langle y\rangle \leq Z(H)$ contrary to Lemma 3.2. Hence H is abelian and by the symmetry between x and y, $\mathcal{C}_G(y)$ is also abelian. Since x commutes with y and both $\mathcal{C}_G(x)$ and $\mathcal{C}_G(x)$ are abelian, it follows that $\mathcal{C}_G(x) = \mathcal{C}_G(y)$. This completes the proof. □

Theorem 4.5 *Let G be a non-abelian CTI-p-group such that* $\exp(Z(G)) = p$ *and* $\exp(G) > p$. *If $p > 2$, then* $\exp(G) = p^2$ *and $G' = G^p$ is of order p. In particular, $\Phi(G) = G' = G^p$ is a central subgroup of G of order p.*

Proof We first prove that $\exp(G) = p^2$. Suppose, for a contradiction, that $g \in G$ is of order p^3. By Theorem 4.3, $\langle g\rangle \trianglelefteq G$ and so $\langle g^p\rangle \trianglelefteq G$. Since $\exp(Z(G)) = p$, $g^p \notin Z(G)$ and so $\mathcal{C}_G(g^p)$ is a maximal subgroup of G. Assume that x is an element of G. Suppose that $|x| = p$. Since $p > 2$, $\langle g\rangle \trianglelefteq G$, and $|g| = p^3$, $g^x = g^{1+kp^2}$ for some $k \in \{1, \ldots, p\}$. It follows that $[g^p, x] = 1$. Now assume that $|x| \geq p^2$. By Theorem 4.3 $\langle x\rangle \trianglelefteq G$ and so $[g, x] \in \langle g\rangle \cap \langle x\rangle$. Since $\langle g\rangle \cap \langle x\rangle \in \{\langle g^p\rangle, \langle x\rangle, \langle g^{p^2}\rangle, 1\}$. It follows that $|[g, x]| \leq p$ and so $[g^p, x] = [g, x]^p = 1$. Therefore, $g^p \in Z(G)$ which is a contradiction as $\exp(Z(G)) = p$. Hence $\exp(G) = p^2$.

Now let $x, y \in G$ be of the same order p^2. We show that $\langle x^p\rangle = \langle y^p\rangle$. The latter is equivalent to say $\langle x\rangle \cap \langle y\rangle \neq 1$. It follows from Lemma 4.4 that $\mathcal{C}_G(x) = \mathcal{C}_G(y)$ is abelian. We can find an h element of order p^2 in $G \setminus \mathcal{C}_G(x)$: for if $g \notin \mathcal{C}_G(x)$ and $|g| = p$, then $|xg| = p^2$ and $xg \notin \mathcal{C}_G(x)$. Since $1 \neq [x, h] \in \langle x\rangle \cap \langle h\rangle$, $\langle x^p\rangle = \langle g^p\rangle$, similarly $\langle y^p\rangle = \langle g^p\rangle$. Therefore $\langle x^p\rangle = \langle y^p\rangle$.

Hence G^p is cyclic of order p, since $\exp(G) = p^2$.

Now we show that $G' = G^p$. We show that $[x, y] \in G^p$ which follows that $G' = G^p$. Suppose a is an element of G of order p^2. Since $\mathcal{C}_G(a)$ is maximal, $G = \mathcal{C}_G(a)\langle b\rangle$ for some $b \in G$. It follows that $G' = \langle [m, b], [m, m'] \mid m, m' \in \mathcal{C}_G(a)\rangle^G$.

Let x and y be elements of G. If $|x|$ or $|y|$ is greater than p, then by Theorem 4.3, $[x, y] \in \langle x^p\rangle$ or $\langle y^p\rangle$ and so $[x, y] \in G^p$.

If $\mathcal{C}_G(a)$ is not abelian, it follows from Theorem 3.4 (ii) that $\mathcal{C}_G(a)' \leq G^p$. The latter is clearly valid if $\mathcal{C}_G(a)$ is abelian. Thus $[m, m'] \in G^p$ for all $m, m' \in \mathcal{C}_G(a)$. By the previous paragraph, it remains to show that $[m, b] \in G^p$ for $m \in \mathcal{C}_G(a)$ whenever

$|m| = |b| = p$. Since $|ab| = p^2$ and $[m, ab] = [m, b]$, it follows from the previous paragraph that $[m, b] \in G^p$. This shows that $G' \leq G^p$. □

The following result is well known and easy to prove. We need it in the sequel. We give it for the reader's convenience.

Lemma 4.6 *Let G be a finite CTI-2-group. Let x and $y \in G$ such that $|x| = 2$ and $|y| = 2^n \geq 4$. Then $y^x \in \{y, y^{-1}, y^{1+2^{n-1}}\}$.*

Proof Suppose that $y^x \neq y$. By Theorem 4.3, $\langle y \rangle \trianglelefteq G$ and so $y^x = y^r$ for some odd integer $r \in \{2, \ldots, 2^n - 1\}$. Thus 2^n divides $r^2 - 1$. If $n = 2$, then $y^x = x^{-1}$ easily follows. Suppose that $n > 2$. Since 2^n divides $(r-1)(r+1)$ and $\gcd(r-1, r+1) = 2$, 2^{n-1} divides exactly one of $r - 1$ or $r + 1$. Therefore $r = 1 + 2^{n-1}$ or $r = -1 + 2^{n-1}$ as $1 < r < 2^n$. We now show that the latter case does not happen. Since $|xy| = 4$ and $(xy)^x = yx$ and G is CTI, $\langle xy \rangle = \langle yx \rangle$. Thus $xy = (yx)^{-1}$ and so $x^2 = y^{-2}$, a contradiction. This completes the proof. □

Theorem 4.7 *Let G be a non-abelian CTI-2-group of exponent 2^e. If $Z(G)$ is of exponent 2, then*

(i) *any 2-generated non-abelian subgroup of G is either a dihedral group or Q_8;*
(ii) *$G' = G^2 = \Phi(G)$. In addition, if* $\exp(G) = 4$ *and G does not contain any subgroup isomorphic to $\mathbb{Z}_4 \times \mathbb{Z}_4$, then G^2 is of order 2.*

Proof If G is Dedekindian, then (i) and (ii) obviously occur. Hence we consider non-Dedekindian case.

(i) First we note that for any two elements g, x such that $[g, x] \neq 1$, if $|g| \leq 4$ and $|x| = 2$, then the subgroup $\langle g, x \rangle$ is a dihedral group.

Let $g \in G$ be of order $2^n \geq 4$ and $Z \leqslant \langle g \rangle$ be of order 4. By Theorem 4.3, $\mathcal{C}_G(Z)$ is a maximal subgroup of G. Assume that $x \in G \backslash \mathcal{C}_G(Z)$, then $G = \mathcal{C}_G(Z)\langle x \rangle$.

First suppose that $|x| = 2$, by Lemma 4.6, $g^x = g^{-1}$ or $g^{1+2^{n-1}}$. In the latter case, since $[g, x] = g^{2^{n-1}} \in Z(G)$ and $g^2 \in Z(\mathcal{C}_G(Z))$ so $g^2 \in Z(G)$ which implies $n = 2$. Therefore $\langle g, x \rangle \cong D_{2^{n+1}}$.

If $|x| \neq 2$, then $[g, x] \in \langle g^2 \rangle \cap \langle x^2 \rangle \leqslant Z(G)$, as $g^2 \in Z(\mathcal{C}_G(Z))$ by Theorem 3.4. Therefore $[g^2, x] = 1$ hence $g^2 \in Z(G)$ and so $|g| = 4$ which implies $\langle g \rangle = Z$. Since $x^4 \in Z(\mathcal{C}_G(Z))$, $x^4 \in Z(G)$ is of order at most 2, so $|x| \leq 8$ and $|x| = 4$ if $\mathcal{C}_G(Z)$ is abelian.

Assume that $|x| = 8$. Then $\mathcal{C}_G(Z)$ is non-abelian. Since $Z(\mathcal{C}_G(Z)) = \langle g \rangle \times E$, where E is elementary abelian, it follows from Theorem 3.4 (ii) that $(\mathcal{C}_G(Z))^2 \leqslant \langle g \rangle$. Now let y be a non-involution element of $\mathcal{C}_G(Z)$. Then

$$[y, x] \leqslant \langle y^2 \rangle \cap \langle x^2 \rangle \leqslant \langle g \rangle \cap \langle x^2 \rangle = \langle g^2 \rangle \leqslant Z(G).$$

So $[y, x^2] = 1$. If y is a non-central involution of $\mathcal{C}_G(Z)$, since $|gy| = 4$, it follows that $[x^2, y] = [x^2, gy] = 1$. Therefore $x^2 \in Z(G)$ which is impossible. Hence $|x| = 4$ and $\langle g, x \rangle \cong Q_8$.

(ii) By the previous part, for any $g \in G$ of order $2^n \geq 4$, if there exists $x \in G \backslash \mathcal{C}_G(g)$ of order 2, then $\langle g, x\rangle \cong D_{2^{n+1}}$. So $g^2 = [x, g] \in G'$. Otherwise $\langle g, x\rangle \cong Q_8$ and so $g^2 = [g, x]$. Therefore $G^2 \leqslant G'$.

Let $x, y \in G$ be of the same order 4. By Theorem 4.3, $\langle x\rangle$ and $\langle y\rangle$ are normal in G. If $\langle x\rangle \cap \langle y\rangle = 1$, then $\langle x\rangle \times \langle y\rangle \cong \mathbb{Z}_4 \times \mathbb{Z}_4$, a contradiction. Thus $\langle x\rangle \cap \langle y\rangle \neq 1$ and so $\langle x^2\rangle = \langle y^2\rangle$. It follows that $G^2 = \langle a^2\rangle$ for any element a of order 4 of G. This completes the proof. □

Lemma 4.8 *Let G be a CTI-2-group such that* $\exp(Z(G)) = 2$ *and* $\exp(G) = 2^e \geq 8$. *Then* $\mathcal{C}_G\left(g^{2^{e-2}}\right)$ *is abelian for any element g of order* 2^e.

Proof Let $a = g^{2^{e-2}}$ so that $|a| = 4$. Suppose, for a contradiction, that $\mathcal{C}_G(a)$ is non-abelian. Since $g \in \mathcal{C}_G(a)$, it follows from Theorem 3.4 that $g^2 \in Z(\mathcal{C}_G(a))$ and so $\mathcal{C}_G(a) \leq \mathcal{C}_G(g^2)$. By Theorem 4.3, $\mathcal{C}_G(a)$ is maximal in G. It follows that $\mathcal{C}_G(a) = \mathcal{C}_G(g^2)$, since $g^2 \notin Z(G)$. Now suppose that there exists an element $h \in G \setminus \mathcal{C}_G(g^2)$ of order greater than 2. Then $G = \mathcal{C}_G(g^2)\langle h\rangle$. By Theorem 4.3, $\langle g\rangle$ and $\langle h\rangle$ are both normal subgroups of G and so $[g, h] \leq \langle g^2\rangle \cap \langle h^2\rangle$. Since $\langle g^2\rangle \cap \langle h^2\rangle \leq Z(G)$ and $\exp(Z(G)) = 2$, $[g, h]^2 = 1$ and so $[g^2, h] = 1$ which implies that $g^2 \in Z(G)$, a contradiction. Therefore, all elements in $G \setminus \mathcal{C}_G(a)$ are of order 2. Now fix an element h of $G \setminus \mathcal{C}_G(a)$ and let $b \in \mathcal{C}_G(a)$. Then $|bh| = |h| = 2$ and so $b^h = b^{-1}$ for all $b \in \mathcal{C}_G(a)$. This implies that $\mathcal{C}_G(a)$ is abelian. This completes the proof. □

Theorem 4.9 *Let G be a CTI-2-group such that* $\exp(Z(G)) = 2$. *Then one of the following holds:*

(1) *G is Dedekindian,*
(2) $G' = G^2$ *is of order* 2, *and*
(3) $G = A\langle x\rangle$ *for some abelian subgroup A and involution* $x \in G \setminus A$ *such that* $a^x = a^{-1}$ *for all* $a \in A$.

Proof Suppose that (1) and (2) do not hold. We show that G has the structure described in (3).

- We first show that there exists an element y of order 4 such that $A = \mathcal{C}_G(y)$ is abelian.

Since G is not abelian and we are assuming that (2) does not hold, it follows from Theorem 4.7 that either $\exp(G) \geq 8$ or G contains a subgroup isomorphic to $\mathbb{Z}_4 \times \mathbb{Z}_4$. If the former happens, then such an element y exists according to Lemma 4.8 and if the latter happens, Lemma 4.4 guaranties the existence of an element y of order 4 with abelian centralizer in G.

- Now we prove that there exists an involution $x \in G \setminus A$.

By Theorem 4.3, A is a maximal subgroup of G. Note that if $g \in A \setminus Z(G)$, then $A = \mathcal{C}_G(g)$ and if $h \in A \setminus Z(G)$ and $[t, h] = 1$ for some $t \in G$, then $t \in A$. We use the latter note in the sequel of the proof.

Since G is non-Dedekindian, it follows from Theorem 4.3 that there exists a non-central element z of order 2. If $z \notin A$, we take $x = z$. Now assume that $z \in A$ and there exists an element $t \in G \setminus A$ of order $2^n \geq 4$. If $t^z = t$, then by the above note $t \in A$, which is impossible. Thus $t^z \neq t$. It follows from Lemma 4.6 that $t^z = t^{-1}$ or $t^z = t^{1+2^{n-1}}$. If the latter happens, then $[t^2, z] = 1$ and so $t^2 \in A$. If $t^2 \notin Z(G)$, then $t \in A$, a contradiction. Thus $t^2 \in Z(G)$ and so $|t| = 4$. Hence $t^z = t^3 = t^{-1}$. Now take $x = tz$ which is of order 2 and $x \notin A$.

- Now we prove that $a^x = a^{-1}$ for all $a \in A = \mathcal{C}_G(y)$.

By Lemma 4.6, $y^x = y^{-1}$ since $|y| = 4$. Let $a \in A$ be of order 2. Then $|ay| = 4$ and so by Lemma 4.6, $(ay)^x = (ay)^{-1}$ or $(ay)^x = ay$. The latter does not happen, otherwise $x \in A$ by the above note. Therefore $a^x = a^{-1} = a$. If $a \in A$ is of order 4, then by Lemma 4.6 $a^x = a^{-1}$ or $a^x = a$. The latter does not happen according to the above note. Now assume that $a \in A$ of order $2^e \geq 8$. Then by Lemma 4.6 and the above note, $a^x = a^{-1}$ or $a^x = a^{1+2^{e-1}}$. If the latter happens, then $[a^2, x] = 1$ and so by the above note $a^2 \in Z(G)$ which is not possible as $\exp(Z(G)) = 2$. This completes the proof. □

5 Proof of the Main Theorem

In this section, we give the proof of Theorem 1.2. Let us restate the statement of Theorem 1.2.

Theorem 1.2 *Let G be a finite nilpotent group. Then G is a CTI-group if and only if one of the following holds:*

(1) *G is Dedekindian;*
(2) *G is a p-group of exponent p for some prime p;*
(3) *G is a p-group for some prime p such that $G' = \Omega_1(G^p)$ is of order p and $\Phi(G) = G^p$ is a central cyclic subgroup of G; and*
(4) *G is a 2-group such that $G = A\langle x\rangle$, where A is an abelian subgroup of G and x is an involution in $G \setminus A$ such that $a^x = a^{-1}$.*

Proof Suppose G satisfies one (1), (2), (3), or (4).

If G is a p-group of exponent p, every non-trivial cyclic C subgroup is of order p and so $C^x \cap C$ is clearly equal to C or 1. Therefore G is CTI.

If G is a Dedekindian group, then every subgroup is normal in G and so G is CTI.

Let G be a p-group such that $G' = \Omega_1(G^p)$ is of order p. If $a \in G$ is of order at least p^2, then it follows that $G' \leq \langle a\rangle$ and so $\langle a\rangle \trianglelefteq G$. Elements of order p obviously satisfy the condition of being CTI. This implies that G is a CTI-group.

Suppose G is of the form described in (3). Then every element in $G \setminus A$ is of order 2 and so if $|g| \geq 4$, $g \in A$. Since $g^h = g$ or g^{-1} for all $g \in G$, $\langle g\rangle \trianglelefteq G$. Hence G is CTI.

Now assume that G is a nilpotent CTI-group which is not Dedekindian. By Lemma 3.1, G is a non-Dedekindian p-group for some prime p. Suppose further that G is not of exponent p. If $\exp(Z(G)) > p$, then it follows from Theorem 3.4 that G satisfies (3). If $\exp(Z(G)) = p > 2$, then it follows from Theorem 4.5 that $G' = G^p$ of order p. Therefore G satisfies (3), since $\Omega_1(G^p) = G^p$ in this case. If

$\exp(Z(G)) = 2$, then it follows from Theorem 4.9 that $G' = G^2$ is of order 2 or G is of the form described in (4). This completes the proof. □

We note that any extra-special group is CTI, since the central factor of such a group is elementary abelian. Let G be a finite p-group such that $\exp(Z(G)) \neq p$. In this case, $\Phi(G)$ is cyclic by Theorem 3.4. If $\Phi(G)$ is of order p, then G has the following structure.

Theorem 5.1 *Let G be a non-Dedekindian CTI-p-group such that* $\exp(G) \geq p^2$ *and* $\exp(Z(G)) > p$. *If $|\Phi(G)| = p$, then $G = AZ(G)$, where $A = A_1 * \cdots * A_s$ is an extra-special p-group and $|A_1| = \cdots = |A_s| = p^3$.*

Proof Since in this case $|G'| = p$ and G/G' is elementary abelian, then [10, Lemma 4.2] completes the proof. □

Theorem 5.2 *Let G be a non-Dedekindian CTI-p-group of exponent p^e such that $\exp(Z(G)) > p$. If $|\Phi(G)| > p$ and $Z(G) = \langle z\rangle \times E$, where E is elementary abelian. Then $G = K \times E$, where K is a non-abelian CTI-group with cyclic center. Also $K \cong M \rtimes \langle x\rangle$ for some maximal subgroup M of K and some prime order element $x \notin \Phi(G)$.*

- (i) *If M is abelian, then either $M \cong \langle z\rangle \times \langle y\rangle$ for some prime order element $y \in M$ or M is maximal cyclic subgroup of G such that $Z(K) = M^p$. The subgroup K has one of the following presentations:*
  - (1) $\langle z, y, x \mid z^{p^e} = y^p = x^{p^e} = [z, y] = [z, x] = 1, y^x = yz^{p^{e-1}}\rangle$;
  - (2) $M_{p^{e+1}} = \langle z, x \mid z^{p^e} = x^p = 1, z^x = z^{1+p^{e-1}}\rangle$.
- (ii) *If M is not abelian then $K = AZ = (A_1 * \cdots * A_s)Z = Z(A)TZ$ where Z is a maximal cyclic subgroup of K containing $\Phi(G)$ and A_i is minimal non-abelian for all i and T is extra-special.*

Proof By Theorem 3.4, $\Phi(G) \cap E = 1$. It follows from [11, Hilfssatz 4.4] that $G = H \times E$ and $\Phi(G) = \Phi(H) \leqslant \langle z\rangle$. Since H is non-abelian, it contains a non-normal cyclic subgroup $\langle x\rangle$ of order p. Obviously, $x \notin \Phi(H)$ and so H has a maximal subgroup M such that $x \notin M$. Therefore $H \cong M \rtimes \langle x\rangle$.

- (i) Since $Z(K) = \langle z\rangle$ and $|z| > p$, it follows from Corollary 3.4 that $\Phi(K) \leqslant Z(K)$. Now let x be a non-central element of order p. By Lemma 3.2 (ii), $C = \mathcal{C}_K(x)$ is maximal and $M \cap C$ is an abelian maximal subgroup of C. Since $x \in Z(C)$ and $x \notin M \cap C$, $C = (M \cap C)\langle x\rangle$ is abelian. Therefore, $Z(K) = M \cap C$ is a maximal cyclic subgroup of M. So either $M \cong \langle z\rangle \times \langle y\rangle$ for some $y \in M$ of order p or M is cyclic and $Z(K) = M^p$.
- (ii) In this case, $G' = K'$ and $\Phi(G) = \Phi(K) \leqslant Z(K) \leqslant M$. Let $Z \leqslant K$ be cyclic of maximal order such that $\Phi(G) \leqslant Z$. Then $|Z| = p|\Phi(G)|$. Assume that H is a subgroup of K such that $|H : Z| = p$. Thus $H = Z\langle y\rangle$ for some prime order $y \in K$ so $T(H) = \langle G', y\rangle \trianglelefteq G$. Now for any non-central element y of K, $\langle Z, y\rangle$ has such properties and $T(\langle Z, y\rangle) \trianglelefteq G$. Let A be generated by $T(H)$ where $|H : Z| = p$. So A contains any non-central element of prime order. If A is abelian, then for any two non-central elements y_1 and y_2, $\mathcal{C}_G(y_1) = \mathcal{C}_G(y_2)$ is

abelian by Lemma 3.2 (ii), (iv). Since M is non-abelian, we can choose a non-central element y of M so that $\mathcal{C}_M(y) = \mathcal{C}_M(x)$ is a maximal in M. Now let $g \in M \backslash \mathcal{C}_M(x)$, so $[g, y] = [g, x]$ is central. Thus $[g, xy^i] = 1$ for some i and so xy^i is central, since $|xy^i| = p$. Then $xy^i \in Z(K) \leqslant M$, which is impossible as $x \notin M$. Therefore, A is non-abelian and H satisfies the hypothesis of [10, Theorem 4.4]. This completes the proof. □

Remark 5.3 Let G be a non-Dedekindian CTI-p-group of exponent p^e such that $\exp(Z(G)) = p$. If $e \geq 3$ or G contains an abelian subgroup of type $(4, 4)$, then $p = 2$ and G is of type (4) in the Theorem 1.2. Otherwise $\exp(G) \leq p^2$ and G does not contain any abelian subgroup of type $(4, 4)$. In this case, we have the following theorem:

Theorem 5.4 *Let G be a non-Dedekindian CTI-p-group of exponent p^2 such that* $\exp(Z(G)) = p$. *If G does not contain any abelian subgroup isomorphic to $\mathbb{Z}_4 \times \mathbb{Z}_4$, then $G \cong E \times H$, where H is an extra-special p-group and $Z(G) = E \times \Phi(G)$ for some elementary abelian subgroup E.*

Proof By Theorems 4.5 and 4.7, $|\Phi(G)| = p$. So we may write $Z(G) = \Phi(G) \times E$, where E is an elementary abelian subgroup of G. By [11, Hilfssatz 4.4], $G \cong E \times H$. Hence $\Phi(G) = \Phi(H)$. This completes the proof. □

6 Some Applications in the Structure of Solvable CTI-Groups with Trivial Center

Let G be a finite solvable CTI-group with trivial center which is not isomorphic to the symmetric group of degree 4. Then by [8, Proposition 3.2 and Theorem 3.4], $G = F(G)H$ is a Frobenius group whose kernel is the Fitting subgroup $F(G)$ and H is the complement. By [8, Theorem 3.5], $F(G)$ is abelian if $F(G)$ is not of prime power order or $|H|$ is even. Assume that $n = |H|$ is odd. Then H is cyclic and $F(G)$ is of prime power order. Thus $F(G)$ is the Sylow p-subgroup of G for some prime p such that $p \nmid n$.

Corollary 6.1 *Under the above assumptions and notations, suppose further that $F(G)$ is not abelian. Now*

(i) *if p is odd and* $\exp(F(G)) \neq p$, *then n divides $p - 1$;*
(ii) *if $p = 2$ then $F(G) = A\langle x \rangle$, where A is an abelian subgroup and $x \in F(G) \setminus A$ of order 2 such that $a^x = a^{-1}$ for all $a \in A$.*

Proof (i) If $p > 2$ and $\exp(F(G)) \neq p$, then it follows from Theorem 1.2 (3) that $F(G)$ contains a characteristic subgroup of order p. Thus H is embedded in the automorphism group of $\mathbb{Z}_p$. Hence $n \mid p - 1$.

(ii) If $\exp(Z(F(G))) \neq 2$, then by Theorem 3.4 $F(G)'$ is of order 2 and so $F(G)' \leq Z(G)$, a contradiction. Thus $\exp(Z(F(G)) = 2$. Now Theorem 4.9 implies that $F(G)$ satisfies one of the cases (1), (2), or (3). If $F(G)$ satisfies (2), then $Z(G) \neq 1$ which is not possible. Then $F(G)$ satisfies (1) or (3). If $F(G)$ is Dedekindian, $F(G) \cong Q_8 \times E$ for some elementary abelian 2-group E. It follows that

$F(G)'$ is of order 2 and it is contained in $Z(G)$, a contradiction. This completes the proof. □

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