Bull. Malays. Math. Sci. Soc. (2016) 39:773–793
DOI 10.1007/s40840-015-0139-8

BULLETIN *of the* MALAYSIAN MATHEMATICAL SCIENCES SOCIETY
http://www.springer.com/mathematics/journal/40840

Viscosity Approximation Methods for Zeros of Accretive Operators and Fixed Point Problems in Banach Spaces

Pongsakorn Sunthrayuth[1] · Yeol Je Cho[2] · Poom Kumam[1]

Received: 1 March 2014 / Revised: 23 June 2014 / Published online: 16 May 2015

Abstract In this paper, we introduce iterative algorithms for finding a zero of set-valued accretive operators by the viscosity approximation method based on Meir–Keeler-type contractions in a reflexive Banach space which admits a weakly continuous duality mapping. We obtain some strong convergence theorems under suitable conditions. As applications, we apply our results for finding common fixed point of nonexpansive semigroups and for solving equilibrium problem, optimization problem, and variational inequalities.

Keywords Viscosity approximation · Accretive operator · Strong convergence · Meir–Keeler-type contraction · Reflexive Banach spaces

Mathematics Subject Classification 47H09 · 47H10 · 47H20 · 47J20 · 47J25

Communicated by Mohammad Sal Mosleihan.

✉ Pongsakorn Sunthrayuth
sci_math@hotmail.com

Yeol Je Cho
yjcho@gnu.ac.kr

Poom Kumam
poom.kum@kmutt.ac.th

1 Department of Mathematics, Faculty of Science, King Mongkut's University of Technology Thonburi (KMUTT), Bang Mod, Thrung Khru, Bangkok 10140, Thailand

2 Department of Mathematics Education and the RINS, Gyeongsang National University, Jinju 660-701, Republic of Korea

1 Introduction

Throughout this paper, we denote by X and X^* a real Banach space and the dual space of X, respectively. The *duality mapping* $J : X \longrightarrow 2^{X^*}$ is defined by

$$J(x) = \{f \in X^* : \langle x, f\rangle = \|x\|^2, \|f\| = \|x\|\},$$

where $\langle\cdot,\cdot\rangle$ denotes the duality pairing between X and X^*. If $X := H$ is a real Hilbert space, then $J = I$ where I is the identity mapping. It is well known that if X is smooth, then J is single-valued, which is denoted by j (see [30]). Let C be a nonempty, closed and convex subset of X and T be a self-mapping on C. We denote the fixed points set of the mapping S by $Fix(T) = \{x \in C : Tx = x\}$. A mapping $T : C \longrightarrow C$ is said to be *L-Lipschitzian* if there exists a constant $L > 0$ such that

$$\|Tx - Ty\| \leq L\|x - y\|, \quad \forall x, y \in C.$$

If $0 < L < 1$, then T is a contraction and if $L = 1$, then T is a nonexpansive mapping.

Let $A : X \longrightarrow 2^X$ be a set-valued mapping. We denote $D(A)$ by domain of A, that is, $D(A) = \{x \in X : Ax \neq \emptyset\}$. A set-valued mapping $A : D(A) \subset X \longrightarrow 2^X$ is said to be *accretive* if for all $x, y \in D(A)$ there exist $j(x-y) \in J(x-y)$ such that

$$\langle u - v, j(x-y)\rangle \geq 0 \text{ for } u \in Ax \text{ and } v \in Ay.$$

In a Hilbert space, an accretive operator is also called *monotone*. Let $A : D(A) \subset X \longrightarrow 2^X$ be an accretive mapping, we can define a single-valued mapping $J_r^A : X \longrightarrow D(A)$ by $J_r^A = (I + rA)^{-1}$, which is called the *resolvent operator* associated with A, where $r > 0$ and also denote $A^{-1}0$ by the set of zeros of A, that is, $A^{-1}0 = \{x \in D(A) : 0 \in Ax\}$. It is well known that J_r^A is nonexpansive and $Fix(J_r^A) = \{x \in X : J_r^A x = x\}$ (see [30]). An operator A is called *m-accretive* if it is accretive and $R(I+rA)$, range of $I+rA$, is X for all $r > 0$; and A is said to satisfy the range condition if $\overline{D(A)} = R(I + rA)$, $\forall r > 0$, where $\overline{D(A)}$ denotes the closure of the domain of A.

Interest in accretive mappings stems mainly from their firm connection with equations of evolution. It is known (see, e.g., [1]) that many physically significant problems can be modeled by initial-value problems of the form

$$x'(t) + Ax(t) = 0, \quad x(0) = x_0, \tag{1.1}$$

where A is an accretive operator in an appropriate Banach space. Typical examples where such evolution equations occur can be found in the heat, wave, or Schrödinger equations. One of the fundamental results in the theory of accretive operators, due to Browder [2], states that if A is locally Lipschitzian and accretive then A is m-accretive. This result was subsequently generalized by Martin [3] to the continuous accretive operators. If in (1.1) $x(t)$ is independent of t, then (1.1) reduces to $Au = 0$ whose solutions correspond to the equilibrium points of system (1.1). Consequently, considerable research effects have been devoted, especially within the past 20 years or so, to iterative methods for approximating these equilibrium points.

In recent years, many authors have constructed the several iterative methods which is related fixed points problems in several settings, see, e.g., ([4–13,16–21]).

In 1974, Bruck [22] introduced an iteration process and proved, in Hilbert space setting, the convergence of the process to a zero of a maximal monotone operator. In [23], Reich extended this result to uniformly smooth Banach spaces provided that the operator is m-accretive. By the inspiration of the regularization method for Rockafellars proximal point algorithm [40] and the iterative methods of Halpern [24], in 2003, Benavides et al. [25] studied the Halpern type iteration process (1.2) to find a zero of an m-accretive operator A in a uniformly smooth Banach space with a weakly continuous duality mapping J_φ with gauge function φ in virtue of the resolvent $J_r = (I + rA)^{-1}$ of A for all $r > 0$:

$$x_{n+1} = \alpha_n u + (1 - \alpha_n) J_{r_n} x_n, \quad \forall n \geq 1. \tag{1.2}$$

On the other hand, Takahashi [14] introduced the following proximal point algorithm in a reflexive Banach space with a uniformly Gâteaux differentiable norm by the viscosity approximation method:

$$x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n) J_{r_n} x_n, \quad \forall n \geq 1, \tag{1.3}$$

where $f : C \longrightarrow C$ is a contraction and $J_r = (I + rA)^{-1}$ is a resolvent of A for all $r > 0$. Under some mild conditions on the parameters $\{\alpha_n\}$ and $\{r_n\}$, he proved that the sequence $\{x_n\}$ defined by (1.3) converges strongly to a point in $A^{-1}0$.

Later, Petruşel and Yao [15] also studied strong convergence theorem of proximal point algorithm (1.3) by the viscosity approximation method with a generalized contraction mapping f.

Recently, Song et al. [17] studied the following strong convergence of the proximal point algorithm in a reflexive Banach space which admits a weakly sequentially continuous duality mapping:

$$x_{n+1} = J_{r_n}(\alpha_n u + (1 - \alpha_n) x_n), \quad \forall n \geq 1, \tag{1.4}$$

where $\{\alpha_n\} \subset (0, 1)$, $\{r_n\} \subset (0, \infty)$ and $J_r = (I + rA)^{-1}$ is a resolvent of A for all $r > 0$.

In this paper, motivated by Petruşel and Yao [15] and Song et al. [17], we introduce iterative algorithms for finding a zero of set-valued accretive operators by the viscosity approximation method based on Meir–Keeler-type contractions in a reflexive Banach space which admits a weakly continuous duality mapping. We obtain some strong convergence theorems under suitable conditions. As applications, we apply our results for finding common fixed point of nonexpansive semigroups and for solving equilibrium problem, optimization problem, and variational inequalities.

2 Preliminaries

A Banach space X is said to be *strictly convex* if $\frac{\|x+y\|}{2} < 1$ for all $x, y \in X$ with $\|x\| = \|y\| = 1$ and $x \neq y$. A Banach space X is said to be *uniformly convex* if for each $\epsilon > 0$, there exists $\delta > 0$ such that for $x, y \in X$ with $\|x\|, \|y\| \leq 1$ and $\|x - y\| \geq \epsilon$, $\frac{\|x+y\|}{2} \leq 1 - \delta$ holds. It is well known that a uniformly convex Banach space is reflexive and strictly convex (see [30]). Let $S(X) = \{x \in X : \|x\| = 1\}$ denote the unit sphere of a Banach space X. The norm of X is said to be *Gâteaux differentiable* (or X is said to be smooth) if the limit

$$\lim_{t \longrightarrow 0} \frac{\|x + ty\| - \|x\|}{t} \tag{2.1}$$

exists for each $x, y \in S(X)$. The norm of X is said to be *uniformly Gâteaux differentiable*, if for each $y \in S(X)$, the limit (2.1) is attained uniformly for $x \in S(X)$.

Let $\varphi : [0, \infty) := \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ be a continuous strictly increasing function such that $\varphi(0) = 0$ and $\varphi(t) \longrightarrow \infty$ as $t \longrightarrow \infty$. This function φ is called a *gauge function*. The duality mapping $J_\varphi : X \longrightarrow 2^{X^*}$ associated with a gauge function φ is defined by

$$J_\varphi(x) = \{f^* \in X^* : \langle x, f^* \rangle = \|x\|\varphi(\|x\|),\ \|f^*\| = \varphi(\|x\|),\ \ \forall x \in X\},$$

where $\langle \cdot, \cdot \rangle$ denotes the generalized duality paring. In particular, the duality mapping with the gauge function $\varphi(t) = t$, denoted by J is referred to as the normalized duality mapping. Clearly, there holds the relation $J_\varphi(x) = \frac{\varphi(\|x\|)}{\|x\|} J(x)$ for each $x \neq 0$ (see [31]).

Browder [31] initiated the study of certain classes of nonlinear operators by means of the duality mapping J_φ. Following Browder [31], we say Banach space X has a *weakly continuous duality mapping* if there exits a gauge function φ for which the duality mapping $J_\varphi(x)$ is single-valued and continuous from the weak topology to the weak* topology, that is, for each $\{x_n\}$ with $x_n \rightharpoonup x$, the sequence $\{J(x_n)\}$ converges weakly* to $J_\varphi(x)$.

A Banach space X is said to satisfy *Opials condition* if, for any sequence $\{x_n\}$ in X, $x_n \rightharpoonup x$ implies

$$\limsup_{n \longrightarrow \infty} \|x_n - x\| < \limsup_{n \longrightarrow \infty} \|x_n - y\|, \ \ \forall y \in X \ \text{ with } \ x \neq y.$$

By Theorem 3.2.8 of [29], we know that, if X admits the weakly continuous duality mapping J_φ with gauge function φ, then X satisfy Opial's condition.

Lemma 2.1 (Cioranescu [32]) *Assume that a Banach space X has a weakly continuous duality mapping J_φ with gauge φ. For all $x, y \in X$, the following inequality holds*

$$\Phi(\|x + y\|) \leq \Phi(\|x\|) + \langle y, J_\varphi(x + y) \rangle.$$

A mapping $\psi : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ is said to be an *L-function* if $\psi(0) = 0,\ \psi(t) > 0$ for each $t > 0$ and for every $s > 0$, there exists $u > s$ such that $\psi(t) \leq s$ for each $t \in [s, u]$. As a consequence, every L-function ψ satisfies $\psi(t) < t$, for each $t > 0$.

Definition 2.2 Let (X, d) be a metric space. A mapping $f : X \longrightarrow X$ is said to be

(1) a *(ψ, L)-contraction* if $\psi : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ is an L-function and $d(f(x), f(y)) < \psi(d(x, y)),\ \forall x, y \in X$, with $x \neq y$,

(2) a *Meir–Keeler-type mapping* if for each $\epsilon > 0$ there exists $\delta = \delta(\epsilon) > 0$ such that for each $x, y \in X$, with $\epsilon \leq d(x, y) < \epsilon + \delta$, we have $d(f(x), f(y)) < \epsilon$.

Lemma 2.3 (Lim [27]) *Let (X, d) be a metric space and $f : X \longrightarrow X$ be a mapping. The following assertions are equivalent:*

(i) *f is a Meir–Keeler-type mapping;*

(ii) *there exists an L-function $\psi : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ such that f is a (ψ, L)-contraction.*

Lemma 2.4 (Petrusel and Yao [28]) *Let C be a convex subset of a Banach space X. Let $T : C \longrightarrow C$ be a nonexpansive mapping and f be a (ψ, L)-contraction. Then the following assertions hold*

(i) *$T \circ f$ is a (ψ, L)-contraction on C and has a unique fixed point in C;*

(ii) *for each $\alpha \in (0, 1)$, the mapping $x \longrightarrow \alpha f(x) + (1 - \alpha)Tx$ is Meir–Keeler-type and it has a unique fixed point in C.*

Lemma 2.5 (Suzuki [26]) *Let C be a convex subset of a Banach space X. Let $f : C \longrightarrow C$ be a Meir–Keeler-type contraction. Then for each $\epsilon > 0$ there exists $k \in (0, 1)$ such that*

$$\textit{for each } x, y \in C \textit{ with } \|x - y\| \geq \epsilon \textit{ we have } \|f(x) - f(y)\| \leq k\|x - y\|.$$

From now on, by a generalized contraction mapping, we mean a Meir–Keeler-type mapping or a (ψ, L)-contraction. In the rest of paper, we suppose that the L-function from the definition of (ψ, L)-contraction is continuous, strictly increasing and $\lim_{t \longrightarrow \infty} \eta(t) = \infty$, where $\eta(t) := t - \psi(t)$, $\forall t \in \mathbb{R}^+$. As a consequence, we have that η is a bijection on $\mathbb{R}^+$.

Lemma 2.6 (Xu [33]) *Assume that $\{a_n\}$ is a sequence of nonnegative real numbers such that*

$$a_{n+1} \leq (1 - \sigma_n)a_n + \delta_n,$$

where $\{\sigma_n\}$ is a sequence in $(0, 1)$ and $\{\delta_n\}$ is a sequence in $\mathbb{R}$ such that

(i) $\sum_{n=0}^{\infty} \sigma_n = \infty$;

(ii) $\limsup_{n \longrightarrow \infty} \frac{\delta_n}{\sigma_n} \leq 0$ *or* $\sum_{n=0}^{\infty} |\delta_n| < \infty$.

Then, $\lim_{n \longrightarrow \infty} a_n = 0$.

3 Main Results

Theorem 3.1 *Let C be a nonempty, closed and convex subset of a reflexive Banach space X which admits a weakly continuous duality mapping J_φ with gauge function φ. Let $T : C \longrightarrow C$ be a nonexpansive mapping such that $Fix(T) \neq \emptyset$ and $f : C \longrightarrow C$ be a Meir–Keeler-type contraction. Then the net $\{x_t\}$ defined by*

$$x_t = tf(x_t) + (1-t)Tx_t. \tag{3.1}$$

converges strongly to an element $x^* \in Fix(T)$, where x^* is the unique solution of the variational inequality

$$\langle f(x^*) - x^*, J_\varphi(z - x^*)\rangle \le 0, \quad \forall z \in Fix(T). \tag{3.2}$$

Proof Note that from Lemma 2.4(i), we have $\{x_t\}$ defined by (3.1) is well defined. First, we show the uniqueness of a solution of the variational inequality (3.2). Suppose that $\tilde{x}, x^* \in Fix(T)$ are solutions of (3.2). Since f is a Meir–Keeler-type contraction, then for each $\epsilon > 0$ such that $\|\tilde{x} - x^*\| \ge \epsilon$. By Lemma 2.5, there exists $k_\epsilon \in (0, 1)$ such that $\|f(\tilde{x}) - f(x^*)\| \le k_\epsilon \|\tilde{x} - x^*\|$. Then we have

$$\langle f(x^*) - x^*, J_\varphi(\tilde{x} - x^*)\rangle \le 0. \tag{3.3}$$

Interchange x^* and $\tilde{x}$ to obtain

$$\langle f(\tilde{x}) - \tilde{x}, J_\varphi(x^* - \tilde{x})\rangle \le 0. \tag{3.4}$$

Adding up (3.3) and (3.4), we have

$$\begin{aligned}
0 &\ge \langle (f(x^*) - x^*) - (f(\tilde{x}) - \tilde{x}), J_\varphi(\tilde{x} - x^*)\rangle \\
&= \langle \tilde{x} - x^*, J_\varphi(\tilde{x} - x^*)\rangle - \langle f(\tilde{x}) - f(x^*), J_\varphi(\tilde{x} - x^*)\rangle \\
&\ge \|\tilde{x} - x^*\|\varphi(\|\tilde{x} - x^*\|) - \|f(\tilde{x}) - f(x^*)\|\|J_\varphi(\tilde{x} - x^*)\| \\
&\ge \|\tilde{x} - x^*\|\varphi(\|\tilde{x} - x^*\|) - k\|\tilde{x} - x^*\|\varphi(\|\tilde{x} - x^*\|) \\
&= (1-k)\Phi(\|\tilde{x} - x^*\|),
\end{aligned}$$

which is a contradiction, we must have $\tilde{x} = x^*$ and the uniqueness is proved. Below, we use x^* to denote the unique solution of the variational inequality (3.2).

Next, we show that $\{x_t\}$ is bounded. Take $p \in Fix(T)$, fixed ϵ_0, for each $t \in (0, 1)$.

Case 1 $\|x_t - p\| < \epsilon_0$. In this case, it easy see that $\{x_t\}$ is bounded.

Case 2 $\|x_t - p\| \ge \epsilon_0$. By Lemma 2.5, there exists $k_{\epsilon_0} \in (0, 1)$ such that $\|f(x_t) - f(p)\| \le k_{\epsilon_0}\|x_t - p\|$. Then, we have

$$\begin{aligned}
\|x_t - p\| &= \|t(f(x_t) - p) + (1-t)(Tx_t - p)\| \\
&\le t\|f(x_t) - p\| + (1-t)\|Tx_t - p\| \\
&\le t\|f(x_t) - f(p)\| + t\|f(p) - p\| + (1-t)\|Tx_t - p\| \\
&\le \big(1 - (1 - k_{\epsilon_0})t\big)\|x_t - p\| + t\|f(p) - p\|.
\end{aligned}$$

It follows that

$$\|x_t - p\| \le \frac{1}{1 - k_{\epsilon_0}}\|f(p) - p\|.$$

Thus, $\{x_t\}$ is bounded, so are $\{f(x_t)\}$ and $\{Tx_t\}$.

By definition of $\{x_t\}$, we have

$$\begin{aligned}\|x_t - Tx_t\| &= \|tf(x_t) + (1-t)Tx_t - Tx_t\| \\ &= t\|f(x_t) - Tx_t\| \longrightarrow 0 \text{ as } t \longrightarrow 0.\end{aligned} \tag{3.5}$$

Assume that $\{t_n\} \subset (0, 1)$ is a sequence such that $t_n \longrightarrow 0$ as $n \longrightarrow \infty$. Put $x_n := x_{t_n}$. By reflexivity of a Banach space X and boundedness of $\{x_n\}$, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $x_{n_i} \rightharpoonup \tilde{x} \in C$ as $i \longrightarrow \infty$. Let us to show $\tilde{x} \in Fix(T)$. Suppose that $\tilde{x} \notin Fix(T)$, i.e., $\tilde{x} \neq T\tilde{x}$. By the Opial's condition and (3.5), we have

$$\begin{aligned}\liminf_{n\longrightarrow\infty} \|x_{n_i} - \tilde{x}\| &< \liminf_{n\longrightarrow\infty} \|x_{n_i} - T\tilde{x}\| \\ &\leq \liminf_{n\longrightarrow\infty} \left(\|x_{n_i} - Tx_{n_i}\| + \|Tx_{n_i} - T\tilde{x}\|\right) \\ &\leq \liminf_{n\longrightarrow\infty} \|x_{n_i} - \tilde{x}\|,\end{aligned}$$

which is a contradiction. Thus, we obtain $\tilde{x} \in Fix(T)$.

Next, we show that $\{x_n\}$ is relatively sequentially compact. For each $z \in Fix(T)$, suppose the contrary, there exists $\epsilon > 0$ and a subsequence $\{x_{m_i}\}$ of $\{x_{n_i}\}$ such that $\|x_{m_i} - z\| \geq \epsilon$. By Lemma 2.5, there exists $k_\epsilon \in (0, 1)$ such that $\|f(x_{m_i}) - f(z)\| \leq k_\epsilon \|x_{m_i} - z\|$. Then we have

$$\begin{aligned}\Phi(\|x_{m_i} - z\|) &= \Phi(\|(1-t_{m_i})(Tx_{m_i} - z) + t_{m_i}(f(x_{m_i}) - z)\|) \\ &= \Phi(\|(1-t_{m_i})(Tx_{m_i} - z) + t_{m_i}(f(x_{m_i}) - f(z)) + t_{m_i}(f(z) - z\|) \\ &\leq \Phi(\|(1-t_{m_i})(Tx_{m_i} - z) + t_{m_i}(f(x_{m_i}) - f(z))\|) + t_{m_i}\langle f(z) \\ &\quad - z, J_\varphi(x_{m_i} - z)\rangle \\ &\leq \left(1 - (1-k_\epsilon)t_{m_i}\right)\Phi(\|x_{m_i} - z\|) + t_{m_i}\langle f(z) - z, J_\varphi(x_{m_i} - z)\rangle.\end{aligned}$$

It follows that

$$\Phi(\|x_{m_i} - z\|) \leq \frac{1}{1-k_\epsilon}\langle f(z) - z, J_\varphi(x_{m_i} - z)\rangle. \tag{3.6}$$

Since J_φ is single-valued and weakly continuous duality mapping, it follows that from (3.6) that $\Phi(\|x_{m_i} - \tilde{x}\|) \longrightarrow 0$. By the properties Φ implies that $x_{m_i} \longrightarrow \tilde{x}$, that is $\|x_{m_i} - \tilde{x}\| < \epsilon$, which is a contradiction. Thus, we obtain $x_{n_i} \longrightarrow \tilde{x}$ as $i \longrightarrow \infty$.

Next, we show that $\tilde{x}$ solves the variational inequality (3.2). Since

$$x_t = tf(x_t) + (1-t)Tx_t,$$

we can derive that

$$(I - f)x_t = -\frac{1}{t}(I - T)x_t + (I - T)x_t. \tag{3.7}$$

Since T is nonexpansive, we have that $I - T$ is accretive. Note that for all $z \in Fix(T)$, it follows from (3.7) that

$$\begin{aligned}\langle (I-f)x_t, J_\varphi(x_t-z)\rangle &= -\frac{1}{t}\langle (I-T)x_t-(I-T)z, J_\varphi(x_t-z)\rangle \\ &\quad + \langle (I-T)x_t, J_\varphi(x_t-z)\rangle \\ &\le \langle (I-T)x_t, J_\varphi(x_t-z)\rangle \\ &\le \|x_t-Tx_t\|M, \end{aligned} \tag{3.8}$$

where $M>0$ is a constant such that $M=\sup_{t\in(0,1)}\{\|J_\varphi(x_t-z)\|\}$. Now, replacing t in (3.8) with t_n and taking the limit as $n\longrightarrow\infty$, we noticing that $x_{t_n}-Tx_{t_n}\longrightarrow \tilde{x}-T\tilde{x}=0$ for $\tilde{x}\in Fix(T)$, we obtain $\langle (f-I)\tilde{x}, J_\varphi(z-\tilde{x})\rangle\le 0$. Hence $\tilde{x}\in Fix(T)$ is the solution of the variational inequality (3.2). Consequently, $x^*=\tilde{x}$ by uniqueness. Therefore $x_t\longrightarrow x^*$ as $t\longrightarrow 0$. This completes the proof. □

Using Theorem 3.1, we get the following strong convergence theorems for proximal point algorithm by the viscosity approximation method based on Meir–Keeler-type contractions for zeros of accretive operators in Banach spaces. The proofs are similarly related to Theorems 3.2 and 3.3 in [17]; see also, [21], [14] and [15].

Theorem 3.2 *Let C be a nonempty, closed and convex subset of a reflexive Banach space X which admits a weakly continuous duality mapping J_φ with gauge function φ. Let $A: D(A)\subset X\longrightarrow 2^X$ be an accretive operator such that $A^{-1}0\neq\emptyset$ which satisfies the condition $\overline{D(A)}\subset C\subset\bigcap_{r>0}R(I+rA)$. Let J_r be the resolvent of A for all $r>0$ and $f: C\longrightarrow C$ be a Meir–Keeler-type contraction. For given $x_1\in C$, let $\{x_n\}$ be a sequence defined by*

$$x_{n+1}=\alpha_n f(x_n)+(1-\alpha_n)J_{r_n}x_n,\quad \forall n\ge 1, \tag{3.9}$$

where $\{\alpha_n\}\subset(0,1)$ and $\{r_n\}\subset(0,\infty)$ are sequences which satisfy the following conditions:

(C1) $\lim_{n\longrightarrow\infty}\alpha_n=0$ *and* $\sum_{n=1}^{\infty}\alpha_n=\infty$;
(C2) $\lim_{n\longrightarrow\infty}r_n=\infty$.

Then the sequence $\{x_n\}$ defined by (3.9) converges strongly to an element $x^*\in A^{-1}0$, where x^* is the unique solution of the variational inequality

$$\langle f(x^*)-x^*, J_\varphi(z-x^*)\rangle\le 0,\quad \forall z\in A^{-1}0. \tag{3.10}$$

Proof First, we show that $\{x_n\}$ is bounded. Take $p\in A^{-1}0$, fixed ϵ_0, for all $n\ge 1$.

Case 1 $\|x_n-p\|<\epsilon_0$. In this case, it easy see that $\{x_n\}$ is bounded.
Case 2 $\|x_n-p\|\ge\epsilon_0$. By Lemma 2.5, there exists $k_{\epsilon_0}\in(0,1)$ such that $\|f(x_n)-f(p)\|\le k_{\epsilon_0}\|x_n-p\|$.

Then, we have

$$\begin{aligned}\|x_{n+1}-p\| &= \|\alpha_n(f(x_n)-p)+(1-\alpha_n)(J_{r_n}x_n-p)\| \\ &\le \alpha_n\|f(x_n)-p\|+(1-\alpha_n)\|J_{r_n}x_n-p)\| \\ &\le \alpha_n\|f(x_n)-f(p)\|+\alpha_n\|f(p)-p\|+(1-\alpha_n)\|x_n-p\|\end{aligned}$$

$$
\begin{aligned}
&\leq \big(1-(1-k_{\epsilon_0})\alpha_n\big)\|x_n-p\|+\alpha_n\|f(p)-p\| \\
&= \big(1-(1-k_{\epsilon_0})\alpha_n\big)\|x_n-p\|+(1-k_{\epsilon_0})\alpha_n\frac{\|f(p)-p\|}{1-k_{\epsilon_0}} \\
&\leq \max\left\{\|x_n-p\|, \frac{\|f(p)-p\|}{1-k_{\epsilon_0}}\right\}.
\end{aligned}
$$

By induction, we have

$$
\|x_n-p\| \leq \max\left\{\|x_1-p\|, \frac{\|f(p)-p\|}{1-k_{\epsilon_0}}\right\}, \quad \forall n \geq 1.
$$

Thus $\{x_n\}$ is bounded, so are $\{f(x_n)\}$ and $\{J_{r_n}x_n\}$.

Next, we show that $\lim_{n\longrightarrow\infty}\|x_n - J_r x_n\| = 0$. By the condition $(C1)$, we have

$$
\|x_{n+1}-J_{r_n}x_n\| = \alpha_n\|f(x_n)-J_{r_n}x_n\| \longrightarrow 0 \text{ as } n \longrightarrow \infty. \tag{3.11}
$$

For $r > 0$, we note that

$$
\begin{aligned}
\|J_r J_{r_n}x_n - J_{r_n}x_n\| &= \|(I-J_r)J_{r_n}x_n\| = r\|A_r J_{r_n}x_n\| \leq r|AJ_{r_n}x_n| \leq r\|A_{r_n}x_n\| \\
&= r\frac{\|x_n-J_{r_n}x_n\|}{r_n}.
\end{aligned}
$$

It follows from the condition $(C2)$ that

$$
\lim_{n\longrightarrow\infty}\|J_r J_{r_n}x_n - J_{r_n}x_n\| = 0. \tag{3.12}
$$

We observe that

$$
\begin{aligned}
\|x_{n+1}-J_r x_{n+1}\| &\leq \|x_{n+1}-J_{r_n}x_n\| + \|J_{r_n}x_n - J_r J_{r_n}x_n\| + \|J_r J_{r_n}x_n - J_r x_{n+1}\| \\
&\leq 2\|x_{n+1}-J_{r_n}x_n\| + \|J_{r_n}x_n - J_r J_{r_n}x_n\|.
\end{aligned}
$$

It follows from (3.11) and (3.12) that $\lim_{n\longrightarrow\infty}\|x_{n+1}-J_r x_{n+1}\| = 0$, and hence

$$
\lim_{n\longrightarrow\infty}\|x_n - J_r x_n\| = 0, \quad \forall r > 0. \tag{3.13}
$$

Next, we show that

$$
\limsup_{n\longrightarrow\infty}\langle f(x^*)-x^*, J_\varphi(x_n-x^*)\rangle \leq 0,
$$

where x^* is the same as in Theorem 3.1. To show this, we take a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that

$$
\limsup_{n\longrightarrow\infty}\langle f(x^*)-x^*, J_\varphi(x_n-x^*)\rangle = \lim_{i\longrightarrow\infty}\langle f(x^*)-x^*, J_\varphi(x_{n_i}-x^*)\rangle.
$$

By reflexivity of a Banach space X and boundedness of $\{x_n\}$, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $x_{n_i} \rightharpoonup z$ as $i \longrightarrow \infty$. Let us to show $z \in A^{-1}0$. Suppose that $z \notin A^{-1}0$, i.e., $z \neq J_r z$. By the Opial's condition and (3.13), we have

$$\begin{aligned}\liminf_{n\longrightarrow\infty} \|x_{n_i} - z\| &< \liminf_{n\longrightarrow\infty} \|x_{n_i} - J_r z\| \\ &\le \liminf_{n\longrightarrow\infty} \big(\|x_{n_i} - J_r x_{n_i}\| + \|J_r x_{n_i} - J_r z\|\big) \\ &\le \liminf_{n\longrightarrow\infty} \|x_{n_i} - z\|,\end{aligned}$$

which is a contradiction. Thus, we obtain $z \in A^{-1}0$. Since J_φ is single-valued and weakly continuous duality mapping, we obtain that

$$\begin{aligned}\limsup_{n\longrightarrow\infty}\langle f(x^*) - x^*, J_\varphi(x_n - x^*)\rangle &= \lim_{i\longrightarrow\infty}\langle f(x^*) - x^*, J_\varphi(x_{n_i} - x^*)\rangle \\ &= \langle f(x^*) - x^*, J_\varphi(z - x^*)\rangle \le 0. \qquad (3.14)\end{aligned}$$

Finally, we show that $x_n \longrightarrow x^*$ as $n \longrightarrow \infty$. Suppose the contrary, $\{x_n\}$ does not converge strongly to $x^* \in A^{-1}0$. Then there exist $\epsilon > 0$ and a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $\|x_{n_j} - x^*\| \ge \epsilon$ for all $j \in \mathbb{N}$. By Lemma 2.5, there exists $k_\epsilon \in (0, 1)$ such that

$$\|f(x_{n_j}) - f(x^*)\| \le k_\epsilon \|x_{n_j} - x^*\| \quad \text{for all } \; j \in \mathbb{N}.$$

By Lemma 2.1, we have

$$\begin{aligned}\Phi(\|x_{n_j+1} - x^*\|) &= \Phi(\|(1-\alpha_{n_j})(J_{r_{n_j}}x_{n_j} - x^*) + \alpha_{n_j}(f(x_{n_j}) - x^*)\|) \\ &= \Phi(\|(1-\alpha_{n_j})(J_{r_{n_j}}x_{n_j} - x^*) + \alpha_{n_j}(f(x_{n_j}) - f(x^*)) \\ &\quad + \alpha_{n_j}(f(x^*) - x^*)\|) \\ &\le \Phi(\|(1-\alpha_{n_j})(J_{r_{n_j}}x_{n_j} - x^*) + \alpha_{n_j}(f(x_{n_j}) - f(x^*))\|) \\ &\quad + \alpha_{n_j}\langle f(x^*) - x^*, J_\varphi(x_{n_j+1} - x^*)\rangle \\ &\le \big(1 - (1-k_\epsilon)\alpha_{n_j}\big)\Phi(\|x_{n_j} - x^*\|) + \alpha_{n_j}\langle f(x^*) \\ &\quad - x^*, J_\varphi(x_{n_j+1} - x^*)\rangle. \qquad (3.15)\end{aligned}$$

Put $\sigma_{n_j} := (1-k_\epsilon)\alpha_{n_j}$ and $\delta_{n_j} := \frac{1}{1-k_\epsilon}\langle f(x^*) - x^*, J_\varphi(x_{n_j+1} - x^*)\rangle$. Then (3.15) reduces to formula

$$\Phi(\|x_{n_j+1} - x^*\|) \le (1-\sigma_{n_j})\Phi(\|x_{n_j} - x^*\|) + \sigma_{n_j}\delta_{n_j}.$$

It is easily seen that $\sum_{j=1}^{\infty}\sigma_{n_j} = \infty$ and (using (3.15))

$$\limsup_{j\longrightarrow\infty}\delta_{n_j} = \limsup_{j\longrightarrow\infty}\frac{1}{1-k_\epsilon}\langle f(x^*) - x^*, J_\varphi(x_{n_j+1} - x^*)\rangle \le 0.$$

Then by Lemma 2.6, we have $\Phi(\|x_{n_j} - x^*\|) \longrightarrow 0$. This implies that $x_{n_j} \longrightarrow x^*$, that is $\|x_{n_j} - x^*\| < \epsilon_0$, which is contradiction. Therefore, we conclude that $x_n \longrightarrow x^*$. This completes the proof. □

Theorem 3.3 *Let C be a nonempty, closed and convex subset of a reflexive Banach space X which admits a weakly continuous duality mapping J_φ with gauge function φ. Let $A : D(A) \subset X \longrightarrow 2^X$ be an accretive operator such that $A^{-1}0 \neq \emptyset$ which satisfies the condition $\overline{D(A)} \subset C \subset \bigcap_{r>0} R(I + rA)$. Let J_r be the resolvent of A for all $r > 0$ and $f : C \longrightarrow C$ be a Meir–Keeler-type contraction. For given $x_1 \in C$, let $\{x_n\}$ be a sequence defined by*

$$x_{n+1} = J_{r_n}(\alpha_n f(x_n) + (1 - \alpha_n)x_n), \quad \forall n \geq 1, \tag{3.16}$$

where $\{\alpha_n\} \subset (0, 1)$ and $\{r_n\} \subset (0, \infty)$ are sequences which satisfy the following conditions:

(C1) $\lim_{n\longrightarrow\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$*;*
(C2) $\lim_{n\longrightarrow\infty} r_n = \infty$.

Then the sequence $\{x_n\}$ defined by (3.16) converges strongly to an element $x^* \in A^{-1}0$, where x^* is the unique solution of the variational inequality (3.2).

Proof By the similar method to the proof technique of Theorem 3.2, we show that the sequence $\{x_n\}$ is bounded firstly. Let $y_n = \alpha_n f(x_n) + (1 - \alpha_n)x_n$. Take $p \in A^{-1}0$, fixed ϵ_0, for all $n \geq 1$.

Case 1 $\|x_n - p\| < \epsilon_0$. In this case, it easy see that $\{x_n\}$ is bounded.
Case 2 $\|x_n - p\| \geq \epsilon_0$. By Lemma 2.5, there exists $k_{\epsilon_0} \in (0, 1)$ such that $\|f(x_n) - f(p)\| \leq k_{\epsilon_0}\|x_n - p\|$. Then, we have

$$\begin{aligned}
\|x_{n+1} - p\| &= \|J_{r_n} y_n - p\| \leq \|y_n - p\| \\
&= \|\alpha_n(f(x_n) - p) + (1 - \alpha_n)(x_n - p)\| \\
&\leq \alpha_n\|f(x_n) - f(p)\| + \alpha_n\|f(p) - p\| + (1 - \alpha_n)\|x_n - p\| \\
&\leq \big(1 - (1 - k_{\epsilon_0})\big)\|x_n - p\| + \alpha_n\|f(p) - p\|.
\end{aligned}$$

By induction, we have

$$\|x_n - p\| \leq \max\left\{\|x_1 - p\|, \frac{\|f(p) - p\|}{1 - k_{\epsilon_0}}\right\}, \quad \forall n \geq 1.$$

Thus $\{x_n\}$ is bounded, so are $\{f(x_n)\}$ and $\{J_{r_n} y_n\}$.

Next, we show that $\lim_{n\longrightarrow\infty} \|y_n - J_r y_n\| = 0$. By the condition $(C1)$, we have

$$\|y_n - x_n\| = \alpha_n\|f(x_n) - x_n\| \longrightarrow 0 \text{ as } n \longrightarrow \infty. \tag{3.17}$$

For $r > 0$, we note that

$$\begin{aligned}\|x_{n+1} - J_r x_{n+1}\| &= \|J_{r_n} y_n - J_r J_{r_n} y_n\| = \|(I - J_r) J_{r_n} y_n\| \\ &= r\|A_r J_{r_n} y_n\| \le r|A J_{r_n} y_n| \le r\|A_{r_n} y_n\| = r\frac{\|y_n - J_{r_n} y_n\|}{r_n}.\end{aligned}$$

It follows from the condition $(C2)$ that $\lim_{n\longrightarrow\infty}\|x_{n+1} - J_r x_{n+1}\| = 0$, and hence

$$\lim_{n\longrightarrow\infty}\|x_n - J_r x_n\| = 0, \quad \forall r > 0. \tag{3.18}$$

Observe that

$$\begin{aligned}\|y_n - J_r y_n\| &\le \|y_n - x_n\| + \|x_n - J_r x_n\| + \|J_r x_n - J_r y_n\| \\ &\le 2\|x_n - y_n\| + \|x_n - J_r x_n\|.\end{aligned}$$

It follows from (3.17) and (3.18) that

$$\lim_{n\longrightarrow\infty}\|y_n - J_r y_n\| = 0. \tag{3.19}$$

Next, we show that

$$\limsup_{n\longrightarrow\infty}\langle f(x^*) - x^*, J_\varphi(y_n - x^*)\rangle \le 0,$$

where x^* is the same as in Theorem 3.1. To show this, we take a subsequence $\{y_{n_i}\}$ of $\{y_n\}$ such that

$$\limsup_{n\longrightarrow\infty}\langle f(x^*) - x^*, J_\varphi(y_n - x^*)\rangle = \lim_{i\longrightarrow\infty}\langle f(x^*) - x^*, J_\varphi(y_{n_i} - x^*)\rangle.$$

By reflexivity of a Banach space X and boundedness of $\{y_n\}$, there exists a subsequence $\{y_{n_i}\}$ of $\{y_n\}$ such that $y_{n_i} \rightharpoonup z$ as $i \longrightarrow \infty$. Let us to show $z \in A^{-1}0$. Suppose that $z \notin A^{-1}0$, i.e., $z \ne J_r z$. By the Opial's condition and (3.19), we have

$$\begin{aligned}\liminf_{n\longrightarrow\infty}\|y_{n_i} - z\| &< \liminf_{n\longrightarrow\infty}\|y_{n_i} - J_r z\| \\ &\le \liminf_{n\longrightarrow\infty}\left(\|y_{n_i} - J_r y_{n_i}\| + \|J_r y_{n_i} - J_r z\|\right) \\ &\le \liminf_{n\longrightarrow\infty}\|y_{n_i} - z\|,\end{aligned}$$

which is a contradiction. Thus, we obtain $z \in A^{-1}0$. Since J_φ is single-valued and weakly continuous duality mapping, we obtain that

$$\begin{aligned}\limsup_{n\longrightarrow\infty}\langle f(x^*) - x^*, J_\varphi(y_n - x^*)\rangle &= \lim_{i\longrightarrow\infty}\langle f(x^*) - x^*, J_\varphi(y_{n_i} - x^*)\rangle \\ &= \langle f(x^*) - x^*, J_\varphi(z - x^*)\rangle \le 0. \end{aligned} \tag{3.20}$$

Finally, we show that $x_n \longrightarrow x^*$ as $n \longrightarrow \infty$. Suppose the contrary, $\{x_n\}$ does not converge strongly to $x^* \in A^{-1}0$. Then there exist $\epsilon > 0$ and a subsequence $\{x_{n_j}\}$ of

$\{x_n\}$ such that $\|x_{n_j} - x^*\| \geq \epsilon$ for all $j \in \mathbb{N}$. By Lemma 2.5, there exists $k_\epsilon \in (0, 1)$ such that

$$\|f(x_{n_j}) - f(x^*)\| \leq k_\epsilon \|x_{n_j} - x^*\| \quad \text{for all} \ \ j \in \mathbb{N}.$$

By Lemma 2.1, we have

$$\begin{aligned}
\Phi(\|x_{n_j+1} - x^*\|) &= \Phi(\|J_{r_{n_j}} y_{n_j} - x^*\|) \leq \Phi(\|y_{n_j} - x^*\|) \\
&= \Phi(\|(1 - \alpha_{n_j})(x_{n_j} - x^*) + \alpha_{n_j}(f(x_{n_j}) - x^*)\|) \\
&= \Phi(\|(1 - \alpha_{n_j})(x_{n_j} - x^*) + \alpha_{n_j}(f(x_{n_j}) - f(x^*)) \\
&\quad + \alpha_{n_j}(f(x^*) - x^*)\|) \\
&\leq \Phi(\|(1 - \alpha_{n_j})(x_{n_j} - x^*) + \alpha_{n_j}(f(x_{n_j}) - f(x^*))\|) \\
&\quad + \alpha_{n_j}\langle f(x^*) - x^*, J_\varphi(y_{n_j} - x^*)\rangle \\
&\leq \big(1 - (1 - k_\epsilon)\alpha_{n_j}\big)\Phi(\|x_{n_j} - x^*\|) + \alpha_{n_j}\langle f(x^*) - x^*, \\
&\quad J_\varphi(y_{n_j} - x^*)\rangle.
\end{aligned} \tag{3.21}$$

Put $\sigma_{n_j} := (1 - k_\epsilon)\alpha_{n_j}$ and $\delta_{n_j} := \frac{1}{1-k_\epsilon}\langle f(x^*) - x^*, J_\varphi(y_{n_j} - x^*)\rangle$. Then (3.21) reduces to formula

$$\Phi(\|x_{n_j+1} - x^*\|) \leq (1 - \sigma_{n_j})\Phi(\|x_{n_j} - x^*\|) + \sigma_{n_j}\delta_{n_j}.$$

It is easily seen that $\sum_{j=1}^{\infty} \sigma_{n_j} = \infty$ and (using (3.20))

$$\limsup_{j\longrightarrow\infty} \delta_{n_j} = \limsup_{j\longrightarrow\infty} \frac{1}{1 - k_\epsilon}\langle f(x^*) - x^*, J_\varphi(y_{n_j} - x^*)\rangle \leq 0.$$

Then by Lemma 2.6, we have $\Phi(\|x_{n_j} - x^*\|) \longrightarrow 0$. This implies that $x_{n_j} \longrightarrow x^*$, that is $\|x_{n_j} - x^*\| < \epsilon_0$, which is contradiction. Therefore, we conclude that $x_n \longrightarrow x^*$. This completes the proof. □

4 Application to Nonexpansive Semigroup

Definition 4.1 Let C be a nonempty, closed and convex subset of a real Banach space X. A one-parameter family $\mathcal{S} = \{T(t) : t > 0\}$ from C into itself is said to be a *nonexpansive semigroup* on C if it satisfies the following conditions:

(i) $T(0)x = x$ for all $x \in C$;
(ii) $T(s + t)x = T(s)T(t)x$ for all $x \in C$ and $s, t > 0$;
(iii) for each $x \in C$ the mapping $t \mapsto T(t)x$ is continuous;
(iv) $\|T(t)x - T(t)y\| \leq \|x - y\|$ for all $x, y \in C$ and $t > 0$.

Remark 4.2 We denote by $Fix(\mathcal{S})$ the set of all common fixed points of $\mathcal{S}$, that is $Fix(\mathcal{S}) := \bigcap_{t>0} Fix(T(t)) = \{x \in C : T(t)x = x\}$. We know that $Fix(\mathcal{S})$ is nonempty if C is bounded (see [44]).

Now, we present the concept of a uniformly asymptotically regular semigroup (see [34–36]).

Definition 4.3 Let C be a nonempty, closed and convex subset of a real Banach space X and $\mathcal{S} = \{T(t) : t > 0\}$ be a semigroup of nonexpansive operators. Then $\mathcal{S}$ is called *uniformly asymptotically regular* (in short, u.a.r.) on C if for all $h \geq 0$ and for any bounded subset B of C such that

$$\lim_{t\longrightarrow\infty} \sup_{x\in B} \|T(h)T(t)x - T(t)x\| = 0.$$

The nonexpansive semigroup $\{\sigma_t : t > 0\}$ defined by the following lemma is an example of u.a.r. operator semigroup. Other examples of u.a.r. operator semigroup can be found in [34]. The following lemma found in [37].

Lemma 4.4 (see [37]) *Let C be a nonempty, closed and convex subset of a smooth Banach space X and let $\mathcal{S} = \{T(h) : h > 0\}$ be a u.a.r. nonexpansive semigroup on C such that $Fix(\mathcal{S}) = \bigcap_{h>0} Fix(T(h)) \neq \emptyset$ and at least there exists a $T(h)$ which is demicompact. Then, for each $x \in C$, there exists a sequence $\{T(t_k) : t_k > 0,\ k \in \mathbb{N}\} \subset \{T(h) : h > 0\}$ such that $\{T(t_k)x\}$ converges strongly to some point in $Fix(\mathcal{S})$, where* $\lim_{k\longrightarrow\infty} t_k = \infty$.

Using Lemma 4.4 and Theorems 3.2 and 3.3, we have the following results.

Theorem 4.5 *Let C be a nonempty, closed and convex subset of a uniformly smooth Banach space X which admits a weakly continuous duality mapping J_φ with gauge function φ. Let $\mathcal{S} = \{T(t) : t > 0\}$ be a u.a.r. nonexpansive semigroup from C into itself such that $Fix(\mathcal{S}) := \bigcap_{h>0} Fix(T(h)) \neq \emptyset$ and least there exists a $T(h)$ which is demicompact and $f : C \longrightarrow C$ be a Meir–Keeler-type contraction. For given $x_1 \in C$, let $\{x_n\}$ be a sequence defined by*

$$x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n)T(t_n)x_n, \quad \forall n \geq 1, \tag{4.1}$$

where $\{\alpha_n\} \subset (0, 1)$ and $\{t_n\} \subset (0, \infty)$ are sequences which satisfy the following conditions:

(C1) $\lim_{n\longrightarrow\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$*;*
(C2) $\lim_{n\longrightarrow\infty} t_n = \infty$.

Then the sequence $\{x_n\}$ defined by (4.1) converges strongly to an element $x^* \in Fix(\mathcal{S})$.

Proof By using the same arguments and techniques as those of Theorem 3.2, we only need show that

$$\lim_{n\longrightarrow\infty} \|x_n - Tx_n\| = 0.$$

By the condition $(C1)$, we have

$$\|x_{n+1} - T(t_n)x_n\| = \alpha_n \|f(x_n) - T(t_n)x_n\| \longrightarrow 0 \text{ as } n \longrightarrow \infty. \tag{4.2}$$

Since $\{T(t) : t > 0\}$ is a u.a.r. nonexpansive semigroup and $\lim_{n\longrightarrow\infty} t_n = \infty$, then for all $h > 0$, and for any bounded subset B of C containing $\{x_n\}$, we obtain that

$$\lim_{n\longrightarrow\infty} \|T(h)T(t_n)x_n - T(t_n)x_n\| \leq \lim_{n\longrightarrow\infty} \sup_{\omega\in B} \|T(h)T(t_n)\omega - T(t_n)\omega\| = 0. \tag{4.3}$$

For all $h > 0$, observe that

$$\begin{aligned}\|x_{n+1} - T(h)x_{n+1}\| &\leq \|x_{n+1} - T(t_n)x_n\| + \|T(t_n)x_n - T(h)T(t_n)x_n\| \\ &\quad + \|T(h)T(t_n) - T(h)x_{n+1}\| \\ &\leq 2\|x_{n+1} - T(t_n)x_n\| + \|T(t_n)x_n - T(h)T(t_n)x_n\|.\end{aligned}$$

It follows from (4.2) and (4.3) that $\lim_{n\longrightarrow\infty} \|x_{n+1} - T(h)x_{n+1}\| = 0$, and hence

$$\lim_{n\longrightarrow\infty} \|x_n - T(h)x_n\| = 0 \text{ for all } h \geq 0. \tag{4.4}$$

Since $\{T(h) : h > 0\}$ is a u.a.r. nonexpansive semigroup, by Lemma 4.4, for each $x \in C$, there exists a sequence $\{T(t_k) : t_k > 0, k \in \mathbb{N}\} \subset \{T(h) : h > 0\}$ such that $\{T(t_k)x\}$ converges strongly to some point in $Fix(\mathcal{S})$, where $t_k \longrightarrow \infty$ as $k \longrightarrow \infty$. Define a mapping $T : C \longrightarrow C$ by

$$Tx = \lim_{k\longrightarrow\infty} T(t_k)x, \quad \forall x \in C.$$

By [37, Remark 3.4], we see that the mapping T is nonexpansive such that $Fix(T) = Fix(\mathcal{S})$. From (4.4), we obtain that

$$\begin{aligned}\lim_{n\longrightarrow\infty} \|x_n - Tx_n\| &= \lim_{n\longrightarrow\infty} \lim_{k\longrightarrow\infty} \|x_n - T(t_k)x_n\| \\ &= \lim_{k\longrightarrow\infty} \lim_{n\longrightarrow\infty} \|x_n - T(t_k)x_n\| = 0.\end{aligned}$$

This completes the proof. □

Theorem 4.6 *Let C be a nonempty, closed and convex subset of a uniformly smooth Banach space X which admits a weakly continuous duality mapping J_φ with gauge function φ. Let $\mathcal{S} = \{T(t) : t > 0\}$ be a u.a.r. nonexpansive semigroup from C into itself such that $Fix(\mathcal{S}) := \bigcap_{h>0} Fix(T(h)) \neq \emptyset$ and least there exists a $T(h)$ which is demicompact and $f : C \longrightarrow C$ be a Meir–Keeler-type contraction. For given $x_1 \in C$, let $\{x_n\}$ be a sequence defined by*

$$x_{n+1} = T(t_n)[\alpha_n f(x_n) + (1 - \alpha_n)x_n], \quad \forall n \geq 1, \tag{4.5}$$

where $\{\alpha_n\} \subset (0, 1)$ and $\{t_n\} \subset (0, \infty)$ are sequences which satisfy the following conditions:

(C1) $\lim_{n\longrightarrow\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$;
(C2) $\lim_{n\longrightarrow\infty} t_n = \infty$.

Then the sequence $\{x_n\}$ defined by (4.5) converges strongly to an element $x^* \in Fix(\mathcal{S})$.

5 Application to Equilibrium Problem

Let C be a nonempty, closed convex subset of a real Hilbert space H and $\Theta : C\times C \longrightarrow \mathbb{R}$ be a bifunction, where $\mathbb{R}$ is the set of all real numbers. The equilibrium problem is to find $x \in C$ such that

$$\Theta(x, y) \geq 0, \quad \forall y \in C. \tag{5.1}$$

The set of solutions of the equilibrium problem (5.1) is denoted by $EP(\Theta)$. Given a mapping $T : C \longrightarrow H$, let $\Theta(x, y) = \langle Tx, y-x\rangle$ for all $x, y \in C$. Then $z \in EP(\Theta)$ if and only if $\langle Tx, y - x\rangle \geq 0$ for all $y \in C$, i.e., z is a solution of the variational inequality.

Numerous problems in physics, optimization and economics reduce to find a solution of the equilibrium problem (5.1). Some methods have been proposed to solve the equilibrium problems (see, for instance, BlumOettli [38] and Combettes and Hirstoaga [39]).

For solving the equilibrium problem, let us assume that a bifunction $\Theta : C\times C \longrightarrow \mathbb{R}$ satisfies the following conditions:

(A1) $\Theta(x, x) = 0$ for all $x \in C$;
(A2) Θ is monotone, i.e., $\Theta(x, y) + \Theta(y, x) \leq 0$ for each $x, y \in C$;
(A3) Θ is upper-semicontinuous, i.e., for each $x, y, z \in C$,

$$\limsup_{t\longrightarrow 0^+} \Theta(tz + (1 - t)x, y) \leq \Theta(x, y);$$

(A4) $\Theta(x, \cdot)$ is convex and weakly lower semicontinuous for each $x \in C$.

The following lemmas were also given in [38] and [39], respectively.

Lemma 5.1 (see [38, corollary 1]) *Let C be a nonempty, closed and convex subset of H and let $\Theta : C \times C \longrightarrow \mathbb{R}$ satisfying the conditions (A1)–(A4). Let $r > 0$ and $x \in H$. Then there exists $z \in C$ such that*

$$\Theta(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \quad \forall y \in C.$$

Lemma 5.2 (see [39, lemma 2.12]) *Assume that $\Theta : C \times C \longrightarrow \mathbb{R}$ satisfies the conditions (A1)–(A4). For $r > 0$ and $x \in H$, define a mapping $T_r : H \longrightarrow C$ as follows:*

$$T_r(x) = \Big\{z \in C : \Theta(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \quad \forall y \in C\Big\}, \quad \forall x \in H.$$

Then the following hold

(1) *T_r is single-valued.*
(2) *T_r is firmly nonexpansive, i.e., for each $x, y \in H$,*

$$\|T_r x - T_r y\|^2 \leq \langle T_r x - T_r y, x - y\rangle.$$

(3) $Fix(T_r) = EP(\Theta)$.
(4) $EP(\Theta)$ *is closed and convex.*

Theorem 5.3 *Let C be a nonempty, closed and convex subset of a real Hilbert space H. Let $\Theta : C \times C \longrightarrow \mathbb{R}$ be a bifunction satisfies the conditions (A1)–(A4) with $EP(\Theta) \neq \emptyset$. Let $f : C \longrightarrow C$ be a Meir–Keeler contraction-type. For given $x_1 \in C$, let $\{x_n\}$ be a sequence defined by*

$$\begin{cases} \Theta(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \\ x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n)u_n, \quad \forall n \geq 1, \end{cases} \tag{5.2}$$

where $\{\alpha_n\} \subset (0, 1)$ and $\{r_n\} \subset (0, \infty)$ are sequences which satisfy the following conditions:

(C1) $\lim_{n\longrightarrow\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$;
(C2) $\lim_{n\longrightarrow\infty} r_n = \infty$.

Then the sequence $\{x_n\}$ defined by (5.2) converges strongly to an element $x^* \in EP(\Theta)$.

Proof By using the same arguments and techniques as those of Theorem 3.2, we only need show that there exists a number $r > 0$ such that

$$\lim_{n\longrightarrow\infty} \|x_n - T_r x_n\| = 0.$$

From definition of T_r, we have

$$\Theta(T_{r_n}x_n, y) + \frac{1}{r_n}\langle y - T_{r_n}x_n, T_{r_n}x_n - x_n\rangle \geq 0, \quad \forall y \in C, \tag{5.3}$$

and

$$\Theta(T_r T_{r_n}x_n, y) + \frac{1}{r_n}\langle y - T_r T_{r_n}x_n, T_r T_{r_n}x_n - x_n\rangle \geq 0, \quad \forall y \in C. \tag{5.4}$$

Substituting $y = T_r T_{r_n}x_n$ in (5.3) and $y = T_{r_n}x_n$ in (5.4). Then, add these two inequalities and from the condition $(A2)$, we obtain

$$\left\langle T_{r_n}x_n - T_r T_{r_n}x_n, \frac{T_r T_{r_n}x_n - x_n}{r} - \frac{T_{r_n}x_n - x_n}{r_n}\right\rangle \geq 0,$$

and hence, for each $r > 0$,

$$\begin{aligned} \frac{\|T_r T_{r_n}x_n - T_{r_n}x_n\|^2}{r} &\leq \left\langle T_{r_n}x_n - T_r T_{r_n}x_n, \frac{1}{r_n}(x_n - T_{r_n}x_n)\right\rangle \\ &\leq \|T_r T_{r_n}x_n - T_{r_n}x_n\|\frac{1}{r_n}\|T_{r_n}x_n - x_n\|, \end{aligned}$$

which implies that

$$\|T_r T_{r_n} x_n - T_{r_n} x_n\| \leq \frac{r\|T_{r_n} x_n - x_n\|}{r_n}.$$

It follows from the condition (C2) that

$$\lim_{n \longrightarrow \infty} \|T_r T_{r_n} x_n - T_{r_n} x_n\| = 0. \tag{5.5}$$

Noticing $u_n = T_{r_n} x_n$, we note that

$$\begin{aligned}\|x_{n+1} - T_r x_{n+1}\| &\leq \|x_{n+1} - T_{r_n} x_n\| + \|T_{r_n} x_n - T_r T_{r_n} x_n\| + \|T_r T_{r_n} x_n - T_r x_{n+1}\| \\ &\leq 2\|x_{n+1} - u_n\| + \|T_{r_n} x_n - T_r T_{r_n} x_n\|\end{aligned}$$

It follows from (5.5) that $\lim_{n \longrightarrow \infty} \|x_{n+1} - T_r x_{n+1}\| = 0$, and hence

$$\lim_{n \longrightarrow \infty} \|x_n - T_r x_n\| = 0, \quad \forall r > 0.$$

This completes the proof. □

6 Application to Optimization Problem

Let H be a real Hilbert space and $\phi : H \longrightarrow (-\infty, +\infty]$ a proper convex lower semicontinuous function. Then the subdifferential $\partial\phi$ of ϕ is defined as follows:

$$\partial\phi = \Big\{y \in H : \phi(z) \geq \phi(x) + \langle z - x, y\rangle, \quad z \in H\Big\} \ \forall x \in H.$$

From [40], we know that $\partial\phi$ is maximal monotone. It is easy to verify that $0 \in \partial\phi$ if and only if $\phi(x) = \min_{y \in H} \phi(y)$ see also [40–42].

Consider a kind of optimization problem with a nonempty set of solutions

$$\min_{x \in C} h(x), \tag{6.1}$$

where $h(x)$ is a convex and lower semicontinuous functional defined on a closed convex subset C of a real Hilbert space H. We denote by $arg\min(h)$ the set of solutions of (6.1). Define a bifunction $\Theta : C \times C \longrightarrow \mathbb{R}$ by $\Theta(x, y) := h(y) - h(x)$. It is obvious that $EP(\Theta) = arg\min(h)$. In addition, it is easy to see that $\Theta(x, y)$ satisfies the conditions (A1)–(A4) in the Sect. 5.

Using Theorem 5.3, we have the following result.

Theorem 6.1 *Let C be a nonempty, closed and convex subset of a real Hilbert space H. Let h be a convex and lower semicontinuous functional on C such that $arg\min(h) \neq \emptyset$. Let $f : C \longrightarrow C$ be a Meir–Keeler-type contraction. For given $x_1 \in C$, let $\{x_n\}$ be a sequence defined by*

$$\begin{cases} u_n = arg\,\min_{y \in C} \left\{ h(y) + \frac{1}{2r_n} \|y - x_n\|^2 \right\}, \\ x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n) u_n, \quad \forall n \geq 1, \end{cases} \tag{6.2}$$

where $\{\alpha_n\} \subset (0, 1)$ *and* $\{r_n\} \subset (0, \infty)$ *are sequences which satisfy the following conditions:*

(C1) $\lim_{n \longrightarrow \infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$;
(C2) $\lim_{n \longrightarrow \infty} r_n = \infty$.

Then the sequence $\{x_n\}$ *defined by* (6.2) *converges strongly to an element* $x^* \in arg\,\min(h)$.

7 Application to Variational Inequalities

Let C be a nonempty, closed and convex subset of a real Hilbert space H and $A : C \longrightarrow H$ be a mapping. The classical variational inequality is to find $x \in C$ such that

$$\langle Ax, y - x \rangle \geq 0, \quad \forall y \in C. \tag{7.1}$$

The set of solutions of the classical variational inequality (7.1) is denoted by $VI(C, A)$. We recall that a mapping $A : C \longrightarrow H$ is monotone if

$$\langle Ax - Ay, x - y \rangle \geq 0, \quad \forall x, y \in C.$$

The following lemmas found in [43].

Lemma 7.1 (Bose [43]) *Let C be a nonempty, closed and convex subset of a real Hilbert space H. Let A be a continuous monotone mapping of C into H. Define a bifunction $\Theta : C \times C \longrightarrow \mathbb{R}$ as follows:*

$$\Theta(x, y) := \langle Ax, y - x \rangle, \quad \forall x, y \in C.$$

Then the following hold

(1) Θ *is satisfies the conditions (A1)–(A4) in Sect. 5 and* $VI(C, A) = EP(\Theta)$.
(2) *for each* $x \in H$, $z \in C$ *and* $r > 0$,

$$\Theta(z, y) + \frac{1}{r} \langle y - z, z - x \rangle \geq 0 \; \forall y \in C, \Longleftrightarrow z = P_C(x - rAx),$$

where P_C is projection operator from H into C.

Using Lemma 7.1 and Theorem 5.3, we have the following result.

Theorem 7.2 *Let C be a nonempty, closed and convex subset of a real Hilbert space H. Let A be a continuous monotone mapping of C such that $VI(C, A) \neq \emptyset$. Let*

$f : C \longrightarrow C$ be a Meir–Keeler-type contraction. For given $x_1 \in C$, let $\{x_n\}$ be a sequence defined by

$$\begin{cases} u_n = P_C(x_n - r_n A x_n), \\ x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)u_n, \quad \forall n \geq 1, \end{cases} \tag{7.2}$$

where $\{\alpha_n\} \subset (0,1)$ and $\{r_n\} \subset (0,\infty)$ are sequences which satisfy the following conditions:

(C1) $\lim_{n\longrightarrow\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$;
(C2) $\lim_{n\longrightarrow\infty} r_n = \infty$.

Then the sequence $\{x_n\}$ defined by (7.2) converges strongly to an element $x^* \in VI(C, A)$.

Acknowledgments This work was supported by the Higher Education Research Promotion and National Research University Project of Thailand , Office of the Higher Education Commission (NRU-CSEC No. 57000621).

References

1. Zeidler, E.: Nonlinear Functional Analysis and its Applications, Part II: Monotone Operators. Springer, Berlin (1985)
2. Browder, F.E.: Nonlinear monotone and accretive operators in Banach space. Proc. Natl. Acad. Sci. USA **61**, 388–393 (1968)
3. Martin Jr, R.H.: A global existence theorem for autonomous differential equations in Banach spaces. Proc. Am. Math. Soc. **26**, 307–314 (1970)
4. Wang, S.H., Li, Q.Y., Fu, J.Y.: Strong vector equilibrium problems on noncompact sets. Bull. Malays. Math. Sci. Soc. (2) **35**(1), 119–132 (2012)
5. Razani, A., Yazdi, M.: A new iterative method for generalized equilibrium and fixed point problems of nonexpansive mappings. Bull. Malays. Math. Sci. Soc. (2) **35**(4), 1049–1061 (2012)
6. Anh, P.N.: A hybrid extragradient method for pseudomonotone equilibrium problems and fixed point problems. Bull. Malays. Math. Sci. Soc. (2) **36**(1), 107–116 (2013)
7. Ofoedu, U.F., Zegeye, H., Shahzad, N.: Approximation of common fixed points of family of asymptotically nonexpansive mappings. Bull. Malays. Math. Sci. Soc. (2) **36**(3), 611–624 (2013)
8. Thuy, N.T., Hieu, P.T.: Implicit iteration methods for variational inequalities in Banach spaces. Bull. Malays. Math. Sci. Soc. (2) **36**(4), 917–926 (2013)
9. Onjai-Uea, N., Kumam, P.: Convergence theorems for generalized mixed equilibrium and variational inclusion problems of strict-pseudocontractive mappings. Bull. Malays. Math. Sci. Soc. (2) **36**(4), 1049–1070 (2013)
10. Wangkeeree, R., Preechasilp, P.: A general iterative method for multi-valued mappings. Bull. Malays. Math. Sci. Soc. **37**(3), 689–701 (2014)
11. Thi, N., Thuy, T.: An iterative method for equilibrium, variational inequality and fixed point problems for a nonexpansive semigroup in Hilbert spaces. Bull. Malays. Math. Sci. Soc. (2014) (in press)
12. Jaiboon, C., Kumam, P., Humphries, U.W.: Weak convergence theorem by an extragradient method for variational inequality, equilibrium and fixed point problems. Bull. Malays. Math. Sci. Soc. (2) **32**(2), 173–185 (2009)
13. Cho, Y.J., Qin, X.: Viscosity approximation methods for a family of m-accretive mappings in reflexive Banach spaces. Positivity **12**, 483–494 (2008)
14. Takahashi, W.: Viscosity approximation methods for resolvents of accretive operators in Banach spaces. J. Fixed Point Theory Appl. **1**, 135–147 (2007)
15. Petruşel, A., Yao, J.C.: Viscosity approximations by generalized contractions for resolvents of accretive operators in Banach spaces. Acta Math. Sin. Engl. Ser. **25**(4), 553–564 (2009)

16. Ceng, L.-C., Ansari, Q.H., Schaible, S., Yao, J.-C.: Hybrid viscosity approximation method for zeros of m-accretive operators in Banach spaces. Numer. Funct. Anal. Optim. **33**(2), 142–165 (2012)
17. Song, Y., Kang, J.I., Cho, Y.J.: On iterations methods for zeros of accretive operators in Banach spaces. Appl. Math. Comput. **216**, 1007–1017 (2010)
18. Hu, L., Liu, L.: A new iterative algorithm for common solutions of a finite family of accretive operators. Nonlinear Anal. **70**, 2344–2351 (2009)
19. Qing, Y., Cho, S.Y., Qin, X.: Convergence of iterative sequences for common zero points of a family of m-accretive mappings in Banach spaces. Fixed Point Theory Appl. **2011**, Article ID 216173: 12
20. He, X.-F., Xu, Y.-C., He, Z.: Iterative approximation for a zero of accretive operator and fixed points problems in Banach space. Appl. Math. Comput. **217**, 4620–4626 (2011)
21. Song, Y.: Iterative solutions for zeros of multivalued accretive operators. Math. Nachr. **284**(23), 370–380 (2011)
22. Bruck Jr, R.E.: A strongly convergent iterative method for the solution of 0 Ux for a maximal monotone operator U in Hilbert space. J. Math. Anal. Appl. **48**, 114–126 (1974)
23. Reich, S.: Strong convergence theorems for resolvents of accretive operators in Banach spaces. J. Math. Anal. Appl. **75**, 287–292 (1980)
24. Halpern, B.: Fixed points of nonexpansive maps. Bull. Am. Math. Soc. **73**, 957–961 (1967)
25. Dominguez Benavides, T., Lopez Acedoand, G., Xu, H.K.: Iterative solutions for zeros of accretive operators, Math. Nachr. **248**–**249**, 62–71 (2003)
26. Suzuki, T.: Moudafi's viscosity approximations with Meir–Keeler contractions. J. Math. Anal. Appl. **325**, 342–352 (2007)
27. Lim, T.C.: On characterizations of Meir–Keeler contractive maps. Nonlinear Anal. **46**(1), 113–120 (2001)
28. Petrusel, A., Yao, J.C.: Viscosity approximation to common fixed points of families of nonexpansive mappings with generalized contractions mappings. Nonlinear Anal. **69**(4), 1100–1111 (2008)
29. Agarwal, R.P., ORegan, D., Sahu, D.R.: Fixed Point Theory for Lipschitzian-type Mappings with Applications. Springer, New York (2009)
30. Takahashi, W.: Nonlinear Functional Analysis. Yokohama Publishers, Yokohama (2000)
31. Browder, F.E.: Convergence theorems for sequences of nonlinear operators in Banach spaces. Mathematische Zeitschrift **100**, 201–225 (1967)
32. Cioranescu, I.: Geometry of Banach Spaces, Duality Mappings and Nonlinear Problems, Vol. 62 of Mathematics and Its Applications. Kluwer Academic Publishers, Dordrecht, The Netherlands (1990)
33. Xu, H.K.: Iterative algorithms for nonlinear operators. J. Lond. Math. Soc. **66**(1), 240–256 (2002)
34. Aleyner, A., Censor, Y.: Best approximation to common fixed points of a semigroup of nonexpansive operator. Nonlinear Convex Anal. **6**(1), 137–151 (2005)
35. Aleyner, A., Reich, S.: An explicit construction of sunny nonexpansive retractions in Banach spaces. Fixed Point Theory Appl. **2005**(3), 295–305 (2005)
36. Benavides, T.D., Acedo, G.L., Xu, H.K.: Construction of sunny nonexpansive retractions in Banach spaces. Bull. Aust. Math. Soc. **66**(1), 9–16 (2002)
37. Nan Li, X., Gu, J.S.: Strong convergence of modified Ishikawa iteration for nonexpansive semigroup in Banach space. Nonlinear Anal. **73**, 1085–1092 (2010)
38. Blum, E., Oettli, W.: From optimization and variational inequalities to equilibrium problems. Math. Stud. **63**, 123–145 (1994)
39. Combettes, P.L., Hirstoaga, S.A.: Equilibrium programming in Hilbert spaces. J. Nonlinear Convex Anal. **6**, 117–136 (2005)
40. Rockafellar, R.T.: Monotone operators and the proximal point algorithm. SIAM J. Control Optim. **14**, 877–898 (1976)
41. Kamimura, S., Takahashi, W.: Weak and strong convergence of solutions to accretive operator inclusions and applications. Set-Valued Anal. **8**, 361–374 (2000)
42. Qin, X., Su, Y.: Approximation of a zero point of accretive operator in Banach spaces. J. Math. Anal. Appl. **329**, 415–424 (2007)
43. Bose, S.C.: Weak convergence to the fixed point of an asymptotically nonexpansive map. Proc. Am. Math. Soc. **68**, 305–308 (1978)
44. Browder, F.E.: Nonexpansive nonlinear operators in a Banach space. Proc. Natl. Acad. Sci. USA **54**(4), 1041–1044 (1965)