Bull. Malays. Math. Sci. Soc. (2016) 39:S361–S371
DOI 10.1007/s40840-015-0145-x

BULLETIN of the MALAYSIAN MATHEMATICAL SCIENCES SOCIETY
http://www.springer.com/mathematics/journal/40840

Minimum Number of Components of 2-Factors in Iterated Line Graphs

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Received: 20 December 2014 / Revised: 28 May 2015 / Published online: 12 June 2015

Abstract It is well known that it is NP-hard to determine the minimum number of components of a 2-factor in a graph, even for iterated line graphs. In this paper, we determine the minimum number of components of 2-factors in iterated line graphs of some special tree-like graphs. It extends some known results.

Keywords 2-Factor · Cut-branch · Iterated line graph · Hamiltonian index

Mathematics Subject Classification 05C38 · 05C45 · 05C76

1 Introduction

Throughout this paper, all graphs considered are simple and finite graphs. We follow the most common graph-theoretical terminology and for concepts and notations not defined here, see [1].

A *2-factor* of a graph G is a spanning subgraph whose components are cycles. In particular, a hamiltonian graph has a 2-factor with exactly one component. There

Communicated by Xueliang Li.

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are many results on the existence of 2-factors with a given number of components, mainly on the existence of hamiltonian graphs, see the survey paper [5]. The *line graph* $L(G)$ of a graph G is the graph with vertex set $E(G)$, in which two vertices are adjacent if, and only if, the corresponding edges have a common end vertex in G. The *n-time iterated line graph* $L^n(G)$ is defined to be $L(L^{n-1}(G))$, and we assume that $E(L^{n-1}(G))$ is not empty. The *hamiltonian index* of a graph G is the minimum nonnegative integer n such that $L^n(G)$ is hamiltonian, denoted by $h(G)$, the interested readers can consult [4]. The *Hamilton-connected index* of a graph G is the minimum nonnegative integer n such that $L^n(G)$ is Hamilton-connected, i.e., any two vertices in $L^n(G)$ are connected by a Hamilton path. We know that the Hamilton problem, i.e., the problem to decide whether a given graph is hamiltonian, is one of the classical NP-complete problems. In [8], the authors have proved that it is NP-hard to determine whether $L^k(G)$ is hamiltonian even for any large integer k. Thus, it is also NP-hard to determine the minimum number of components of a 2-factor in $L^k(G)$ for any large integer k. Wang and Xiong [10] provided an upper bound of minimum number of components of 2-factors in iterated line graph. In present paper, we consider the similar problem and determine the minimum number of components of 2-factors in iterated line graph of some special graphs. Before presenting our main results, we first introduce some additional terminology and notation.

A *branch* is a nontrivial path whose internal vertices have degree two and end vertices have degree other than two. The number of edges in a branch B is said to be its *length*, denoted by $l(B)$. We denote by $\mathcal{B}(G)$ the set of branches of G. Note that a branch of length one has no internal vertex. A branch B of a graph G is called a *cut-branch* if the subgraph obtained from G by deleting all edges and internal vertices of B has more components than G, and we denote by $\mathcal{CB}(G)$ the set of all cut-branches of G. Let $\mathcal{B}_2(G) = \{B \in \mathcal{CB}(G) :$ both end vertices of B have degree at least 3 in $G\}$ and $\mathcal{B}_1(G) = \{B \in \mathcal{CB}(G) :$ at least one end vertex of B has degree 1 in $G\}$, then it is obvious that $\mathcal{CB}(G) = \mathcal{B}_1(G) \cup \mathcal{B}_2(G)$.

For $i \in \{1, 2\}$, define

$$h_i(G) = \begin{cases} max\{l(B): B \in \mathcal{B}_i(G)\}, & \text{if } \mathcal{B}_i(G) \text{ is not empty,} \\ 0, & \text{otherwise.} \end{cases}$$

Now we state the main results as follows.

Theorem 1 *Let G be a connected graph with $h_1(G) \le h_2(G) - 1$ and $h_2(G) \ge 2$, such that every nontrivial component of the graph obtained from G by deleting all cut edges and by attaching at least three pendent edges at all vertices of degree at least 3 in G has hamiltonian index at most $h_2(G) - 1$. Then*

(1) $L^{h_2(G)-2}(G)$ *has no 2-factor;*
(2) if $h_2(G) \ge 3$, *then the minimum number of components of 2-factors in* $L^{h_2(G)-1}(G)$ *is*

$$\left|\left\{B \in \mathcal{B}_2(G): l(B) \in \{h_2(G) - 1, h_2(G)\}\right\}\right| + 1;$$

(3) *the minimum number of components of 2-factors in* $L^{h_2(G)}(G)$ *is*

$$\left|\left\{B \in \mathcal{B}_2(G):l(B) = h_2(G)\right\}\right| + 1;$$

(4) $L^{h_2(G)+1}(G)$ *is hamiltonian.*

Theorem 2 *Let* G *be a connected graph with* $h_1(G) \geq h_2(G) \geq 1$*, such that every nontrivial component of the graph obtained from* G *by deleting all cut edges and by attaching at least three pendent edges at all vertices of degree at least three in* G *has hamiltonian index at most* $h_1(G)$*. Then*

(1) $L^{h_1(G)-1}(G)$ *and* $L^{h_2(G)-1}(G)$ *have no 2-factor;*
(2) *if* $h_1(G) = h_2(G) \geq 2$*, then the minimum number of components of 2-factors in* $L^{h_1(G)}(G)$ *is* $|\{B \in \mathcal{B}_2(G) : l(B) = h_2(G)\}| + 1$*;*
(3) *if* $h_1(G) = h_2(G) \geq 1$*, then* $L^{h_1(G)+1}(G)$ *is hamiltonian;*
(4) *if* $h_1(G) > h_2(G) \geq 1$*, then* $L^{h_1(G)}(G)$ *is hamiltonian.*

The proofs of Theorems 1 and 2 will be given in Sect. 3.

The authors in [4] gave a formula of the hamiltonian index $h(T)$ for a tree T. We shall extend the result. Eminjan and Elkin [3] gave a relation between hamiltonian index and Hamilton-connected index of trees.

Since every nontrivial component of the graph obtained from T by deleting all cut edges and by attaching at least three pendent edges at all vertices of degree at least 3 in G has hamiltonian index at most $h_2(T) - 1$ under the condition $h_1(T) \leq h_2(T) - 1$ and $h_2(T) \geq 2$, which satisfies the condition of Theorem 1. Let $l = max\{l(B):B \in \mathcal{B}_2(T)\}$, and $\mathcal{B}_2(T) = \mathcal{B}(T)\backslash\mathcal{B}_1(T)$, then we consequently have the following conclusion.

Corollary 3 *Let* $l \geq 2$ *and let* T *be a tree with* $l(B) \leq l-1$ *for any* $B \in \mathcal{B}_1(T)$*. Then*

(1) $L^{l-2}(T)$ *has no 2-factor;*
(2) *if* $l \geq 3$*, then the minimum number of components of 2-factors in* $L^{l-1}(T)$ *is*

$$\left|\left\{B \in \mathcal{B}(T) \setminus \mathcal{B}_1(T) : \left|E(B)\right| \in \{l-1, l\}\right\}\right| + 1;$$

(3) *the minimum number of components of 2-factors in* $L^{l}(T)$ *is*

$$\left|\left\{B \in \mathcal{B}(T) \setminus \mathcal{B}_1(T) : \left|E(B)\right| = l\right\}\right| + 1;$$

(4) $L^{l+1}(T)$ *is hamiltonian.*

By Corollary 3 and Theorem 2, we can obtain the following result:

Corollary 4 (Chartrand and Wall [4]) *If* T *is a tree which is not a path, then*

$$h(T) = max\{h_2(T) + 1, h_1(T)\}.$$

Proof By Corollary 3 (4), we have $h(T) = h_2(T) + 1$ if $h_1(T) \le h_2(T) - 1$ and $h_2(T) \ge 2$. Since any tree T with $h_1(T) \ge h_2(T) \ge 1$ satisfies the condition of Theorem 2, then we have $h(T) \le h_2(T)+1$ if $h_1(T) = h_2(T) \ge 1$, and $h(T) \le h_1(T)$ if $h_1(T) > h_2(T) \ge 1$ by (3) and (4) of Theorem 2. Again by Theorem 2 (1), $h(T) > h_1(T) - 1$ and $h(T) > h_2(T) - 1$. Above all, we can obtain $h(T) = max\{h_2(T) + 1, h_1(T)\}$. □

2 Preliminaries and Notations

As noted in the first section, for graph-theoretic notation not explained in this paper, we refer readers to [1]. Let $G = (V(G), E(G))$ be a graph with vertex set $V(G)$ and edge set $E(G)$. For a nonnegative integer k, we define $V_k(G)$ by $V_k(G) = \{x \in V(G) \colon d_G(x) = k\}$, where $d_G(x)$ is the degree of x in G. Given two subgraphs G_1 and G_2, we define the distance $d_G(G_1, G_2)$ between G_1 and G_2 by $d_G(G_1, G_2) = \min\{d_G(x_1, x_2) \colon x_1 \in V(G_1), x_2 \in V(G_2)\}$. For subgraphs $G_1, G_2, \ldots, G_k$, their union $G_1 \cup G_2 \cup \cdots \cup G_k$ is the graph whose vertex set and edge set are $V(G_1) \cup V(G_2) \cup \cdots \cup V(G_k)$ and $E(G_1) \cup E(G_2) \cup \cdots \cup E(G_k)$, respectively. For $S \subseteq V(G)$, we denote by $G[S]$ the subgraph of G induced by S.

A *circuit* is a connected graph with at least three vertices in which every vertex has even degree. A set of vertices S is said to *dominate* G if each edge of G has at least one end vertex in S. A circuit of G is called a *dominating circuit* of G if every edge of G either belongs to the circuit or is adjacent to an edge of the circuit. For a graph G of order at least three, its subgraph H is called a *k-system that dominates* if it comprises k edge-disjoint stars ($K(1, s)$, $s \ge 3$) and circuits, such that each edge of G is either contained in one of the circuits or stars, or is adjacent to one of the circuits. Harary and Nash-Williams [7] showed that for a connected graph G with at least three edges, $L(G)$ has a hamiltonian cycle if and only if G has a dominating circuit. This characterization has been widely employed to study the properties of cycles in line graphs and iterated line graphs, see [2]. In 1999, Gould and Hynds presented a necessary and sufficient condition for the line graph $L(G)$ of a graph G that has a 2-factor with exactly k components.

Lemma 5 (Gould and Hynds [6]) *Let G be a graph such that each component of G has at least three edges. Then $L(G)$ has a 2-factor with exactly k components if and only if G has a k-system that dominates.*

We let $EU_n^k(G)$ denote the set of subgraphs H of G satisfying the following five conditions:

- (I) H is an even graph;
- (II) $V_0(H) \subseteq \bigcup_{i \ge 3} V_i(G) \subseteq V(H)$;
- (III) $|E(B)| \le n + 1$ for any branch B with $E(B) \cap E(H) = \emptyset$;
- (IV) $|E(B)| \le n$ for any branch B in $\mathcal{B}_1(G)$;
- (V) H can be decomposed into at most k pairwise vertex-disjoint subgraphs $H_1, \ldots, H_t$ ($t \le k$) such that for every j and for every induced subgraph F of H_j with $\emptyset \ne V(F) \subsetneq V(H_j)$, it holds $d_G(F, H_j - V(F)) \le n - 1$.

In [11], Xiong and Liu considered iterated line graphs and gave a characterization of the graphs G for which $L^n(G)$ is hamiltonian for $n \geq 2$, which has been used to study the hamiltonian index. We state it as follows.

Lemma 6 (Xiong and Liu [11]) *Let G be a connected graph with at least three edges. Then for $n \geq 2$, $L^n(G)$ is hamiltonian if and only if $EU_n^1(G) \neq \emptyset$.*

Saito and Xiong in [9] showed that the following result, which extends Lemma 6.

Lemma 7 (Saito and Xiong [9]) *Let G be a connected graph with at least three edges and let k be a positive integer. Then for $n \geq 2$, $L^n(G)$ has a 2-factor with at most k components if and only if $EU_n^k(G) \neq \emptyset$.*

Next, we provide the proofs of Theorem 1 and Theorem 2.

3 Proofs of Main Results

Proof of Theorem 1 (1) Let $B \in \mathcal{B}_2(G)$ and $l(B) = h_2(G)$. Then B becomes a branch B' of length 2 in $L^{h_2(G)-2}(G)$, whose end vertices belong to two distinct components. Let u be the vertex of degree 2 in B', then u does not belong to any 2-factor component of $L^{h_2(G)-2}(G)$. Thus, $L^{h_2(G)-2}(G)$ has no 2-factor. □

(2) Assume that the minimum number of components of 2-factors in $L^{h_2(G)-1}(G)$ is k and let $\left|\left\{B \in \mathcal{B}_2(G) : l(B) \in \{h_2(G)-1, h_2(G)\}\right\}\right| = s$.

Let $G_1, G_2, \ldots, G_t$ be the components of the graph obtained from G by deleting all edges and inner vertices of any branch in $\left\{B \in \mathcal{B}_2(G) : l(B) \in \{h_2(G)-1, h_2(G)\}\right\}$. It is obvious that $G_1, G_2, \ldots, G_t$ are pairwise vertex-disjoint and $l(B) \leq h_2(G) - 2$ for any branch $B \in \mathcal{B}_2(G_j) \cap \mathcal{B}_2(G)$, $1 \leq j \leq t$. Contracting G_j to a vertex, the resulting graph becomes a tree, then $t = s + 1$ since $|V(T)| = |E(T)| + 1$ in a tree T. Since any possible 2-factor has at least one component in every $L^{h_2(G)-1}(G_j)$, then $k \geq s + 1$. Now we verify that $k \leq s + 1$. By Lemma 7, we need to prove that $EU^{s+1}_{h_2(G)-1}(G) \neq \emptyset$.

Assume that every G_j $(1 \leq j \leq s+1)$ is composed of i_j nontrivial components $G_{j,1}, G_{j,2}, \ldots, G_{j,i_j}$ of the graph obtained from G by deleting all cut edges and by attaching at least three pendent edges at all vertices of degree at least 3 in G, i.e., $G_j = \bigcup_{k=1}^{i_j} G_{j,k}$. Then $G_{j,1}, G_{j,2}, \ldots, G_{j,i_j}$ are pairwise vertex-disjoint and $h(G_{j,k}) \leq h_2(G) - 1$ for $1 \leq k \leq i_j$, $1 \leq j \leq s+1$ by the hypothesis of Theorem 1. By Lemma 6, $EU^1_{h_2(G)-1}(G_{j,k}) \neq \emptyset$. Suppose $H_{j,k} \in EU^1_{h_2(G)-1}(G_{j,k})$, $1 \leq j \leq s+1$, $1 \leq k \leq i_j$. Then $H_{j,k}$ satisfies (I)-(V) in the definition of $EU^1_{h_2(G)-1}(G_{j,k})$, and $H_{j,1}, H_{j,2}, \ldots, H_{j,i_j}$ are pairwise vertex-disjoint. Let $H_j = \bigcup_{k=1}^{i_j} H_{j,k}$, $1 \leq j \leq s+1$. Next, we verify that $H_j \in EU^1_{h_2(G)-1}(G_j)$, $1 \leq j \leq s+1$.

Since $H_{j,k}$ is even graph and $H_{j,k}$'s are mutually vertex-disjoint subgraphs, then H_j is even graph, and (I) follows. Since $V_0(H_{j,k}) \subseteq \bigcup_{i\geq 3} V_i(G_{j,k}) \subseteq V(H_{j,k})$, $1 \leq k \leq i_j$, $\bigcup_{k=1}^{i_j} V_0(H_{j,k}) \subseteq \bigcup_{k=1}^{i_j} \bigcup_{i\geq 3} V_i(G_{j,k}) \subseteq \bigcup_{k=1}^{i_j} V(H_{j,k})$, then $V_0(H_j) \subseteq \bigcup_{i\geq 3} V_i(G_j) \subseteq V(H_j)$, and (II) follows. Let B' be any branch with $E(B') \cap E(H_j) = \emptyset$. Assume that $|E(B')| \geq h_2(G) + 1$, which contradicts that $H_{j,k}$ satisfies (III) of the definition of $EU^1_{h_2(G)-1}(G_{j,k})$, then $|E(B')| \le h_2(G)$, and (III) follows. By the hypothesis of Theorem 1, we have $|E(B'')| \le h_2(G)-1$ for any branch $B'' \in \mathcal{B}_1(G_j)$, and (IV) follows. For any induced subgraph F of H_j with $\emptyset \ne V(F) \subsetneq V(H_j)$. If $F \cap H_{j,k} \ne \emptyset$, then $d_{G_j}(F, H_j - V(F)) = d_{G_{j,k}}(F, H_{j,k} - V(F)) \le h_2(G) - 2$ by the fact that $H_{j,k}$ satisfies (V) in the definition of $EU^1_{h_2(G)-1}(G_{j,k})$. If $F \cap H_{j,k} = \emptyset$, then $d_{G_j}(F, H_j - V(F)) \le h_2(G) - 2$. Thus, $d_{G_j}(F, H_j - V(F)) \le h_2(G) - 2$ for any induced subgraph F of H_j with $\emptyset \ne V(F) \subsetneq V(H_j)$, and (V) follows. Then $H_j \in EU^1_{h_2(G)-1}(G_j)$, $1 \le j \le s+1$. Then H_j satisfies (I)-(V) in the definition of $EU^1_{h_2(G)-1}(G_j)$, and $H_1, H_2, \ldots, H_{s+1}$ are pairwise vertex-disjoint. Let $H = \bigcup_{j=1}^{s+1} H_j$. Then we shall prove $H \in EU^{s+1}_{h_2(G)-1}(G)$, i.e., we shall show that H satisfies the conditions (I)-(V) for $n = h_2(G) - 1$ and $k = s+1$ in Lemma 7.

Since H_j is even graph and disjoint union of even graphs is still even graph, H is even graph, and (I) follows. Further, $V_0(H_j) \subseteq \bigcup_{i\ge3} V_i(G_j) \subseteq V(H_j)$, $1 \le j \le s+1$, $\bigcup_{j=1}^{s+1} V_0(H_j) \subseteq \bigcup_{j=1}^{s+1}\bigcup_{i\ge3} V_i(G_j) \subseteq \bigcup_{j=1}^{s+1} V(H_j)$, then $V_0(H) \subseteq \bigcup_{i\ge3} V_i(G) \subseteq V(H)$, and (II) follows. Let B be a branch of G with $E(B) \cap E(H) = \emptyset$. Assume that $|E(B)| \ge h_2(G) + 1$, which contradicts $h_2(G) = max\{l(B) : B \in \mathcal{B}_2(G)\}$, hence (III) follows. (IV) is obvious by the hypothesis of Theorem 1.

Next, we verify that H satisfies (V). By the construction of H, H can be decomposed into $s+1$ pairwise vertex-disjoint subgraphs $H_1, H_2, \ldots, H_{s+1}$, and for every j and for every induced subgraph F of H_j with $\emptyset \ne V(F) \subsetneq V(H_j)$, we have $d_{G_j}(F, H_j - V(F)) \le h_2(G) - 2$ by the condition (V) of $EU^1_{h_2(G)-1}(G_j)$. Furthermore, $d_G(F, H_j - V(F)) = d_{G_j}(F, H_j - V(F)) \le h_2(G) - 2$ for every j and the same F, then (V) follows. Hence, $H \in EU^{s+1}_{h_2(G)-1}(G)$. Therefore, $k = s+1$. □

(3) The proof is similar to the proof of (2). Assume that the minimum number of components of 2-factors in $L^{h_2(G)}(G)$ is k' and let $\left|\{B \in \mathcal{B}_2(G) : l(B) = h_2(G)\}\right| = s'$.

Let $G'_1, G'_2, \ldots, G'_t$ be the components of the graph obtained from G by deleting all edges and inner vertices of any branch in $\{B \in \mathcal{B}_2(G) : l(B) = h_2(G)\}$, then $G'_1, G'_2, \ldots, G'_t$ are pairwise vertex-disjoint and $l(B) \le h_2(G) - 1$ for any branch $B \in \mathcal{B}_2(G'_j) \cap \mathcal{B}_2(G)$, $1 \le j \le t$. Contracting G'_j to a vertex, the resulting graph becomes a tree, then $t = s'+1$. Since any possible 2-factor has at least one component in every $L^{h_2(G)}(G'_j)$, then $k' \ge s'+1$. It remains to verify that $k' \le s'+1$. By Lemma 7, we need to prove that $EU^{s'+1}_{h_2(G)}(G) \ne \emptyset$.

Assume that every G'_j $(1 \le j \le s'+1)$ is composed of i_j nontrivial components $G'_{j,1}, G'_{j,2}, \ldots, G'_{j,i_j}$ of the graph obtained from G by deleting all cut edges and by attaching at least three pendent edges at all vertices of degree at least 3 in G, i.e., $G'_j = \bigcup_{q=1}^{i_j} G'_{j,\,q}$. Then $G'_{j,1}, G'_{j,2}, \ldots, G'_{j,i_j}$ are pairwise vertex-disjoint, and $h(G'_{j,q}) \le h_2(G)-1$ for $1 \le q \le i_j$, $1 \le j \le s'+1$ by the hypothesis of Theorem 1. This means that $L^{h_2(G)-1}(G'_{j,q})$ has a hamiltonian cycle. Then $L^{h_2(G)}(G'_{j,q})$ is hamiltonian. By Lemma 6, there exists a subgraph $H'_{j,q}$ of $G'_{j,q}$ such that $H'_{j,q} \in EU^1_{h_2(G)}(G'_{j,q})$, $1 \le q \le i_j$, $1 \le j \le s'+1$. Let $H'_j = \bigcup_{q=1}^{i_j} H'_{j,q}$, $1 \le j \le s'+1$. Now, we shall show that $H'_j \in EU^1_{h_2(G)}(G'_j)$, $1 \le j \le s'+1$.

Since $H'_{j,q}$'s $(1 \le j \le s'+1,\ 1 \le q \le i_j)$ are mutually vertex-disjoint even subgraphs, and hence H'_j is also even subgraph, and (I) follows. Since $V_0(H'_{j,q}) \subseteq \bigcup_{i\ge 3} V_i(G'_{j,q}) \subseteq V(H'_{j,q}),\ 1 \le q \le i_j,\ \bigcup_{q=1}^{i_j} V_0(H'_{j,q}) \subseteq \bigcup_{q=1}^{i_j}\bigcup_{i\ge 3} V_i(G'_{j,q}) \subseteq \bigcup_{q=1}^{i_j} V(H'_{j,q})$, then $V_0(H'_j) \subseteq \bigcup_{i\ge 3} V_i(G'_j) \subseteq V(H'_j)$, and (II) follows. Let B' be any branch with $E(B') \cap E(H'_j) = \emptyset$. Assume that $|E(B')| \ge h_2(G)+2$, which contradicts the fact that $H'_{j,q}$ satisfies (III) of the definition of $EU^1_{h_2(G)}(G'_{j,q})$, then $|E(B')| \le h_2(G)+1$, and hence (III) follows. (IV) is obviously true by the hypothesis of Theorem 1. By the fact that $H'_{j,q}$ satisfies (V) of the definition of $EU^1_{h_2(G)}(G'_{j,q})$, we have $d_{G'_j}(F, H'_j - V(F)) = d_{G'_{j,q}}(F, H'_{j,q} - V(F)) \le h_2(G)-1$ for any induced subgraph F of H'_j with $\emptyset \ne V(F) \subsetneq V(H'_j)$ if $F \cap H'_{j,q} \ne \emptyset$, and $d_{G'_j}(F, H'_j - V(F)) \le h_2(G)-1$ if $F \cap H'_{j,q} = \emptyset$. Thus, $d_{G'_j}(F, H'_j - V(F)) \le h_2(G)-1$ for any induced subgraph F of H'_j with $\emptyset \ne V(F) \subsetneq V(H'_j)$, and (V) follows. Then $H'_j \in EU^1_{h_2(G)}(G'_j),\ 1 \le j \le s'+1$. Let $H' = \bigcup_{j=1}^{s'+1} H'_j$. Then we shall verify that H' satisfies the conditions (I)-(V) in the definition of $EU^{s'+1}_{h_2(G)}(G)$.

H' is even graph since it is composed of those vertex-disjoint even subgraphs H'_j, and (I) follows. Since H'_j satisfies (II) in the definition of $EU^1_{h_2(G)}(G'_j)$, we have $V_0(H'_j) \subseteq \bigcup_{i\ge 3} V_i(G'_j) \subseteq V(H'_j),\ 1 \le j \le s'+1$. Then $\bigcup_{j=1}^{s'+1} V_0(H'_j) \subseteq \bigcup_{j=1}^{s'+1}\bigcup_{i\ge 3} V_i(G'_j) \subseteq \bigcup_{j=1}^{s'+1} V(H'_j)$, that is, $V_0(H') \subseteq \bigcup_{i\ge 3} V_i(G) \subseteq V(H')$, and (II) follows. Let B be any branch of G with $E(B) \cap E(H') = \emptyset$. Then $|E(B)| \le h_2(G) \le h_2(G)+1$, (III) follows. (IV) is obvious by the hypothesis of Theorem 1.

By the construction of H' and H'_j satisfies (V) in the definition of $EU^1_{h_2(G)}(G'_j)$, H' can be decomposed into $s'+1$ pairwise vertex-disjoint subgraphs $H'_1, H'_2, \cdots, H'_{s'+1}$, and for every j and for every induced subgraph F of H'_j with $\emptyset \ne V(F) \subsetneq V(H'_j)$, it holds $d_{G'_j}(F, H'_j - V(F)) \le h_2(G)-1$. Since $d_G(F, H'_j - V(F)) = d_{G'_j}(F, H'_j - V(F)) \le h_2(G)-1$ for every j and the same F, H' satisfies (V). Hence, $H' \in EU^{s'+1}_{h_2(G)}(G)$. □

(4) By Theorem 1 (2), we have that $L^{h_2(G)-1}(G)$ has a 2-factor which has at least $s+1$ components. Let $L^{h_2(G)-1}(G) = G^*$, we need to prove that $EU^1_2(G^*) \ne \emptyset$.

Assume that H_j is a 2-factor component of $L^{h_2(G)-1}(G),\ 1 \le j \le s+1$. Let $H = \bigcup_{j=1}^{s+1} H_j$. Then H is a 2-factor of G^* since $\{H_j,\ 1 \le j \le s+1\}$ is pairwise vertex-disjoint. It is obvious that H is an even graph of G^*, and (I) follows. Since $V_0(H) = \emptyset$ and H is a 2-factor of G^*, $\emptyset = V_0(H) \subseteq \bigcup_{i\ge 3} V_i(G^*) \subseteq V(H)$, and (II) follows. Let B be a branch with $E(B) \cap E(H) = \emptyset$. Then $|E(B)| = 1 \le 2+1$ by H is a 2-factor of G^*, hence (III) follows. (IV) holds by the hypothesis of Theorem 1. Now, we prove (V). Assume, by contradiction, that $d_{G^*}(F, H - V(F)) \ge 2$ for some induced subgraph F of H with $\emptyset \ne V(F) \subsetneq V(H)$. This implies that there exists a branch B of length at least 2 between F and $H - F$. Let x be the vertex of degree 2 in B, then $x \in V(B) \setminus V(H)$. This contradicts the fact that H is a 2-factor of G^*, hence H satisfies (V). Therefore, $H \in EU^1_2(G^*)$. By Lemma 6, $L^2(G^*) = L^{h_2(G)+1}(G)$ is hamiltonian. □

Proof of Theorem 2 (1) Let $B \in \mathcal{B}_1(G)$ and $l(B) = h_1(G)$. Then B becomes a branch B' with an end vertex of degree 1 in $L^{h_1(G)-1}(G)$ and $L^{h_2(G)-1}(G)$, respectively. Therefore, $L^{h_1(G)-1}(G)$ and $L^{h_2(G)-1}(G)$ have no 2-factor. □

(2) Assume that the minimum number of components of 2-factors in $L^{h_1(G)}(G)$ is k'' and let $\left|\{B \in \mathcal{B}_2(G) : l(B) = h_2(G) = h_1(G)\}\right| = s''$.

Let $G_1'', G_2'', \dots, G_t''$ be the components of the graph obtained from G by deleting all edges and internal vertices of any branch in $\{B \in \mathcal{B}_2(G) : l(B) = h_2(G) = h_1(G) \geq 2\}$. Then $G_1'', G_2'', \dots, G_t''$ are pairwise vertex-disjoint, and $l(B) \leq h_1(G) - 1$ for any branch $B \in \mathcal{B}_2(G_j'') \cap \mathcal{B}_2(G)$, $1 \leq j \leq t$. Contracting G_j'' to a vertex, the resulting graph becomes a tree, then $t = s''+1$ since $|V(T)| = |E(T)|+1$ in a tree T. Since any possible 2-factor has at least one component in every $L^{h_1(G)}(G_j'')$, then $k'' \geq s'' + 1$. Next, we shall prove that $k'' \leq s'' + 1$. According to Lemma 7, we shall prove that $EU^{s''+1}_{h_1(G)}(G) \neq \emptyset$.

Assume that every G_j'' $(1 \leq j \leq s'' + 1)$ comprises i_j nontrivial components $G''_{j,1}, G''_{j,2}, \dots, G''_{j,i_j}$ of the graph obtained from G by deleting all cut edges and by attaching at least three pendent edges at all vertices of degree at least 3 in G, that is, $G_j'' = \bigcup_{q=1}^{i_j} G''_{j,q}$. It is obvious that $G''_{j,1}, G''_{j,2}, \dots, G''_{j,i_j}$ are pairwise vertex-disjoint. By the hypothesis of Theorem 2, $h(G''_{j,q}) \leq h_1(G)$ for $1 \leq q \leq i_j$, $1 \leq j \leq s'' + 1$. By Lemma 6, $EU^1_{h_1(G)}(G''_{j,q}) \neq \emptyset$. We may take $H''_{j,q} \in EU^1_{h_1(G)}(G''_{j,q})$, $1 \leq j \leq s''+1$, $1 \leq q \leq i_j$, then $H''_{j,q}$ satisfies (I)-(V) in the definition of $EU^1_{h_1(G)}(G''_{j,q})$, and $H''_{j,q}$'s are mutually vertex-disjoint subgraphs. Let $H_j'' = \bigcup_{q=1}^{i_j} H''_{j,q}$, $1 \leq j \leq s''+1$. Next, we shall show that $H_j'' \in EU^1_{h_1(G)}(G_j'')$, $1 \leq j \leq s'' + 1$.

Since $H''_{j,q}$'s are mutually vertex-disjoint even subgraphs, H_j'' is even graph, and (I) follows. Since $V_0(H''_{j,q}) \subseteq \bigcup_{i\geq 3} V_i(G''_{j,q}) \subseteq V(H''_{j,q})$, $1 \leq q \leq i_j$, and $\bigcup_{q=1}^{i_j} V_0(H''_{j,q}) \subseteq \bigcup_{q=1}^{i_j} \bigcup_{i\geq 3} V_i(G''_{j,q}) = \bigcup_{i\geq 3} V_i(G_j'') \subseteq \bigcup_{q=1}^{i_j} V(H''_{j,q})$, then $V_0(H_j'') \subseteq \bigcup_{i\geq 3} V_i(G_j'') \subseteq V(H_j'')$, and (II) follows. Let B' be any branch with $E(B') \cap E(H_j'') = \emptyset$. Assume that $|E(B')| \geq h_1(G)+2$, which contradicts the fact that $H''_{j,q}$ satisfies (III) of $EU^1_{h_1(G)}(G''_{j,q})$, then $|E(B')| \leq h_1(G)+1$, and (III) follows. (IV) is obvious by the hypothesis of Theorem 2. By the fact that $H''_{j,q}$ satisfies (V) of the definition of $EU^1_{h_1(G)}(G''_{j,q})$, we have $d_{G_j''}(F, H_j'' - V(F)) = d_{G''_{j,q}}(F, H''_{j,q} - V(F)) \leq h_1(G) - 1$ for any induced subgraph F of H_j'' with $\emptyset \neq V(F) \subsetneq V(H_j'')$ when $F \cap H''_{j,q} \neq \emptyset$, and $d_{G_j''}(F, H_j'' - V(F)) \leq h_1(G) - 1$ when $F \cap H''_{j,q} = \emptyset$. Then $d_{G_j''}(F, H_j'' - V(F)) \leq h_1(G) - 1$ for any induced subgraph F of H_j'' with $\emptyset \neq V(F) \subsetneq V(H_j'')$, and hence (V) follows. Then $H_j'' \in EU^1_{h_1(G)}(G_j'')$, $1 \leq j \leq s''+1$. Let $H'' = \bigcup_{j=1}^{s''+1} H_j''$. Next, we show that $H'' \in EU^{s''+1}_{h_1(G)}(G)$.

Since H_j'' satisfies (I) and (II) of the definition of $EU^1_{h_1(G)}(G_j'')$, it is easy to deduce H'' is even graph and $V_0(H'') \subseteq \bigcup_{i\geq 3} V_i(G) \subseteq V(H'')$, then (I) (II) follow. Let B be a branch of G with $E(B) \cap E(H'') = \emptyset$. Then $|E(B)| \leq h_2(G) \leq h_2(G) + 1 = h_1(G) + 1$, and (III) follows. It is obvious that $|E(B)| \leq h_1(G)$ for any branch B in $\mathcal{B}_1(G)$, (IV) follows.

Now, we verify that (V) holds. By the construction of H'', H'' can be decomposed into $s''+1$ pairwise vertex-disjoint subgraphs $H_1'', H_2'', \cdots, H_{s''+1}''$, and for every j and for every induced subgraph F of H_j'' with $\emptyset \neq V(F) \subsetneq V(H_j'')$, we have $d_{G_j''}(F, H_j'' - V(F)) \leq h_1(G) - 1$ by the fact that H_j'' satisfies (V) of the definition of $EU^1_{h_1(G)}(G_j'')$. Furthermore, $d_G(F, H_j'' - V(F)) = d_{G_j''}(F, H_j'' - V(F)) \leq h_1(G) - 1$ for every j and the same F, then (V) follows. Hence, $H'' \in EU^{s''+1}_{h_1(G)}(G)$. □

(3) According to Lemma 6, we need to prove that $EU^1_{h_1(G)+1}(G) \neq \emptyset$ when $h_1(G) = h_2(G) \geq 1$. Let $G_1, G_2, \ldots, G_q$ be those nontrivial components of the graph obtained from G by deleting all cut edges and by attaching at least three pendent edges at all vertices of degree at least 3 in G. Then we have $h(G_i) \leq h_1(G)$ by the hypothesis of Theorem 2. It implies that $L^{h_1(G)+1}(G_i)$ is hamiltonian. By Lemma 6, there exists a subgraph $H_i \in EU^1_{h_1(G)+1}(G_i), 1 \leq i \leq q$. Then $H_1, \ H_2, \ldots, H_q$ (some of these subgraphs maybe isolated vertices because there maybe some vertices v_k's in G such that $d_G(v_k) \geq 3$ and the edges incident to v_k are all cut edges) are pairwise vertex-disjoint. Let $H = \bigcup_{i=1}^q H_i$. Then H is even graph since H_i's are mutually vertex-disjoint even subgraphs, and (I) follows. Since $V_0(H_i) \subseteq \bigcup_{i\geq 3} V_i(G_i) \subseteq V(H_i)$ $(1 \leq i \leq q)$, $\bigcup_{i=1}^q V_0(H_i) \subseteq \bigcup_{i=1}^q \bigcup_{i\geq 3} V_i(G_i) = \bigcup_{i\geq 3} V_i(G) \subseteq \bigcup_{i=1}^q V(H_i)$, then $V_0(H) \subseteq \bigcup_{i\geq 3} V_i(G) \subseteq V(H)$, (II) follows. Since $|E(B')| \leq h_1(G) + 2$ for any branch B' with $E(B') \cap E(H_i) = \emptyset$ by H_i satisfies (III) in the definition of $EU^1_{h_1(G)+1}(G_i)$, and $|E(B'')| \leq h_2(G) = h_1(G) \leq h_1(G) + 2$ for any branch $B'' \in \mathcal{B}_2(G)$, then $|E(B)| \leq h_1(G) + 2$ for any branch B of G with $E(B) \cap E(H) = \emptyset$, and (III) follows. (IV) follows from the fact that $|E(B)| \leq h_1(G) \leq h_1(G) + 1$ for any branch B in $\mathcal{B}_1(G)$. For any induced subgraph F of H with $\emptyset \neq V(F) \subsetneq V(H)$, we have $d_G(F, H - V(F)) = d_{G_i}(F, H_i - V(F)) \leq h_1(G)$ if $F \cap H_i \neq \emptyset$ by the fact that H_i (Here H_i is the subgraph which is not an isolated vertex) satisfies (V) of the definition of $EU^1_{h_1(G)+1}(G_i)$. If $F \cap H_i = \emptyset$, then $d_G(F, H - V(F)) \leq h_2(G) = h_1(G)$. Thus, $d_G(F, H - V(F)) \leq h_1(G)$ for any induced subgraph F of H with $\emptyset \neq V(F) \subsetneq V(H)$, and (V) follows. Then $H \in EU^1_{h_1(G)+1}(G)$. □

(4) Similar to the proof of Theorem 2 (3), we shall show that $EU^1_{h_1(G)}(G) \neq \emptyset$ when $h_1(G) > h_2(G) \geq 1$. We use G_i' $(1 \leq i \leq q)$ to denote those components of the graph obtained from G by deleting all cut edges and by attaching at least three pendent edges at all vertices of degree at least 3 in G. Then $h(G_i') \leq h_1(G)$ by the hypothesis of Theorem 2. This means that $L^{h_1(G)}(G_i')$ is hamiltonian. By Lemma 6, $EU^1_{h_1(G)}(G_i') \neq \emptyset$. Let $H_i' \in EU^1_{h_1(G)}(G_i')$, $1 \leq i \leq q$, then $H_1', \ H_2', \ldots, H_q'$ (some of these subgraphs maybe isolated vertices) are pairwise vertex-disjoint. Let $H' = \bigcup_{i=1}^q H_i'$. Then H' is even graph and $V_0(H') \subseteq \bigcup_{i\geq 3} V_i(G) \subseteq V(H')$ by the fact that H_i' satisfies the conditions (I) and (II) in the definition of $EU^1_{h_1(G)}(G_i')$, and hence (I), (II) follow. Since $|E(B')| \leq h_1(G) + 1$ for any branch B' with $E(B') \cap E(H_i') = \emptyset$ by the fact that H_i' satisfies (III) in the definition of $EU^1_{h_1(G)}(G_i')$, and $|E(B'')| \leq h_2(G) < h_1(G) + 1$ for any branch $B'' \in \mathcal{B}_2(G)$, then $|E(B)| \leq h_1(G) + 1$ for any branch B of G with $E(B) \cap E(H') = \emptyset$, and (III) follows. (IV) is obviously true. By the fact that

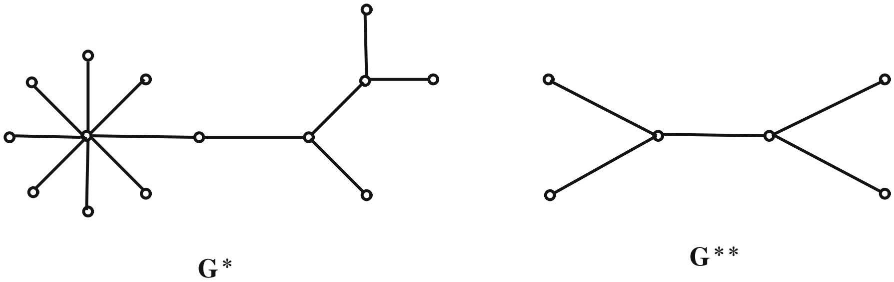

Fig. 1 Condition (2) in both Theorems 1 and 2 is sharp

H_i' (Here H_i' is the subgraph which is not an isolated vertex) satisfies the condition (V) in the definition of $EU^1_{h_1(G)}(G_i')$, we have $d_{G_i'}(F, H_i' - V(F)) \leq h_1(G) - 1$ for every induced subgraph F of H_i' with $\emptyset \neq V(F) \subsetneq V(H_i')$. Then $d_G(F, H' - V(F)) = d_{G_i'}(F, H_i' - V(F)) \leq h_1(G) - 1$ for any induced subgraph F of H' if $F \cap H_i' \neq \emptyset$. If $F \cap H_i' = \emptyset$, then $d_G(F, H' - V(F)) \leq h_2(G) \leq h_1(G) - 1$. Thus, $d_G(F, H' - V(F)) \leq h_1(G) - 1$ for any induced subgraph F of H' with $\emptyset \neq V(F) \subsetneq V(H')$, and hence (V) follows. Therefore, $H' \in EU^1_{h_1(G)}(G)$. □

4 Remark

Our results in this paper provide two classes of graphs, such that as long as they satisfy the condition of Theorems 1 or 2, their iterated line graph has a 2-factor, and we determine the minimum number of components of 2-factors. Note that Theorem 1 is best possible in the sense: (1) The condition $h_2(G) \geq 2$ can not be replaced by $h_2(G) = 1$. Otherwise, we have $h_1(G) = h_2(G) - 1 = 0$. However, the graphs satisfying the condition "$h_1(G) = 0,\ h_2(G) = 1$" could not reach the condition "every nontrivial component of the graph obtained from G by deleting all cut edges and by attaching at least three pendent edges at all vertices of degree at least 3 in G has hamiltonian index at most $h_2(G) - 1$". (2) The condition $h_2(G) \geq 3$ in Theorem 1 (2) can not be replaced by $h_2(G) \leq 2$. This can be seen from the graph G^* in Fig. 1 with $h_2(G^*) = 2$, but there is no 2-factor in $L^{h_2(G^*)-1}(G^*) = L(G^*)$. In addition, Theorem 2 is also best possible in the sense: The condition $h_1(G) = h_2(G) \geq 2$ in Theorem 2 (2) can not be replaced by $h_1(G) = h_2(G) = 1$. This can be seen from the graph G^{**} in Fig. 1 with $h_1(G^{**}) = h_2(G^{**}) = 1$, but there is no 2-factor in $L(G^{**})$.

Acknowledgments This work has been supported by the National Natural Science Foundation (No.11471037 and No.11171129) and Specialized Research Fund for the Doctoral Program of Higher Education (No.20131101110048).

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