Bull. Malays. Math. Sci. Soc. (2017) 40:1465–1478
DOI 10.1007/s40840-015-0143-z

BULLETIN of the MALAYSIAN MATHEMATICAL SCIENCES SOCIETY
http://www.springer.com/mathematics/journal/40840

Additive Results for the Generalized Drazin Inverse in a Banach Algebra

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Received: 30 January 2015 / Revised: 21 May 2015 / Published online: 2 June 2015

Abstract Under new conditions on Banach algebra elements a and b, we derive explicit expressions for the generalized Drazin inverse of the sum $a + b$. As an application of our results, we present new representations for the generalized Drazin inverse of a block matrix in a Banach algebra.

Keywords Generalized Drazin inverse · Additive properties · Banach algebras

Mathematics Subject Classification 46H05 · 47A05 · 15A09

1 Introduction

Let $\mathcal{A}$ be a complex unital Banach algebra with unit 1. The sets of all invertible, nilpotent, and quasinilpotent elements of $\mathcal{A}$ will be denoted by $\mathcal{A}^{-1}$, $\mathcal{A}^{nil}$, and $\mathcal{A}^{qnil}$, respectively.

Let us recall that the generalized Drazin inverse of $a \in \mathcal{A}$ (or Koliha–Drazin inverse of a [9]) is the unique element $a^d \in \mathcal{A}$ which satisfies

$$a^d a a^d = a^d, \quad a a^d = a^d a, \quad a - a^2 a^d \in \mathcal{A}^{qnil}.$$

We use $a^\pi = 1 - aa^d$ to denote the spectral idempotent of a corresponding to the set $\{0\}$. Notice that, if $a \in \mathcal{A}^{qnil}$, then $a^d = 0$. The set $\mathcal{A}^d$ consists of all generalized

Communicated by Mohammad Sal Moslehian.

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Drazin invertible elements of $\mathcal{A}$. If $a - a^2a^d \in \mathcal{A}^{nil}$ in the above definition, then $a^d = a^D$ is ordinary Drazin inverse. The group inverse is a special case of the Drazin inverse for which $a = aa^da$ instead of $a - a^2a^d \in \mathcal{A}^{nil}$. We use $a^\#$ to denote the group inverse of a, and $\mathcal{A}^\#$ to denote the set of all group invertible elements of $\mathcal{A}$.

The next auxiliary result, which is proved in [7, Lemma 2.1] for bounded linear operators, has been extended to Banach algebra elements in [4].

Lemma 1.1 [4, Lemma 2.1] *Let $a, b \in \mathcal{A}^{qnil}$. If $ab = ba$ or $ab = 0$, then $a + b \in \mathcal{A}^{qnil}$.*

Let $p = p^2 \in \mathcal{A}$ be an idempotent. Then we can represent element $a \in \mathcal{A}$ as

$$a = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix},$$

where $a_{11} = pap$, $a_{12} = pa(1-p)$, $a_{21} = (1-p)ap$, and $a_{22} = (1-p)a(1-p)$.
If $a \in \mathcal{A}^d$, we can write

$$a = \begin{bmatrix} a_1 & 0 \\ 0 & a_2 \end{bmatrix}$$

relative to $p = aa^d$, where $a_1 \in (p\mathcal{A}p)^{-1}$ and $a_2 \in ((1-p)\mathcal{A}(1-p))^{qnil}$. Then the generalized Drazin inverse of a can be expressed as

$$a^d = \begin{bmatrix} a^d & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} a_1^{-1} & 0 \\ 0 & 0 \end{bmatrix}.$$

We use the following lemma.

Lemma 1.2 [2, Theorem 2.3] *Let $x = \begin{bmatrix} a & 0 \\ c & b \end{bmatrix} \in \mathcal{A}$ relative to the idempotent $p \in \mathcal{A}$ and let $y = \begin{bmatrix} b & c \\ 0 & a \end{bmatrix} \in \mathcal{A}$ relative to the idempotent $1 - p$.*

(i) *If $a \in (p\mathcal{A}p)^d$ and $b \in ((1-p)\mathcal{A}(1-p))^d$, then $x, y \in \mathcal{A}^d$ and*

$$x^d = \begin{bmatrix} a^d & 0 \\ u & b^d \end{bmatrix}, \quad y^d = \begin{bmatrix} b^d & u \\ 0 & a^d \end{bmatrix},$$

where

$$u = \sum_{n=0}^{\infty} (b^d)^{n+2} ca^n a^\pi + \sum_{n=0}^{\infty} b^\pi b^n c (a^d)^{n+2} - b^d c a^d.$$

(ii) *If $x \in \mathcal{A}^d$ and $a \in (p\mathcal{A}p)^d$, then $b \in ((1-p)\mathcal{A}(1-p))^d$ and x^d is given as in part* (i).

In this paper, we studied additive properties of the generalized Drazin inverse in a Banach algebra. Precisely, we find explicit formulae for the generalized Drazin inverse $(a+b)^d$ in the cases that $ab = b^\pi bab^\pi$ or $ab = a^\pi b^\pi bab^\pi$. Applying these expressions, some representations for the generalized Drazin inverse of a block matrix are presented.

2 Main Results

We start with an important special case of our main theorem.

Theorem 2.1 *If $a \in \mathcal{A}^{qnil}$, $b \in \mathcal{A}^d$, and $ab = b^\pi bab^\pi$, then $a+b \in \mathcal{A}^d$ and*

$$(a+b)^d = \sum_{n=0}^{\infty} (b^d)^{n+1} a^n. \tag{1}$$

Proof If $b \in \mathcal{A}^{qnil}$, by $b^\pi = 1$ and $ab = b^\pi bab^\pi$, we get $ab = ba$. Applying Lemma 1.1, $a+b \in \mathcal{A}^{qnil}$ and the formula (1) is satisfied.

Suppose that $b \notin \mathcal{A}^{qnil}$. Then, relative to $p = bb^d$, we have the following representations of b and a:

$$b = \begin{bmatrix} b_1 & 0 \\ 0 & b_2 \end{bmatrix} \quad \text{and} \quad a = \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix},$$

where $b_1 \in (p\mathcal{A}p)^{-1}$ and $b_2 \in ((1-p)\mathcal{A}(1-p))^{qnil}$. So

$$b^d = \begin{bmatrix} b_1^{-1} & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad b^\pi = \begin{bmatrix} 0 & 0 \\ 0 & 1-p \end{bmatrix}.$$

The equalities

$$\begin{bmatrix} a_1 b_1 & a_2 b_2 \\ a_3 b_1 & a_4 b_2 \end{bmatrix} = ab = b^\pi bab^\pi = \begin{bmatrix} 0 & 0 \\ 0 & b_2 a_4 \end{bmatrix}$$

imply $a_1b_1 = 0$, $a_2b_2 = 0$, $a_3b_1 = 0$, and $a_4b_2 = b_2a_4$. Because b_1 is invertible, $a_1 = 0$ and $a_3 = 0$. Now, since $a \in \mathcal{A}^{qnil}$, $a_4 \in ((1-p)\mathcal{A}(1-p))^{qnil}$. By Lemma 1.1, $a_4 + b_2 \in ((1-p)\mathcal{A}(1-p))^{qnil}$ and $(a_4+b_2)^d = 0$.

Using Lemma 1.2, $a_2b_2 = 0$ and $a_4b_2 = b_2a_4$, we conclude that $a+b \in \mathcal{A}^d$ and

$$(a+b)^d = \begin{bmatrix} b_1 & a_2 \\ 0 & a_4+b_2 \end{bmatrix}^d = \begin{bmatrix} b_1^{-1} & \sum\limits_{n=0}^{\infty} b_1^{-(n+2)} a_2 a_4^n \\ 0 & 0 \end{bmatrix}$$

$$= b^d + \sum_{n=0}^{\infty} (b^d)^{n+2} a^{n+1} = \sum_{n=0}^{\infty} (b^d)^{n+1} a^n.$$

□

Castro-González and Koliha obtained the formula (1) for $(a+b)^d$ in [2, Corollary 3.4] when $ab = 0$ instead of $ab = b^\pi bab^\pi$ in Theorem 2.1.

Notice that the conditions $ab = 0$ and $ab = b^\pi bab^\pi$ are independent, but in the both cases we obtain the same expressions for $(a+b)^d$. In the first example, we have that the condition $ab = 0$ holds, but the condition $ab = b^\pi bab^\pi$ is not satisfied.

Example 2.1 Let $\mathcal{A}$ be the algebra of all complex 3×3 matrices and let $a, b \in \mathcal{A}$ such that

$$a = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad b = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

Thus, $ab = 0$. From $b^2 = 0$, we have $b^\pi = 1$ and

$$b^\pi bab^\pi = ba = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \neq ab.$$

In the second example, we consider matrices a and b in the algebra $\mathcal{A}$ of all complex 3×3 matrices such that $ab = b^\pi bab^\pi$ is satisfied but $ab = 0$ does not hold.

Example 2.2 Let $\mathcal{A}$ be the algebra of all complex 3×3 matrices and let $a, b \in \mathcal{A}$ such that

$$a = b = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

Then

$$a^2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

and $a^3 = 0$ implying $a^d = 0$ and $a^\pi = 1$. Hence, $0 \neq ab = a^2 = a^\pi a^2 a^\pi = b^\pi bab^\pi$.

Now, we prove our main theorem.

Theorem 2.2 *If $a, b \in \mathcal{A}^d$ and $ab = a^\pi b^\pi bab^\pi$, then $a + b \in \mathcal{A}^d$ and*

$$(a+b)^d = b^\pi a^d + b^d a^\pi + \sum_{n=1}^{\infty} (b^d)^{n+1} a^n a^\pi + \sum_{n=0}^{\infty} b^\pi (a+b)^n b (a^d)^{n+2}. \tag{2}$$

Proof First, if we assume that $a \in \mathcal{A}^{qnil}$, by Theorem 2.1, we get (2). For $a \in \mathcal{A}^{-1}$, we obtain $ab = 0$ and the formula (2) holds by [2, Example 4.5].

Further, if a is neither invertible nor quasinilpotent, we have the following matrix representations of a and b relative to $p = aa^d$:

$$a = \begin{bmatrix} a_1 & 0 \\ 0 & a_2 \end{bmatrix}, \quad b = \begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix},$$

where $a_1 \in (p\mathcal{A}p)^{-1}$ and $a_2 \in ((1-p)\mathcal{A}(1-p))^{qnil}$.

From $ab = a^\pi b^\pi bab^\pi$, we obtain $a_1 b_1 = 0$ and $a_1 b_2 = 0$. Since a_1 is invertible, then $b_1 = 0$ and $b_2 = 0$. Now

$$b = \begin{bmatrix} 0 & 0 \\ b_3 & b_4 \end{bmatrix}$$

and, using Lemma 1.2, we observe that $b_4 \in ((1-p)\mathcal{A}(1-p))^d$,

$$b^d = \begin{bmatrix} 0 & 0 \\ (b_4^d)^2 b_3 & b_4^d \end{bmatrix} \quad \text{and} \quad b^\pi = \begin{bmatrix} p & 0 \\ -b_4^d b_3 & b_4^\pi \end{bmatrix}.$$

Also, the equalities

$$\begin{bmatrix} 0 & 0 \\ a_2 b_3 & a_2 b_4 \end{bmatrix} = ab = a^\pi b^\pi bab^\pi = \begin{bmatrix} 0 & 0 \\ b_4^\pi b_3 a_1 - b_4^\pi b_4 a_2 b_4^d b_3 & b_4^\pi b_4 a_2 b_4^\pi \end{bmatrix}$$

give $a_2 b_3 = b_4^\pi b_3 a_1 - b_4^\pi b_4 a_2 b_4^d b_3$ and $a_2 b_4 = b_4^\pi b_4 a_2 b_4^\pi$. By Theorem 2.1, we conclude that $a_2 + b_4 \in ((1-p)\mathcal{A}(1-p))^d$ and

$$(a_2 + b_4)^d = \sum_{n=0}^{\infty} (b_4^d)^{n+1} a_2^n.$$

Now, from Lemma 1.2, $a + b \in \mathcal{A}^d$ and

$$(a+b)^d = \begin{bmatrix} a_1 & 0 \\ b_3 & a_2 + b_4 \end{bmatrix}^d = \begin{bmatrix} a_1^{-1} & 0 \\ u & (a_2 + b_4)^d \end{bmatrix}, \tag{3}$$

where

$$u = \sum_{n=0}^{\infty} (a_2 + b_4)^\pi (a_2 + b_4)^n b_3 a_1^{-(n+2)} - (a_2 + b_4)^d b_3 a_1^{-1}.$$

Because $a_2 b_4 = b_4^\pi b_4 a_2 b_4^\pi$, we get $a_2 b_4^d = 0$ and $b_4^d a_2 b_4 = 0$. By $a_2 b_3 = b_4^\pi b_3 a_1 - b_4^\pi b_4 a_2 b_4^d b_3$, we obtain $a_2 b_3 = b_4^\pi b_3 a_1$ and $b_4^d a_2 b_3 = 0$. Since

$$b_4^d a_2 a_2 b_3 = b_4^d a_2 b_4^\pi b_3 a_1 = b_4^d a_2 b_3 a_1 = 0$$

and

$$b_4^d a_2 a_2 b_4 = b_4^d a_2 b_4^\pi b_4 a_2 b_4^\pi = b_4^d a_2 b_4 a_2 b_4^\pi = 0,$$

then $b_4^d a_2^n b_3 = 0$ and $b_4^d a_2^n b_4 = 0$, for all $n \geq 1$, which imply $(b_4^d)^{k+1} a_2^{k+1} (a_2 + b_4)^n b_3 = 0$, for all $k, n \geq 0$. Thus,

$$\begin{aligned}(a_2 + b_4)^\pi &= (1-p) - (a_2+b_4)(a_2+b_4)^d = (1-p) - b_4 \sum_{n=0}^{\infty} (b_4^d)^{n+1} a_2^n \\ &= b_4^\pi - \sum_{n=0}^{\infty} (b_4^d)^{n+1} a_2^{n+1}\end{aligned}$$

which yields

$$\begin{aligned}u &= \sum_{n=0}^{\infty} b_4^\pi (a_2+b_4)^n b_3 a_1^{-(n+2)} - \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} (b_4^d)^{k+1} a_2^{k+1} (a_2+b_4)^n b_3 a_1^{-(n+2)} \\ &\quad - b_4^d b_3 a_1^{-1} - \sum_{n=1}^{\infty} (b_4^d)^{n+1} a_2^n b_3 a_1^{-1} \\ &= \sum_{n=0}^{\infty} b_4^\pi (a_2+b_4)^n b_3 a_1^{-(n+2)} - b_4^d b_3 a_1^{-1}.\end{aligned}$$

Using the equalities

$$\begin{aligned}X_1 &= b^d a^\pi + \sum_{n=1}^{\infty} (b^d)^{n+1} a^n a^\pi = \begin{bmatrix} 0 & 0 \\ 0 & b_4^d \end{bmatrix} + \sum_{n=1}^{\infty} \begin{bmatrix} 0 & 0 \\ 0 & (b_4^d)^{n+1} a_2^n \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 \\ 0 & \sum_{n=0}^{\infty} (b_4^d)^{n+1} a_2^n \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & (a_2+b_4)^d \end{bmatrix}, \\ X_2 &= \sum_{n=0}^{\infty} b^\pi (a+b)^n b (a^d)^{n+2} = \begin{bmatrix} 0 & 0 \\ \sum_{n=0}^{\infty} b_4^\pi (a_2+b_4)^n b_3 a_1^{-(n+2)} & 0 \end{bmatrix}, \\ X_3 &= b^\pi a^d = \begin{bmatrix} a_1^{-1} & 0 \\ -b_4^d b_3 a_1^{-1} & 0 \end{bmatrix}\end{aligned}$$

and (3), we deduce that

$$(a+b)^d = X_1 + X_2 + X_3$$

and the formula (2) is satisfied. □

Note that under the conditions of Theorem 2.2, it can be verified that $b^d ab = 0$. Precisely, $b_4^d a_2^n b_3 = 0$ and $b_4^d a_2^n b_4 = 0$ imply $b^d a^n b = 0$ for all $n \geq 1$.

In [2, Theorem 3.5], Castro-González and Koliha presented the expression for the generalized Drazin inverse of $a + b$ in the case that $a^\pi b = b$, $ab^\pi = a$, and $b^\pi aba^\pi = 0$. Observe that matrices a and b introduced in Example 2.2 satisfy the conditions of Theorem 2.2, but the assumption $b^\pi aba^\pi = 0$ of [2, Theorem 3.5] does not hold.

Liu and Qin [10] derived a formula for $(a+b)^d$ under the conditions $ab = a^\pi bab^\pi$. The following example describes two matrices a and b in the algebra of all complex 2×2 matrices which do not satisfy the conditions of [10, Theorem 2.2], whereas the conditions of Theorem 2.2 are met, which allows us to compute $(a + b)^d$.

Example 2.3 Let $\mathcal{A}$ be the algebra of all complex 2×2 matrices and let $a, b \in \mathcal{A}$ such that

$$a = \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix}, \quad b = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}.$$

Since $a^2 = 0$ and $b^2 = b$, then $a^\pi = 1$ and

$$b^\pi = 1 - b = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

Thus, $0 = ab \neq a^\pi bab^\pi = a$ and $ab = 0 = a^\pi b^\pi bab^\pi$.

3 Applications

The problem of finding explicit representations for the Drazin inverse of a 2×2 block matrix in terms of its blocks was posed by Campbell and Meyer [1]. There have been many papers on this subject, under different conditions [3,5,8,11–14], but it is still a hard problem to find an explicit formula for the Drazin inverse of a block matrix.

In this section, as an application of Theorem 2.2, we obtain representations for the generalized Drazin inverse of a block matrix $x \in \mathcal{A}$ given by

$$x = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \tag{4}$$

relative to the idempotent $p \in \mathcal{A}$, where $a \in (p\mathcal{A}p)^d$ and $d \in ((1-p)\mathcal{A}(1-p))^d$. Throughout this section, if the lower limit of a sum is greater than its upper limit, we always define the sum to be 0. For example, the sum $\sum_{k=1}^{0} * = 0$.

Theorem 3.1 *Let x be defined as in (4). If*

$$a^\pi bc = 0, \quad a^\pi bd = ab, \quad \text{and} \quad \sum_{n=0}^{\infty} (d^d)^n ca^n b = 0, \tag{5}$$

then $x \in \mathcal{A}^d$ *and*

$$x^d = \begin{bmatrix} a^d & 0 \\ u & d^d \end{bmatrix} + \sum_{n=0}^{\infty} x^n \begin{bmatrix} i_n - \sum_{k=1}^{n} b(d^d)^{k+1}c(a^d)^{n+2-k} & b(d^d)^{n+2} \\ 0 & 0 \end{bmatrix}, \tag{6}$$

where

$$\begin{aligned} u &= \sum_{n=0}^{\infty}(d^d)^{n+2}ca^n a^\pi + \sum_{n=0}^{\infty} d^\pi d^n c(a^d)^{n+2} - d^d ca^d, \\ i_n &= \sum_{k=0}^{\infty} bd^\pi d^k c(a^d)^{n+k+3} - bd^d c(a^d)^{n+2} + \sum_{k=0}^{\infty} b(d^d)^{n+k+3}ca^k a^\pi - b(d^d)^{n+2}ca^d, \end{aligned} \tag{7}$$

for $n \geq 0$.

Proof Let

$$x = \begin{bmatrix} a & 0 \\ c & d \end{bmatrix} + \begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix} := y + z. \tag{8}$$

Applying Lemma 1.2, we have that $y \in \mathcal{A}^d$,

$$y^d = \begin{bmatrix} a^d & 0 \\ u & d^d \end{bmatrix} \quad \text{and} \quad y^\pi = \begin{bmatrix} a^\pi & 0 \\ -ca^d - du & d^\pi \end{bmatrix},$$

where u is defined as in (7). Observe that $z^2 = 0$ implies $z^d = 0$ and $z^\pi = 1$.
Then

$$yz = \begin{bmatrix} 0 & ab \\ 0 & cb \end{bmatrix} \quad \text{and} \quad y^\pi z^\pi zyz^\pi = \begin{bmatrix} a^\pi bc & a^\pi bd \\ (-ca^d - du)bc & (-ca^d - du)bd \end{bmatrix}.$$

The hypothesis $a^\pi bd = ab$ gives $a^d b = 0$ and $bd = ab$. Now, from $a^\pi bc = 0$, we get $bc = 0$. By the third equality in (5), we obtain

$$\begin{aligned} cb + dubd &= cb + \sum_{n=0}^{\infty}(d^d)^{n+1}ca^n bd = cb + \sum_{n=0}^{\infty}(d^d)^{n+1}ca^{n+1}b \\ &= \sum_{n=0}^{\infty}(d^d)^n ca^n b = 0. \end{aligned}$$

Hence, $yz = y^\pi z^\pi zyz^\pi$ which yields, by Theorem 2.2, $x \in \mathcal{A}^d$ and

$$
\begin{aligned}
x^d &= y^d + \sum_{n=0}^{\infty} x^n z(y^d)^{n+2} \\
&= \begin{bmatrix} a^d & 0 \\ u & d^d \end{bmatrix} + \sum_{n=0}^{\infty} x^n \begin{bmatrix} \sum_{k=0}^{n+1} b(d^d)^k u(a^d)^{n+1-k} & b(d^d)^{n+2} \\ 0 & 0 \end{bmatrix} \\
&= \begin{bmatrix} a^d & 0 \\ u & d^d \end{bmatrix} \\
&\quad + \sum_{n=0}^{\infty} x^n \begin{bmatrix} bu(a^d)^{n+1} - \sum_{k=1}^{n} b(d^d)^{k+1} c(a^d)^{n+2-k} + b(d^d)^{n+1} u & b(d^d)^{n+2} \\ 0 & 0 \end{bmatrix}
\end{aligned}
$$

implying that the equality (6) is satisfied. □

Observe that the third equality in (5) can be replaced by weaker assumption $cb = 0$, since the equalities $cb = 0$ and $bd = ab$ give $ca^n b = 0$, for $n \geq 0$.

In the following theorem, we derive a formula for the generalized Drazin inverse of x under some rather complicated conditions, but the theorem itself will have useful consequences which will include much simpler conditions.

Theorem 3.2 *Let x be defined as in (4) and let u be defined as in (7). If*

$$
a^\pi abd^\pi = bd, \quad (-ca^d - du)ab + d^\pi cb = 0, \quad \text{and} \quad \sum_{n=0}^{\infty} bd^n c(a^d)^n = 0, \tag{9}
$$

then $x \in \mathcal{A}^d$ and

$$
x^d = \begin{bmatrix} a^d & (a^d)^2 b \\ u & d^d + i \end{bmatrix}, \tag{10}
$$

where

$$
i = \sum_{n=0}^{\infty} d^\pi d^n c(a^d)^{n+3} b - d^d c(a^d)^2 b + \sum_{n=0}^{\infty} (d^d)^{n+3} ca^n a^\pi b - (d^d)^2 ca^d b.
$$

Proof If we suppose that x is represented as in (8) and denote by $t = (-ca^d - du)ab + d^\pi cb$, we have

$$
zy = \begin{bmatrix} bc & bd \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad y^\pi yzy^\pi = \begin{bmatrix} a^\pi ab(-ca^d - du) & a^\pi abd^\pi \\ t(-ca^d - du) & td^\pi \end{bmatrix}.
$$

From the condition $a^\pi abd^\pi = bd$, we obtain $bd^d = 0$ and $a^\pi ab = bd$. So, using the assumptions (9), note that $t = 0$,

$$a^\pi ab(-ca^d - du) = -bdca^d - bd^2u = -bdca^d - \sum_{n=1}^{\infty} bd^{n+1}c(a^d)^{n+1}$$
$$= -\sum_{n=1}^{\infty} bd^n c(a^d)^n = bc$$

and $zy = z^\pi y^\pi yzy^\pi$. Applying Theorem 2.1, we conclude that $x \in \mathcal{A}^d$ and

$$x^d = y^d + (y^d)^2 z = \begin{bmatrix} a^d & (a^d)^2 b \\ u & d^d + ua^d b + d^d ub \end{bmatrix},$$

which yields that (10) holds. □

The following corollary presents conditions weaker than those given in Theorem 3.2 under which we have simpler expression for x^d.

Corollary 3.1 *Let x be defined as in* (4) *and let u be defined as in* (7).

(i) *If $a^\pi abd^\pi = bd$, $ca^d ab = d^\pi cb$, $dca = 0$, and $bc = 0$, then $x \in \mathcal{A}^d$ and*

$$x^d = \begin{bmatrix} a^d & (a^d)^2 b \\ (d^d)^2 c + c(a^d)^2 & d^d + c(a^d)^3 b + (d^d)^3 cb \end{bmatrix}.$$

(ii) *If $a^\pi abd^\pi = bd$, $ca^\pi b = 0$, $dc = 0$, and $bc = 0$, then $x \in \mathcal{A}^d$ and*

$$x^d = \begin{bmatrix} a^d & (a^d)^2 b \\ c(a^d)^2 & d^d + c(a^d)^3 b \end{bmatrix}.$$

(iii) *If $ab = 0$, $bd = 0$, $d^\pi cb = 0$, and $bc = 0$, then $x \in \mathcal{A}^d$ and*

$$x^d = \begin{bmatrix} a^d & 0 \\ u & d^d + (d^d)^3 cb \end{bmatrix}.$$

(iv) *If $ab = 0$, $bd = 0$, $cb = 0$, and $bc = 0$, then $x \in \mathcal{A}^d$ and*

$$x^d = \begin{bmatrix} a^d & 0 \\ u & d^d \end{bmatrix}.$$

The formula for the generalized Drazin inverse given in part (ii) of Corollary 3.1 was obtained for operator matrices in [6, Theorem 5.3] in the case that $bd = 0$, $dc = 0$, and $bc = 0$.

If we suppose that a and d are group invertible in Theorems 3.1 and 3.2, we get the following representations of x^d.

Corollary 3.2 *Let x be defined as in (4), $a \in (p\mathcal{A}p)^{\#}$ and $d \in ((1-p)\mathcal{A}(1-p))^{\#}$.*

(i) *If the equalities (5) hold, then $x \in \mathcal{A}^d$ and*

$$x^d = \begin{bmatrix} a^{\#} & 0 \\ (d^{\#})^2ca^{\pi} + d^{\pi}c(a^{\#})^2 - d^{\#}ca^{\#} & d^{\#} \end{bmatrix} + \sum_{n=0}^{\infty} x^n \begin{bmatrix} i'_n - \sum_{k=1}^{n} b(d^{\#})^{k+1}c(a^{\#})^{n+2-k} & b(d^{\#})^{n+2} \\ 0 & 0 \end{bmatrix},$$

where $i'_n = bd^{\pi}c(a^{\#})^{n+3} - bd^{\#}c(a^{\#})^{n+2} + b(d^{\#})^{n+3}ca^{\pi} - b(d^{\#})^{n+2}ca^{\#}$, for $n \geq 0$.

(ii) *If $bd = 0$, $(-ca^{\#} - dd^{\#}ca^{\#})ab + d^{\pi}cb = 0$, and $bc = 0$, then $x \in \mathcal{A}^d$ and*

$$x^d = \begin{bmatrix} a^{\#} & (a^{\#})^2b \\ (d^{\#})^2ca^{\pi} + d^{\pi}c(a^{\#})^2 - d^{\#}ca^{\#} & d^{\#} + i' \end{bmatrix},$$

where $i' = d^{\pi}c(a^{\#})^3b - d^{\#}c(a^{\#})^2b + (d^{\#})^3ca^{\pi}b - (d^{\#})^2ca^{\#}b$.

In a similar way as it was done in the previous theorems, using the another splitting, we present new expressions for the generalized Drazin inverse of a block matrix in a Banach algebra.

Theorem 3.3 *Let x be defined as in (4). If*

$$d^{\pi}cb = 0, \quad d^{\pi}ca = dc \quad \text{and} \quad \sum_{n=0}^{\infty}(a^d)^n bd^n c = 0, \tag{11}$$

then $x \in \mathcal{A}^d$ and

$$x^d = \begin{bmatrix} a^d & v \\ 0 & d^d \end{bmatrix} + \sum_{n=0}^{\infty} x^n \begin{bmatrix} 0 & 0 \\ c(a^d)^{n+2} & j_n - \sum_{k=1}^{n} c(a^d)^{n+2-k}b(d^d)^{k+1} \end{bmatrix}, \tag{12}$$

where

$$\begin{aligned} v &= \sum_{n=0}^{\infty}(a^d)^{n+2}bd^nd^{\pi} + \sum_{n=0}^{\infty} a^{\pi}a^nb(d^d)^{n+2} - a^dbd^d, \\ j_n &= \sum_{k=0}^{\infty} c(a^d)^{n+k+3}bd^kd^{\pi} - c(a^d)^{n+2}bd^d + \sum_{k=0}^{\infty} ca^{\pi}a^kb(d^d)^{n+k+3} - ca^db(d^d)^{n+2}, \end{aligned} \tag{13}$$

for $n \geq 0$.

Proof If we write

$$x = \begin{bmatrix} a & b \\ 0 & d \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ c & 0 \end{bmatrix} := y + z, \tag{14}$$

then $z^2 = 0 = z^d$, $z^\pi = 0$,

$$y^d = \begin{bmatrix} a^d & v \\ 0 & d^d \end{bmatrix} \text{ and } y^\pi = \begin{bmatrix} a^\pi & -av - bd^d \\ 0 & d^\pi \end{bmatrix}.$$

The equalities (11) give $yz = y^\pi z^\pi zyz^\pi$. By Theorem 2.2, similarly as in the proof of Theorem 3.1, we have that $x \in \mathcal{A}^d$ and x^d is represented as in (12). □

Instead of the third condition of (11), we can assume that weaker condition $bc = 0$ holds.

Theorem 3.4 *Let x be defined as in* (4) *and let v be defined as in* (13). *If*

$$d^\pi dca^\pi = ca, \quad (-av - bd^d)dc + a^\pi bc = 0, \quad \text{and} \quad \sum_{n=0}^{\infty} ca^n b(d^d)^n = 0,$$

then $x \in \mathcal{A}^d$ and

$$x^d = \begin{bmatrix} a^d + j & v \\ (d^d)^2 c & d^d \end{bmatrix},$$

where

$$j = \sum_{n=0}^{\infty} (a^d)^{n+3} bd^n d^\pi c - (a^d)^2 bd^d c + \sum_{n=0}^{\infty} a^\pi a^n b(d^d)^{n+3} c - a^d b(d^d)^2 c.$$

Proof Using the decomposition (14) of x, as Theorem 3.2, we prove this result. □

Remark that, if $b = 0$ in Theorem 3.1 (or Theorem 3.2) and $c = 0$ in Theorem 3.3 (or Theorem 3.4), we obtain Lemma 1.2 (i).

As a consequence of Theorem 3.4, we get the next result.

Corollary 3.3 *Let x be defined as in* (4) *and let v be defined as in* (13).

(i) *If $d^\pi dca^\pi = ca$, $bd^d dc = a^\pi bc$, $abd = 0$, and $cb = 0$, then $x \in \mathcal{A}^d$ and*

$$x^d = \begin{bmatrix} a^d + (a^d)^3 bc + b(d^d)^3 c & (a^d)^2 b + b(d^d)^2 \\ (d^d)^2 c & d^d \end{bmatrix}.$$

(ii) *If $d^\pi dca^\pi = ca$, $bd^\pi c = 0$, $ab = 0$, and $bc = 0$, then $x \in \mathcal{A}^d$ and*

$$x^d = \begin{bmatrix} a^d + b(d^d)^3 c & b(d^d)^2 \\ (d^d)^2 c & d^d \end{bmatrix}.$$

(iii) *If* $dc = 0$, $ca = 0$, $a^\pi bc = 0$, *and* $cb = 0$, *then* $x \in \mathcal{A}^d$ *and*

$$x^d = \begin{bmatrix} a^d + (a^d)^3 bc & v \\ 0 & d^d \end{bmatrix}.$$

(iv) *If* $dc = 0$, $ca = 0$, $bc = 0$, *and* $cb = 0$, *then* $x \in \mathcal{A}^d$ *and*

$$x^d = \begin{bmatrix} a^d & v \\ 0 & d^d \end{bmatrix}.$$

Applying Theorems 3.3 and 3.4, we verify the following corollary.

Corollary 3.4 *Let* x *be defined as in* (4), $a \in (p\mathcal{A}p)^\#$ *and* $d \in ((1-p)\mathcal{A}(1-p))^\#$.

(i) *If the equalities* (11) *hold, then* $x \in \mathcal{A}^d$ *and*

$$x^d = \begin{bmatrix} a^\# & (a^\#)^2 bd^\pi + a^\pi b(d^\#)^2 - a^\# bd^\# \\ 0 & d^\# \end{bmatrix} + \sum_{n=0}^{\infty} x^n \begin{bmatrix} 0 & 0 \\ c(a^\#)^{n+2} & j'_n - \sum_{k=1}^{n} c(a^\#)^{n+2-k} b(d^\#)^{k+1} \end{bmatrix},$$

where $j'_n = c(a^\#)^{n+3} bd^\pi - c(a^\#)^{n+2} bd^\# + ca^\pi b(d^\#)^{n+3} - ca^\# b(d^\#)^{n+2}$, *for* $n \geq 0$.

(ii) *If* $ca = 0$, $(-aa^\# bd^\# - bd^\#)dc + a^\pi bc = 0$, *and* $cb = 0$, *then* $x \in \mathcal{A}^d$ *and*

$$x^d = \begin{bmatrix} a^\# + j' & (a^\#)^2 bd^\pi + a^\pi b(d^\#)^2 - a^\# bd^\# \\ (d^\#)^2 c & d^\# \end{bmatrix},$$

where $j' = (a^\#)^3 bd^\pi c - (a^\#)^2 bd^\# c + a^\pi b(d^\#)^3 c - a^\# b(d^\#)^2 c$.

Acknowledgements The author is grateful to the referee for constructive comments towards improvement of the original version of this paper. The author is supported by the Ministry of Education and Science, Republic of Serbia, Grant No. 174007.

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