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Existence and Multiplicity of Solutions for $p(x)$-Laplacian Equations in $\mathbb{R}^N$

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Abstract This article concerns the existence and multiplicity of solutions to a class of $p(x)$-Laplacian-like equations. We introduce a revised Ambrosetti–Rabinowitz condition, and show that the problem has a nontrivial solution and infinitely many solutions, respectively.

Keywords $p(x)$-Laplacian-like operator · Variational method · Radial solution · Ambrosetti–Rabinowitz condition

Mathematics Subject Classification 35J60 · 35J20 · 47J30 · 58E30

1 Introduction

Capillarity can be briefly explained by considering the effects of two opposing forces: adhesion, i.e. the attractive (or repulsive) force between the molecules of the liquid and those of the container; and cohesion, i.e. the attractive force between the molecules of the liquid. The study of capillary phenomena has gained some attention recently.

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This increasing interest is motivated not only by fascination in naturally occurring phenomena such as motion of drops, bubbles and waves but also its importance in applied fields ranging from industrial and biomedical and pharmaceutical to microfluidic systems. Ni and Serrin [13] initiated the study of ground states for equations of the form

$$-\mathbf{div}\left(\frac{|\nabla u|}{\sqrt{1+|\nabla u|^2}}\right)=f(u),\quad \text{in } \mathbb{R}^N$$

with very general right-hand side f.

Recently, the study of various mathematical problems with variable exponent growth condition has received considerable attention in recent years; see e.g. [6,7,11,12,14]. For background information, we refer the reader to [17,18]. The aim of this paper is to discuss the existence and multiplicity of solutions of the following $p(x)$-Laplacian-like equation in $\mathbb{R}^N$:

$$-\mathbf{div}\left(\left(1+\frac{|\nabla u|^{p(x)}}{\sqrt{1+\left|\nabla u\right|^{2p(x)}}}\right)\left|\nabla u\right|^{p(x)-2}\nabla u\right)+|u|^{p(x)-2}u=K(x)f(u),$$
$$\text{in } \mathbb{R}^N,\quad u\in W^{1,p(x)}\left(\mathbb{R}^N\right), \tag{1.1}$$

where $p(x)=p(|x|)\in C\left(\mathbb{R}^N\right)$ with $2\leq N<p^-:=\inf_{\mathbb{R}^N}p(x)\leq p^+:=\sup_{\mathbb{R}^N}p(x)<+\infty$, $K:\mathbb{R}^N\to\mathbb{R}$ is a measurable function and $f\in C(\mathbb{R},\mathbb{R})$. Recently, the following equation also has been studied very well:

$$-\Delta_{p(x)}u+|u|^{p(x)-2}u=f(x,u),\quad \text{in } \mathbb{R}^N, u\in W^{1,p(x)}\left(\mathbb{R}^N\right). \tag{1.2}$$

when $p(x)=p(|x|)\in C\left(\mathbb{R}^N\right)$ with $2\leq N<p^-\leq p^+<+\infty$, the authors in [2] proved the existence of infinitely many distinct homoclinic radially symmetric solutions for (1.2), under adequate hypotheses about the nonlinearity at zero (and at infinity). For $p(x)$-Laplacian-like operator, Rodrigues [16] established the existence of nontrivial solutions for problem (1.1) on bounded area under the case of superlinear, by assuming the following key condition:

(F1′) there exist $\theta>p^+$ and $M>0$ such that

$$0<\theta F(t):=\theta\int_0^t f(s)\mathrm{d}s\leq f(t)t,\quad \forall|t|\geq M.$$

This condition is originally due to Ambrosetti and Rabinowitz [1] in the case $p(x)\equiv 2$. Actually, condition (F1′) is quite natural and important not only to ensure that the Euler–Lagrange functional associated to problem (1.2) has a mountain pass geometry, but also to guarantee that Palais–Smale sequence of the Euler–Lagrange functional is bounded. But this condition is very restrictive eliminating many nonlinearities. In this paper, we introduce a new condition (F1) (motivated by [10]), below, which is different from the Ambrosetti–Rabinowitz-type condition (F1′).

(F1) there exist a constant $M \geq 0$ and a decreasing function τ in the space $C(\mathbb{R} \setminus (-M, M), \mathbb{R})$, such that

$$0 < (p^+ + \tau(t))F(t) := (p^+ + \tau(t)) \int_0^t f(s)\mathrm{d}s \leq f(t)t, \quad |t| \geq M,$$

where $\tau(t) > 0$, $\lim_{|t|\to+\infty} |t|\tau(t) = +\infty$ and $\lim_{|t|\to+\infty} \int_M^{|t|} \frac{\tau(s)}{s}\mathrm{d}s = +\infty$.

Remark 1.1 Obviously, when $\inf_{|t|\geq M} \tau(t) > 0$, conditions (F1) and (F1′) are equivalent. However, condition (F1) is weaker than (F1′) when $\inf_{|t|\geq M} \tau(t) = 0$. For example, let $|t| \geq M = 2$, and assume that $F(t) = |t|^{p^+} \ln|t|$. Then $f(t) = (p^+ + \tau(t))\mathrm{sgn}(t)|t|^{p^+-1}\ln|t|$ satisfies condition (F1) not (F1′), where $\tau(t) = \frac{1}{\mathrm{lnt}} \in C(\mathbb{R} \setminus (-M, M), \mathbb{R})$.

Remark 1.2 Condition (F1) was introduced in [10] to study p-Laplacian equation in $\mathbb{R}^N$. We can see that this new condition (F1) can also study $p(x)$-Laplacian-like equation and another situation with $p^- > N$ when compared with the reference [16].

The aim of this paper is twofold. First, we want to handle the case when $p^- > N$ and the unbounded area $\mathbb{R}^N$. Although important problems can be treated within this framework, only a few works are available in this direction, see [2]. The main difficulty in studying problem (1.1) lies in the fact that no compact embedding is available for $W^{1,p(x)}(\mathbb{R}^N) \hookrightarrow L^\infty(\mathbb{R}^N)$. However, the subspace of radially symmetric functions of $W^{1,p(x)}(\mathbb{R}^N)$, denoted further by $W_r^{1,p(x)}(\mathbb{R}^N)$, can be embedded compactly into $L^\infty(\mathbb{R}^N)$ whenever $N < p^- \leq p^+ < +\infty$ (cf. [2, Theorem 2.1]). Second, instead of some usual assumption on the nonlinear term f, we assume that it satisfies a modified Ambrosetti–Rabinowitz-type condition (F1).

To state our results, we first introduce the following assumptions:

(**H1**) $K \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ is radial, $K(x) \geq 0$ for any $x \in \mathbb{R}^N$ and $\sup_{d>0} \operatorname{ess\,inf}_{|x|\leq d} K(x) > 0$.

(**H2**) $f(t) = o(t^{p^+-1})$ for t near 0.

Now, we are ready to state the main result of this paper.

Theorem 1.3 *Suppose that* (H1)*,* (H2) *and* (F1) *hold. Then problem* (1.1) *has a nontrivial radially symmetric solution. Furthermore, if $f(t) = f(-t)$, then problem* (1.1) *has infinitely many pairs of radially symmetric solutions.*

In the remainder of this section, we recall some definitions and basic properties of variable spaces $L^{p(x)}(\mathbb{R}^N)$ and $W^{1,p(x)}(\mathbb{R}^N)$. For a deeper treatment on these spaces, we refer to [4,5].

Let $p \in L^\infty(\mathbb{R}^N)$, $p^- > 1$. The variable exponent Lebesgue space $L^{p(x)}(\mathbb{R}^N)$ is defined by

$$L^{p(x)}(\mathbb{R}^N) = \left\{u\colon \mathbb{R}^N \to \mathbb{R}\colon u \text{ is measurable and } \int_{\mathbb{R}^N} |u|^{p(x)}\mathrm{d}x < +\infty\right\}$$

endowed with the norm $|u|_{p(x)} \left\{\lambda > 0\colon \int_{\mathbb{R}^N} |\frac{u}{\lambda}|^{p(x)} \mathrm{d}x \le 1\right\}$. Then we define the variable exponent Sobolev space

$$W^{1,p(x)}(\mathbb{R}^N) = \left\{u \in L^{p(x)}(\mathbb{R}^N) : |\nabla u| \in L^{p(x)}(\mathbb{R}^N)\right\}$$

with the norm $\|u\| = |u|_{p(x)} + |\nabla u|_{p(x)}$.

Proposition 1.4 *([3]) Set* $\psi(u) = \int_{\mathbb{R}^N} (|\nabla u(x)|^{p(x)} + |u(x)|^{p(x)})\mathrm{d}x$. *If* $u, u_k \in W^{1,p(x)}(\mathbb{R}^N)$, *then*

1. $\|u\| < 1 (= 1; > 1) \Leftrightarrow \psi(u) < 1 (= 1; > 1)$;
2. *If* $\|u\| > 1$, *then* $\|u\|^{p^-} \le \psi(u) \le \|u\|^{p^+}$;
3. *If* $\|u\| < 1$, *then* $\|u\|^{p^+} \le \psi(u) \le \|u\|^{p^-}$;
4. $\lim_{k\to+\infty} \|u_k\| = 0 \Leftrightarrow \lim_{k\to+\infty} \psi(u_k) = 0$;

2 Proof of Theorem 1.3

In this section, we prove Theorem 1.3 when $\inf_{|t|\ge M} \tau(t) = 0$. If $\inf_{|t|\ge M} \tau(t) > 0$, then conditions (F1′) and (F1) are equivalent, and the proof is rather standard. We may assume that $M \ge 1$, and that there is constant $N_0 > 0$ such that

$$|\tau(t)| \le N_0 \tag{2.1}$$

for all $t \in \mathbb{R} \setminus (-M, M)$.

We introduce the energy functional φ associated to problem (1.1) defined by

$$\begin{aligned} \varphi(u) = &\int_{\mathbb{R}^N} \frac{1}{p(x)} \left(|\nabla u(x)|^{p(x)} + \sqrt{1 + |\nabla u(x)|^{2p(x)}} + |u(x)|^{p(x)}\right) \mathrm{d}x \\ &- \int_{\mathbb{R}^N} K(x)F(u)\mathrm{d}x \quad u \in W_r^{1,p(x)}(\mathbb{R}^N). \end{aligned}$$

Due to the principle of symmetric criticality of Palais (see [8]), the critical points of $\varphi|_{W_r^{1,p(x)}(\mathbb{R}^N)}$ are critical points of φ as well, so radially symmetric weak solutions of problem (1.1).

Claim 2.1 *Let* $W = \{w \in W_r^{1,p(x)}(\mathbb{R}^N)\colon \|w\| = 1\}$. *Then, for any* $w \in W$, *there exist* $\delta_w > 0$ *and* $\lambda_w > 0$, *such that*

$$\varphi(\lambda v) < 0, \quad \forall v \in W \cap B(w, \delta_w), \forall |\lambda| \ge \lambda_w,$$

where $B(w, \delta_w) = \{v \in W_r^{1,p(x)}(\mathbb{R}^N)\colon \|v - w\| < \delta_w\}$.

Proof Since the embedding $W_r^{1,p(x)}(\mathbb{R}^N) \hookrightarrow L^\infty(\mathbb{R}^N)$ is compact, there is constant $C > 0$ such that $|u|_\infty \le C\|u\|$. Thus, for all $w \in W$ and a.e. $x \in \mathbb{R}^N$, we have $|w(x)| \le C$. By the definition of $\tau(t)$ and decreasing property of $\tau(t)$, we deduce that there exists $t_\lambda \in \{t \in \mathbb{R}\colon M \le |t| \le |\lambda|C\}$ such that $\tau(t_\lambda) = \min_{M\le|t|\le|\lambda|C} \tau(t)$.

Then $|\lambda| \geq \frac{t_\lambda}{C}$ and $\lim_{|\lambda|\to+\infty}|t_\lambda| \to +\infty$. From condition (F1), we conclude that $F(t) \geq C_1|t|^{p^+}H(|t|)$ for all $|t| \geq M$, where $H(t) = \exp\left(\int_M^{|t|}\frac{\tau(s)}{s}\mathrm{d}s\right)$. Hence, using $\lim_{|t|\to+\infty}\int_M^{|t|}\frac{\tau(s)}{s}\mathrm{d}s = +\infty$, it follows that $H(|t|)$ increases when $|t|$ increases, and $\lim_{|t|\to+\infty} H(|t|) = +\infty$.

Fix $w \in W$. By $\|w\| = 1$, we deduce that $\mu(\{x \in \mathbb{R}^N\colon w(x) \neq 0\}) > 0$, and that there exists a $\bar{t}_w > M$ such that $\mu(\{x \in \mathbb{R}^N\colon |\bar{t}_w w(x)| \geq M\}) > 0$, where μ is the Lebesgue measure.

Set $\Omega_1 := \{x \in \mathbb{R}^N : |\bar{t}_w w(x)| \geq M\}$ and $\Omega_2 := \mathbb{R}^N\backslash\Omega_1$. Then $\mu(\Omega_1) > 0$. Therefore, for any $x \in \Omega_1$, we have that $|w(x)| \geq \frac{M}{\bar{t}_w}$. Now take $\delta_w = \frac{M}{2C\bar{t}_w}$. Then, for any $v \in W\cap B(w, \delta_w)$, $|v-w|_\infty \leq C\|v-w\| < \frac{M}{2\bar{t}_w}$. Hence, for all $x \in \Omega_1$, we deduce that $|v(x)| \geq \frac{M}{2\bar{t}_w}$ and $|\lambda v(x)| \geq M$ for any $x \in \Omega_1$ and $\lambda \in \mathbb{R}$ with $|\lambda| \geq 2\bar{t}_w$. Thus, for $|\lambda| \geq 2\bar{t}_w$, by the above estimates and $H(|t|)$ increases when $|t|$ increases, we have

$$\begin{aligned}\int_{\Omega_1} K(x)F(\lambda v(x))\mathrm{d}x &\geq C_1|\lambda|^{p^+}\int_{\Omega_1} K(x)|v(x)|^{p^+}H(|\lambda v(x)|)\mathrm{d}x\\ &\geq C_1|\lambda|^{p^+}\left(\frac{M}{2\bar{t}_w}\right)^{p^+} H\left(|\lambda|\frac{M}{2\bar{t}_w}\right)\int_{\Omega_1} K(x)\mathrm{d}x. \end{aligned}\tag{2.2}$$

On the other hand, by continuity, we deduce that there exists a $C_2 > 0$ such that $F(t) \geq -C_2$ when $|t| \leq M$. Note that $F(t) > 0$ if $|t| \geq M$. Hence,

$$\begin{aligned}\int_{\Omega_2} K(x)F(\lambda v(x))\mathrm{d}x &= \int_{\Omega_2\cup\{x\in\mathbb{R}^N:|\lambda v(x)|\geq M\}} K(x)F(\lambda v(x))\mathrm{d}x\\ &\quad+\int_{\Omega_2\cup\{x\in\mathbb{R}^N:|\lambda v(x)|\leq M\}} K(x)F(\lambda v(x))\mathrm{d}x\\ &\geq \int_{\Omega_2\cup\{x\in\mathbb{R}^N:|\lambda v(x)|\leq M\}} K(x)F(\lambda v(x))\mathrm{d}x\\ &\geq -C_2|K|_1. \end{aligned}\tag{2.3}$$

Hence, for $v \in W \cap B(w, \delta_w)$ and $|\lambda| > 1$, from (2.2) to (2.3), we have

$$\begin{aligned}\varphi(\lambda v) &= \int_{\mathbb{R}^N}\frac{|\lambda|^{p(x)}}{p(x)}\left(\left|\nabla v\right|^{p(x)} + \sqrt{1+\left|\nabla v\right|^{2p(x)}} + |v|^{p(x)}\right)\mathrm{d}x\\ &\quad-\int_{\mathbb{R}^N} K(x)F(\lambda v(x))\mathrm{d}x\\ &\leq 2|\lambda|^{p^+} - C_1|\lambda|^{p^+}\left(\frac{M}{2\bar{t}_w}\right)^{p^+} H\left(|\lambda|\frac{M}{2\bar{t}_w}\right)\int_{\Omega_1} K(x)\mathrm{d}x + C_2|K|_1\\ &= |\lambda|^{p^+}\left[2 - C_1\left(\frac{M}{2\bar{t}_w}\right)^{p^+} H\left(|\lambda|\frac{M}{2\bar{t}_w}\right)\int_{\Omega_1} K(x)\mathrm{d}x\right] + C_2|K|_1 \to -\infty,\end{aligned}$$

as $|\lambda| \to +\infty$, because $\lim_{|t|\to+\infty} H(|t|) = +\infty$. □

Claim 2.2 *There exist* $\nu > 0$ *and* $\rho > 0$ *such that* $\inf_{\|u\|=\nu} \varphi(u) \geq \rho > 0$.

Proof Note that $|u|_\infty \to 0$ if $\|u\| \to 0$. Then, by hypothesis (H2), we have

$$\int_{\mathbb{R}^N} K(x)F(u)\mathrm{d}x = |K|_1 o\big(|u|_\infty^{p^+}\big) = |K|_1 o\big(\|u\|^{p^+}\big),$$

which implies

$$\begin{aligned}\varphi(u) = &\int_{\mathbb{R}^N} \frac{1}{p(x)}\left(\left|\nabla u(x)\right|^{p(x)} + \sqrt{1+\left|\nabla u\right|^{2p(x)}} + |u(x)\|^{p(x)}\right)\mathrm{d}x \\ &- \int_{\mathbb{R}^N} K(x)F(u)\mathrm{d}x \geq \frac{2}{p^+}\|u\|^{p^+} - |K|_1 o\big(\|u\|^{p^+}\big).\end{aligned}$$

Therefore, there exist $1 > \nu > 0$ and $\rho > 0$ such that $\inf_{\|u\|=\nu} \varphi(u) \geq \rho > 0$. □

Claim 2.3 *The functional* φ *satisfies the (PS) condition.*

Proof Let $\{u_n\} \subset W_r^{1,p(x)}(\mathbb{R}^N)$ be a (PS) sequence of the functional φ; that is, $|\varphi(u_n)| \leq c$ and $|\langle \varphi'(u_n), h\rangle| \leq \varepsilon_n \|h\|$ with $\varepsilon_n \to 0$, for all $h \in W_r^{1,p(x)}(\mathbb{R}^N)$. We will prove that the sequence $\{u_n\}$ is bounded in $W_r^{1,p(x)}(\mathbb{R}^N)$. Indeed, if $\{u_n\}$ is unbounded in $W_r^{1,p(x)}(\mathbb{R}^N)$, we may assume that $\|u_n\| \to \infty$ as $n \to \infty$. Let $u_n = \lambda_n w_n$, where $\lambda_n \in \mathbb{R}$, $w_n \in W$. It follows that $|\lambda_n| \to \infty$.

Let $\Omega_1^n := \{x \in \mathbb{R}^N : |\lambda_n w_n(x)| \geq M\}$ and $\Omega_2^n := \mathbb{R}^N \backslash \Omega_1^n$. Then

$$\begin{aligned}-\varepsilon_n|\lambda_n| = -\varepsilon_n\|u_n\| &\leq \langle \varphi'(u_n), u_n\rangle \\ &= \int_{\mathbb{R}^N}\left(\left|\nabla u_n\right|^{p(x)} + \frac{\left|\nabla u_n\right|^{2p(x)}}{\sqrt{1+|\nabla u_n|^{2p(x)}}} + |u_n|^{p(x)}\right)\mathrm{d}x \\ &\quad - \int_{\mathbb{R}^N} K(x) f(u_n)u_n \mathrm{d}x \\ &\leq \int_{\mathbb{R}^N} |\lambda_n|^{p(x)}\left(|\nabla w_n|^{p(x)} + \frac{|\nabla w_n|^{2p(x)}}{\sqrt{1+|\nabla w_n|^{2p(x)}}} + |w_n|^{p(x)}\right) \\ &\quad - \int_{\Omega_1^n} K(x) f(\lambda_n w_n)\lambda_n w_n \mathrm{d}x - \int_{\Omega_2^n} K(x) f(\lambda_n w_n)\lambda_n w_n \mathrm{d}x,\end{aligned}$$

which implies that

$$\begin{aligned}&\int_{\Omega_1^n} K(x) f(\lambda_n w_n)\lambda_n w_n \,\mathrm{d}x \\ &\quad \leq \int_{\mathbb{R}^N} |\lambda_n|^{p(x)}\left(\left|\nabla w_n\right|^{p(x)} + \frac{\left|\nabla w_n\right|^{2p(x)}}{\sqrt{1+|\nabla w_n|^{2p(x)}}} + |w_n|^{p(x)}\right)\mathrm{d}x \\ &\quad\quad + \varepsilon_n|\lambda_n| - \int_{\Omega_2^n} K(x) f(\lambda_n w_n)\lambda_n w_n \mathrm{d}x.\end{aligned}$$

Note that $0 < (p^+ + \tau(t_{\lambda_n}))F(\lambda_n w_n) \le f(\lambda_n w_n)\lambda_n w_n$ in Ω_1^n. So,

$$\int_{\Omega_1^n} K(x)F(\lambda_n w_n)\mathrm{d}x \le \frac{1}{p^+ + \tau(t_{\lambda_n})}\int_{\Omega_1^n} K(x)f(\lambda_n w_n)\lambda_n w_n \mathrm{d}x.$$

Then, by (2.1), it follows that

$$\begin{aligned}
\varphi(u_n) &= \varphi(\lambda_n w_n)\\
&= \int_{\mathbb{R}^N} \frac{|\lambda_n|^{p(x)}}{p(x)}\left(\left|\nabla w\right|^{p(x)} + \sqrt{1+|\nabla w|^{2p(x)}} + |w|^{p(x)}\right)\mathrm{d}x\\
&\quad - \int_{\mathbb{R}^N} K(x)F(\lambda_n w_n)\mathrm{d}x\\
&= \int_{\mathbb{R}^N} \frac{|\lambda_n|^{p(x)}}{p(x)}\left(\left|\nabla w\right|^{p(x)} + \sqrt{1+\left|\nabla w\right|^{2p(x)}} + |w|^{p(x)}\right)\mathrm{d}x\\
&\quad - \int_{\Omega_1^n} K(x)F(\lambda_n w_n)\mathrm{d}x - \int_{\Omega_2^n} K(x)F(\lambda_n w_n)\mathrm{d}x\\
&\ge \frac{1}{p^+}\int_{\mathbb{R}^N} |\lambda_n|^{p(x)}\left(\left|\nabla w\right|^{p(x)} + \sqrt{1+\left|\nabla w\right|^{2p(x)}} + |w|^{p(x)}\right)\mathrm{d}x\\
&\quad - \frac{1}{p^+ + \tau(t_{\lambda_n})}\int_{\Omega_1^n} K(x)f(\lambda_n w_n)\lambda_n w_n\mathrm{d}x - \int_{\Omega_2^n} K(x)F(\lambda_n w_n)\mathrm{d}x\\
&\ge \frac{1}{p^+}\int_{\mathbb{R}^N} |\lambda_n|^{p(x)}\left(2\left|\nabla w_n\right|^{p(x)} + |w_n|^{p(x)}\right)\mathrm{d}x\\
&\quad - \frac{1}{p^+ + \tau(t_{\lambda_n})}\left[\int_{\mathbb{R}^N} |\lambda_n|^{p(x)}\left(2\left|\nabla w_n\right|^{p(x)} + |w_n|^{p(x)}\right)\mathrm{d}x + \varepsilon_n|\lambda_n|\right]\\
&\quad + \frac{1}{p^+ + \tau(t_{\lambda_n})}\int_{\Omega_2^n} K(x)f(\lambda_n w_n)\lambda_n w_n\mathrm{d}x - \int_{\Omega_2^n} K(x)F(\lambda_n w_n)\mathrm{d}x\\
&= \frac{\tau(t_{\lambda_n})}{p^+(p^+ + \tau(t_{\lambda_n}))}\int_{\mathbb{R}^N} |\lambda_n|^{p(x)}\left(2\left|\nabla w_n\right|^{p(x)} + |w_n|^{p(x)}\right)\mathrm{d}x\\
&\quad - \frac{1}{p^+ + \tau(t_{\lambda_n})}\varepsilon_n|\lambda_n| + T(\lambda_n w_n)\\
&\ge \frac{\tau(t_{\lambda_n})}{p^+(p^+ + N_0)}|\lambda_n|^{p^-} - \frac{1}{p^+}\varepsilon_n|\lambda_n| + T(\lambda_n w_n)\\
&= |\lambda_n|\left[\frac{|\lambda_n|^{p^- -1}\tau(t_{\lambda_n})}{p^+(p^+ + N_0)} - \frac{\varepsilon_n}{p^+}\right] + T(\lambda_n w_n)\\
&\ge |\lambda_n|\left[\frac{|\lambda_n|^{p^- -1}\tau(t_{\lambda_n})}{p^+(p^+ + N_0)} - \frac{\varepsilon_n}{p^+}\right] - C_2,
\end{aligned}$$

where

$$T(\lambda_n w_n) = \frac{1}{p^+ + \tau(t_{\lambda_n})}\int_{\Omega_2^n} K(x)f(\lambda_n w_n)\lambda_n w_n\,\mathrm{d}x - \int_{\Omega_2^n} K(x)F(\lambda_n w_n)\,\mathrm{d}x$$

is bounded from below. We know that $|\lambda_n| \to +\infty$, and so $|t_{\lambda_n}| \to +\infty$, as $n \to +\infty$. It follows from (F1) and $p^- > N \geq 2$ that

$$\lim_{n\to+\infty} |\lambda_n|^{p^- -1}\tau(t_{\lambda_n}) \geq \lim_{n\to+\infty} \frac{|t_{\lambda_n}|\tau(t_{\lambda_n})}{M} = +\infty.$$

This means that $\lim_{n\to+\infty} \varphi(u_n) \to +\infty$. This is a contradiction. So, the sequence $\{u_n\}$ is bounded in $W_r^{1,p(x)}(\mathbb{R}^N)$. Note that the embedding $W_r^{1,p(x)}(\mathbb{R}^N) \hookrightarrow L^\infty(\mathbb{R}^N)$ is compact, there exists a $u \in W_r^{1,p(x)}(\mathbb{R}^N)$ such that passing to subsequence, still denoted by $\{u_n\}$, it converges strongly to u in $L^\infty(\mathbb{R}^N)$, and in the same way as the proof of [9, Proposition 3.1], we can conclude that u_n converges strongly also in $W_r^{1,p(x)}(\mathbb{R}^N)$. Thus, φ satisfies the (PS) condition. □

Proof of Theorem 1.3 Due to Claims 2.1, 2.2 and 2.3, we know that φ satisfies the conditions of the classical mountain pass theorem due to Ambrosetti and Rabinowitz [1]. Hence, we obtain a nontrivial critical point, which gives rise to a nontrivial radially symmetric solution to problem (1.1).

Furthermore, if $f(t) = f(-t)$, then φ is even. We will use the following $\mathbb{Z}_2$ version of the mountain pass theo in [15]. □

Theorem 2.4 *Let E be an infinite-dimensional Banach space, and $\varphi \in C(E, \mathbb{R})$ be even, satisfying the (PS) condition, and having $\varphi(0) = 0$. Assume that $E = V \oplus X$, where V is finite dimensional. Suppose that the following hold.*

(a) There are constants $\nu, \rho > 0$ such that $\inf_{\partial B_\nu \cup X} \varphi \geq \rho$.
(b) For each finite-dimensional subspace $\hat{E} \subset E$, there is an $\sigma = \sigma(\hat{E})$ such that $\varphi \leq 0$ on $\hat{E}\backslash B_\sigma$.

Then φ possesses an unbounded sequence of critical values.

From Claims 2.1 and 2.2, φ satisfies (a) and the (PS) condition. For any finite-dimensional subspace $\hat{E} \subset E$, $S \cap \hat{E} = \{w \in \hat{E} : \|w\| = 1\}$ is compact. By Claim 2.1 and the finite covering theo, it is easy to verify that φ satisfies condition (b). Hence, by the $\mathbb{Z}_2$ version of the mountain pass theo, φ has a sequence of critical points $\{u_n\}_{n=1}^\infty$. That is, problem (1.1) has infinitely many pairs of radially symmetric solutions.

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