Bull. Malays. Math. Sci. Soc. (2016) 39:707–721
DOI 10.1007/s40840-015-0136-y

BULLETIN of the
MALAYSIAN MATHEMATICAL
SCIENCES SOCIETY
http://www.springer.com/mathematics/journal/40840

CrossMark

Topological Polygroups

D. Heidari[1] · B. Davvaz[1] · S. M. S. Modarres[1]

Received: 7 June 2013 / Revised: 22 August 2013 / Published online: 11 June 2015

Abstract This paper deals with certain algebraic systems called polygroups. A polygroup is a completely regular, reversible in itself hypergroup. The concept of topological polygroups is a generalization of the concept of topological groups. In this paper, we present the concept of topological hypergroups and prove some properties. Then, we define the notion of topological polygroups. By considering the relative topology on subpolygroups we prove some properties of them. Finally, the topological isomorphism theorems of topological polygroups are proved.

Keywords Topological hypergroup · Topological polygroup · Fundamental relation

Mathematics Subject Classification 20N20 · 22A30

1 Basic Definitions

Let H be a non-empty set. Then, a mapping $\circ : H \times H \to \mathcal{P}^*(H)$ is called a *hyperoperation*, where $\mathcal{P}^*(H)$ is the family of non-empty subsets of H. The couple $(H, \circ)$ is called a *hypergroupoid*. In the above definition, if A and B are two non-empty

Communicated by Rosihan Ali.

✉ B. Davvaz
davvaz@yazd.ac.ir

D. Heidari
dheidari82@gmail.com

S. M. S. Modarres
smodares@yazd.ac.ir

1 Department of Mathematics, Yazd University, Yazd, Iran

subsets of H and $x \in H$, then we define $A \circ B = \bigcup_{a\in A, b\in B} a \circ b$, $x \circ A = \{x\} \circ A$, and $A \circ x = A \circ \{x\}$. A hypergroupoid $(H, \circ)$ is called a *semihypergroup* if for every $x, y, z \in H$, we have $x \circ (y \circ z) = (x \circ y) \circ z$ and is called a *quasihypergroup* if for every $x \in H$, we have $x \circ H = H = H \circ x$. This condition is called the *reproduction axiom*. The couple $(H, \circ)$ is called a *hypergroup* if it is a semihypergroup and a quasihypergroup [5,21].

For all $n > 1$, we define the relation β_n on a semihypergroup H, as follows:

$$a \ \beta_n \ b \Leftrightarrow \exists (x_1, \ldots, x_n) \in H^n \ : \ \{a, b\} \subseteq \prod_{i=1}^{n} x_i,$$

and $\beta = \bigcup_{i=1}^{\infty} \beta_n$, where $\beta_1 = \{(x, x) \mid x \in H\}$ is the diagonal relation on H. This relation was introduced by Koskas [20] and studied mainly by Corsini, Davvaz, Freni, Leoreanu, Vougiouklis, and many others. Suppose that β^* is the smallest equivalence relation on a hypergroup (semihypergroup) H such that the quotient H/β^* is a group (semigroup). If H is a hypergroup, then $\beta = \beta^*$ [13]. The relation β^* is called as the *fundamental relation* on H, and H/β^* is called as the *fundamental group*. Let $(H, \circ)$ be a semihypergroup and A be a non-empty subset of H. We say that A is a *complete part* of H if for any non-zero natural number n and for all $a_1, \cdots, a_n$ of H, the following implication holds:

$$A \cap \prod_{i=1}^{n} a_i \neq \emptyset \Longrightarrow \prod_{i=1}^{n} a_i \subseteq A.$$

The complete parts were introduced for the first time by Koskas [20]. Then, this concept was studied by many authors, for example, see [5,6,10,11,17,22,23]. Till now, only a few papers treated the notion of topological hyperstructures, in the classical and fuzzy case, see [2,7,8,14,16]. Let $(H_1, \circ)$ and $(H_2, *)$ be two hypergroups. A map $f : H_1 \longrightarrow H_2$, is called

- a *homomorphism* if for all x, y of H, we have $f(x \circ y) \subseteq f(x) * f(y)$;
- a *good homomorphism* if for all x, y of H, we have $f(x \circ y) = f(x) * f(y)$;
- an *isomorphism* if it is a homomorphism, and its inverse f^{-1} is a homomorphism, too.

A special subclass of hypergroups is the class of polygroups. We recall the following definition from [3]. A *polygroup* is a system $P =< P, \circ, e, ^{-1} >$, where $\circ : P \times P \longrightarrow \mathcal{P}^*(P)$, $e \in P$, $^{-1}$ is a unitary operation on P and the following axioms hold for all $x, y, z \in P$:

(1) $(x \circ y) \circ z = x \circ (y \circ z)$;
(2) $e \circ x = x \circ e = x$;
(3) $x \in y \circ z$ implies $y \in x \circ z^{-1}$ and $z \in y^{-1} \circ x$.

The following elementary facts about polygroups follow easily from the axioms: $e \in x \circ x^{-1} \cap x^{-1} \circ x$, $e^{-1} = e$, $(x^{-1})^{-1} = x$, and $(x \circ y)^{-1} = y^{-1} \circ x^{-1}$. A non-empty subset K of a polygroup P is a subpolygroup of P if and only if $a, b \in K$ implies $a \circ b \subseteq K$ and $a \in K$ implies $a^{-1} \in K$. The subpolygroup N of P is *normal* in P if and only if $a^{-1} \circ N \circ a \subseteq N$ for all $a \in P$. For a subpolygroup K of P and $x \in P$, denote the right coset of K by $K \circ x$ and let P/K be the set of all right cosets of K in P. If N is a normal subpolygroup of P, then $(P/N, \odot, N, ^{-I})$ is a polygroup where $N \circ x \odot N \circ y = \{N \circ z \mid z \in N \circ x \circ y\}$ and $(N \circ x)^{-I} = N \circ x^{-1}$. For more details about polygroups we refer to [1,9,10,18].

Now, we recall the definition of a topological group from [15]. A *topological group* is a group G together with a topology on G that satisfies the following two properties:

(1) the mapping $p : G \times G \longrightarrow G$ defined by $p(g, h) = gh$ is continuous when $G \times G$ is endowed with the product topology;
(2) the mapping $inv : G \longrightarrow G$ defined by $inv(g) = g^{-1}$ is continuous.

We remark that item (1) is equivalent to the statement that, whenever $U \subseteq G$ is open, and $g_1 g_2 \in U$, then there exist open sets V_1 and V_2 such that $g_1 \in V_1$, $g_2 \in V_2$, and $V_1 V_2 = \{v_1 v_2 \mid v_1 \in V_1, v_2 \in V_2\} \subseteq U$. Also, item (2) is equivalent to showing that whenever $U \subseteq G$ is open, then $U^{-1} = \{g^{-1} \mid g \in U\}$ is open.

Let X be a topological space and $\sim$ an equivalence relation on X. For every $x \in X$, denote by $[x]$ its equivalence class. The quotient space of X modulo $\sim$ is given by the set $X/\sim = \{[x] \mid x \in X\}$. We have the projection map $p : X \longrightarrow X/\sim$, $x \mapsto [x]$ and we equip $X/\sim$ with the topology: $U \subseteq X/\sim$ is open if and only if $p^{-1}(U)$ is an open subset of X.

In this paper, we introduce the concept of topological hypergroups and topological polygroups as a generalization of topological groups. Let $(P, \circ, e, ^{-1})$ be a polygroup and (P, τ) be a topological space such that the mappings $(x, y) \mapsto x \circ y$ from $P \times P$ to $\mathcal{P}^*(P)$ and $x \mapsto x^{-1}$ from P to P are continuous with respect to product topology on $P \times P$ and the topology τ^* on $\mathcal{P}^*(P)$ induced by τ. By considering the relative topology on subpolygroups we prove some properties about them. In the last section, we prove the isomorphism theorems on topological polygroups.

2 Topological Algebraic Hyperstructures

Let (H, τ) be a topological space. The following lemma give us a topology on $\mathcal{P}^*(H)$ induced by τ.

Lemma 2.1 [16] *Let (H, τ) be a topological space. Then, the family $\mathcal{B}$ consisting of all sets $S_V = \{U \in \mathcal{P}^*(H) \mid U \subseteq V, U \in \tau\}$ is a base for a topology on $\mathcal{P}^*(H)$. This topology is denoted by τ^*.*

Let (H, τ) be a topological space. Then, we can consider the product topology on $H \times H$ and the topology τ^* on $\mathcal{P}^*(H)$.

Definition 2.2 Let $(H, \circ)$ be a hypergroup and (H, τ) be a topological space. Then, the system $(H, \circ, \tau)$ is called a *topological hypergroup* if

(1) the mapping $(x, y) \mapsto x \circ y$, from $H \times H \longrightarrow \mathcal{P}^*(H)$ is continuous;
(2) the mapping $(x, y) \mapsto x/y$, from $H \times H \longrightarrow \mathcal{P}^*(H)$ is continuous, where $x/y = \{z \in H \mid x \in z \circ y\}$.

Let H be a hypergroup and A and B be non-empty subsets of H. Then, $A/B = \bigcup\{a/b \mid a \in A, b \in B\}$. Let $(H, \circ, \tau)$ be a topological hypergroup and β^* be the fundamental relation on H. Then, $(H/\beta^*, \overline{\tau})$ is a topological space, where $\overline{\tau}$ is the quotient topology induced by the natural mapping $\pi : H \longrightarrow H/\beta^*$. That is $A \subseteq H/\beta^*$ is open in H/β^* if and only if $\pi^{-1}(A)$ is open in H. Let $(H, \circ, \tau)$ be a topological hypergroup such that every open subset of H is a complete part. Then, the natural mapping $\pi : H \longrightarrow H/\beta^*$ is an open mapping [14].

Theorem 2.3 [14] *Let $(H, \circ, \tau)$ be a topological hypergroup such that every open subset of H is a complete part. Then, $(H/\beta^*, \otimes, \overline{\tau})$ is a topological group.*

Theorem 2.4 *Let $(H, \circ, \tau)$ be a topological hypergroup and $U \in \tau$ such that U is a complete part. Then, $U = \bigcup_{u \in U} \beta^*(u)$.*

Proof Obviously, $U \subseteq \bigcup_{u \in U} \beta^*(u)$. Suppose that $u \in U$ and $x \in \beta^*(u)$. Then, there exist $a_1, \cdots, a_n \in H$ such that $\{x, u\} \subseteq \prod_{i=1}^{n} a_i$. Since U is a complete part, it follows that $x \in \prod_{i=1}^{n} a_i \subseteq U$ and so $\beta^*(u) \subseteq U$. Therefore, $U = \bigcup_{u \in U} \beta^*(u)$. □

Lemma 2.5 *Let $(H, \circ)$ be a hypergroup and β^* be the fundamental relation on H. Then, $\mathcal{B} = \{\beta^*(x) \mid x \in H\}$ is a base for a topology on H and every open subset of H is a complete part.*

Proof Since $H = \bigcup_{x \in H} \beta^*(x)$, it follows that $\mathcal{B}$ is a base for a topology on H. It is easy to see that every open subset of H is a complete part. □

We denote the topology in the previous lemma by τ_β.

Let τ_1 and τ_2 be two topologies on the same set X. Then, we say that τ_1 is stronger or finer than τ_2 if $\tau_1 \supset \tau_2$, and that then τ_2 is weaker or coarser than τ_1.

Theorem 2.6 *Let $(H, \circ)$ be a hypergroup and β^* be the fundamental relation on H. Then, τ_β is the finest topology on H such that H becomes a topological hypergroup and every open subset of H is a complete part.*

Proof Firstly, we prove that $(H, \circ, \tau_\beta)$ is a topological hypergroup. Suppose that $x, y \in H$ such that $x \circ y \subseteq U$ for some open subset U of H. So by Theorem 2.4, we have $U = \bigcup_{u \in U} \beta^*(u)$. Thus, there exists $u \in U$ such that $x \circ y \subseteq \beta^*(u)$. Hence, $\beta^*(x) \circ \beta^*(y) \subseteq \beta^*(u) \subseteq U$ and $\beta^*(x)$ and $\beta^*(y)$ are open subsets of H containing x and y, respectively. Therefore, the hyperoperation $\circ$ is continuous.

Similarly, we can prove that if $x/y \subseteq U$ for some open subset U and $x, y \in H$, then $\beta^*(x)/\beta^*(y) \subseteq U$.

Now, suppose that τ is a topology on H such that every open subset of (H, τ) is a complete part and $(H, \circ, \tau)$ is a topological hypergroup. Let $x \in U$ and $U \in \tau$. Then, by Theorem 2.4, we have $U = \bigcup_{u \in U} \beta^*(u)$. Thus, $\beta^*(x) \subseteq U$ and $\beta^*(x)$ is an open subset of (H, τ_β). Therefore, τ_β is the finest topology on H such that H becomes a topological hypergroup and every open subset of H is a complete part. □

Theorem 2.7 *Let $(H, \circ, \tau)$ be a T_0 topological hypergroup such that every open subset of H is a complete part. Then, H is a group.*

Proof We prove that $|x \circ y| = 1$ for every $x, y \in H$. Assume for the contradiction that $a, b \in x \circ y$ and $a \neq b$. Since H is T_0, it follows that there exists an open subset U of H containing exactly one of a or b. Let $a \in U$ and $b \notin U$. Then, $a \in \beta^*(u)$ for some $u \in U$. Thus, $b \in \beta^*(b) = \beta^*(a) = \beta^*(u)$ hence, $b \in U$, and it is a contradiction. So $|x \circ y| = 1$. Therefore, H is a group. □

Now, we introduce the concept of topological polygroups and prove some properties. Let P be a polygroup and τ a topology on P. Then, as in topological hypergroup we consider a topology τ^* on $\mathcal{P}^*(P)$ which is generated by $\mathcal{B} = \{S_V \mid V \in \tau\}$, where $S_V = \{U \in \mathcal{P}^*(P) \mid U \subseteq V, U \in \tau\}$.

In the following we use the topology τ^* on $\mathcal{P}^*(P)$ and the product topology on $P \times P$.

Definition 2.8 Let $P = (P, \circ, e, ^{-1})$ be a polygroup and (P, τ) be a topological space. Then, the system $P = (P, \circ, e, ^{-1}, \tau)$ is called a *topological polygroup* if the mappings $\circ : P \times P \longrightarrow \mathcal{P}^*(P)$ and $^{-1} : P \longrightarrow P$ are continuous.

Obviously, every topological group is a topological polygroup. Now, we give some other examples of topological polygroups.

Example 1 Every polygroup equipped with discrete or indiscrete topology is a topological polygroup.

Example 2 Let P be a polygroup and β^* be the fundamental relation of P. Then,

$$\tau = \left\{ \bigcup_{u \in U} \beta^*(u) \mid U \subseteq P \right\} \cup \{\emptyset\}$$

is a topology on P, and $(P, \circ, e, ^{-1}, \tau)$ is a topological polygroup.

In [4], an extension of polygroups by polygroups has been introduced in the following way: Suppose A and B are polygroups whose elements have been renamed so that $A \cap B = \{e\}$, where e is the identity of both A and B. A new system $\mathbb{A}[\mathbb{B}] = (M, *, e, ^I)$ called as the extension of A by B, is formed in the following way: Set $M = \{x \mid x \in A, x \neq e\} \cup \{x \mid x \in B, x \neq e\} \cup \{e\}$ and let $e^I = e$, $x^I = x^{-1}$ (in the appropriate system), $e * x = x * e = x$ for all $x \in M$, and for all $x, y \in \{x \mid x \in M, x \neq e\}$:

$$x * y = \begin{cases} x.y & \text{if } x, y \in A \\ x & \text{if } x \in B, y \in A \\ y & \text{if } x \in A, y \in B \\ x.y & \text{if } x, y \in B, y \neq x^{-1} \\ x.y \cup A & \text{if } x, y \in B, y = x^{-1}. \end{cases}$$

The extension $\mathbb{A}[\mathbb{B}]$ is a polygroup.

Theorem 2.9 *Let* $(A, \circ_1, e_1, ^{-1})$ *be a polygroup and* $(B, \circ_2, e_2, ^{-1}, \tau_B)$ *be a topological polygroup. Then, there is a topology on* $\mathbb{A}[\mathbb{B}]$ *such that* $\mathbb{A}[\mathbb{B}]$ *is a topological polygroup.*

Proof We define a topology on $\mathbb{A}[\mathbb{B}]$ as follows

$$\tau_{\mathbb{A}[\mathbb{B}]} = \{U \cup A \mid U \in \tau_B\} \cup \{\emptyset\}.$$

Then, $(\mathbb{A}[\mathbb{B}], \tau_{\mathbb{A}[\mathbb{B}]})$ is a topological space. Suppose that $x \in \mathbb{A}[\mathbb{B}]$ and $U \cup A$ be an open subset of $\mathbb{A}[\mathbb{B}]$ such that $x^{-1} \in U \cup A$ for some open subset U of B. If $x^{-1} \in A$, then $x \in A \subseteq U \cup A$. If $x^{-1} \in U$, then there exists an open subset V of B such that $x \in V$ and $V^{-1} \subseteq U$ hence, $V^{-1} \subseteq U \cup A$. Therefore, the mapping $x \longmapsto x^{-1}$ is continuous. Suppose that $x, y \in \mathbb{A}[\mathbb{B}]$ and $U \in \tau_B$ such that $x * y \subseteq U \cup A$, then we have the following cases:

Case 1 If $x, y \in A$, then $x * y = x \circ_1 y \subseteq A.A \subseteq A \subseteq U \cup A$.
Case 2 If $x \in A$ and $y \in B$, then $x * y = y \in U \subseteq U \cup A$.
Case 3 If $x \in B$ and $y \in A$, then $x * y = x \in U \subseteq U \cup A$.
Case 4 If $x, y \in B$ and $x \neq x^{-1}$, then $x * y = x \circ_2 y \subseteq U$. So there exist open subsets V and W of B containing x and y, respectively, such that $V \cdot W \subseteq U$. Thus, $(V \cup A) * (W \cup A) \subseteq V \cdot W \cup A \subseteq U \cup A$.
Case 5 If $x, y \in B$, then $x * y = x \circ_2 y \cup A \subseteq U \cup A$. Also, we can do the similar way to Case 4.

Thus, the hyperoperation $*$ is continuous. Therefore, $(\mathbb{A}[\mathbb{B}], *, I, \tau_{\mathbb{A}[\mathbb{B}]})$ is a topological polygroup. □

By using the previous theorem we can construct topological polygroups by considering B as a topological group.

Example 3 Consider the topological group $(\mathbb{R}, +)$ with standard topology. Then, $\mathbb{Z}_3[\mathbb{R}]$ is a topological polygroup.

Example 4 Consider symmetric group S_3. Let $\tau = \{\emptyset, A_3, A_3^c, S_3\}$, where A_3 is the set of all even permutations of S_3 and $A_3^c = S_3 \backslash A_3$. Then, (S_3, τ) is a topological group so $\mathbb{Z}_2[S_3]$ is a topological polygroup.

In [14] we prove the next two lemmas for topological hypergroups. In the following we rewrite them for topological polygroups.

Lemma 2.10 *Let P be a topological polygroup. Then, the hyperoperation $\circ : P \times P \longrightarrow \mathcal{P}^*(\mathcal{P})$ is continuous if and only if for every $x, y \in P$ and $U \in \tau$ such that $x \circ y \subseteq U$ then there exist $V, W \in \tau$ such that $x \in V$ and $y \in W$ and $V \circ W \subseteq U$.*

Lemma 2.11 *Let P be a topological polygroup. Then, the mappings*

$$_a\varphi : P \longrightarrow \mathcal{P}^*(P) \text{ by } x \mapsto a \circ x,$$
$$\varphi_a : P \longrightarrow \mathcal{P}^*(P) \text{ by } x \mapsto x \circ a$$

are continuous, for every $a \in P$.

Lemma 2.12 *Let P be a topological polygroup, $A \subseteq P$ and U be an open subset of P. Then, $A \subseteq x^{-1} \circ U$ if and only if $x \circ A \subseteq U$ for all $x \in P$.*

Proof Suppose that $A \subseteq x^{-1} \circ U$ and $t \in x \circ a$ for some $a \in A$. Then, $a \in x^{-1} \circ t \cap x^{-1} \circ U$. So $a \in x^{-1} \circ u$ for some $u \in U$. Thus, $u \in x \circ a \cap U$ hence $x \circ a \subseteq U$. Therefore, $x \circ A \subseteq U$.

Conversely, suppose that $x \in P$ and $x \circ A \subseteq U$. Then, we have $A \subseteq (x^{-1} \circ x) \circ A = x^{-1} \circ (x \circ A) \subseteq x^{-1} \circ U$. □

Lemma 2.13 *Let U be an open subset of a topological polygroup P such that U is a complete part. Then, $a \circ U$ and $U \circ a$ are open subsets of P for every $a \in P$.*

Proof Suppose that U is an open subset of P such that U is a complete part and $a \in P$. Then, by Lemma 2.12 we have

$$\varphi_{a^{-1}}^{-1}(S_U) = \{x \in P \mid a^{-1} \circ x \subseteq U\} = a \circ U.$$

So by Lemma 2.11, the mapping $\varphi_{a^{-1}}$ is continuous; thus, $a \circ U$ is open. Similarly, we can prove that $U \circ a$ is open. □

Theorem 2.14 *Let P be a topological polygroup and A, B be open subsets of P. If A or B is a complete part, then $A \circ B$ is open.*

Proof Suppose that A is a complete part. By Lemma 2.11, $A \circ b$ is open. Since every arbitrary union of open subsets is open, it follows that $A \circ B = \bigcup_{b \in B} A \circ b$ is open. □

Lemma 2.15 *Let P be a topological polygroup such that every open subset of P is a complete part. Let $\mathcal{U}$ be an open basis at e. Then, the families $\{x \circ U\}$ and $\{U \circ x\}$, where x runs through all elements of P and U runs through all elements of $\mathcal{U}$, are open basis for P.*

Proof Suppose that W is an open subset of P and $a \in W$. Since $e \in a^{-1} \circ W$, it follows that there exists $U \in \mathcal{U}$ such that $e \in U \subseteq a^{-1} \circ W$. Since W is a complete part we conclude that $a \in a \circ U \subseteq W$. Thus, W is a union of open subsets $a \circ U$. Therefore, $\{x \circ U\}$ is an open basis for P. Similarly, the family $\{U \circ x\}$ is a basis for P. □

Theorem 2.16 *Let P be a topological polygroup and $\mathcal{U}$ be a basis at e. Then, the following assertions hold:*

(1) *for every $U \in \mathcal{U}$ and $x \in U$ there exists $V \in \mathcal{U}$ such that $x \circ V \subseteq U$;*
(2) *for every $U \in \mathcal{U}$ there exists $V \in \mathcal{U}$ such that $V \circ V \subseteq U$;*
(3) *for every $U \in \mathcal{U}$ there exists $V \in \mathcal{U}$ such that $V^{-1} \subseteq U$.*

Proof The proofs are straightforward. □

As in topological spaces, we use the term "neighborhood" for open subsets. An open subset U of a topological polygroup P is called a symmetric neighborhood if $U^{-1} = U$.

Theorem 2.17 *Every topological polygroup has an open basis at e containing a symmetric open basis at e.*

Proof Suppose that $\mathcal{U}$ is an open basis at e. Then, for every $U \in \mathcal{U}$, put $V = U \cap U^{-1}$. Then, $V = V^{-1}$ and $V \subseteq U$. □

Theorem 2.18 *Let P be a topological polygroup such that every open subset of P is a complete part. Then, for every neighborhood U of e there exists a neighborhood V of e such that $\overline{V} \subseteq U$, where $\overline{V}$ is the closure of V.*

Proof Suppose that V is a symmetric neighborhood of e such that $V \circ V \subseteq U$. Now, if $x \in \overline{V}$, then $x \circ V \cap V \neq \emptyset$. So there exist $v_1, v_2 \in V$ such that $v_2 \in x \circ v_1$. Thus, $x \in v_2 \circ v_1^{-1} \subseteq V \circ V^{-1} = V \circ V \subseteq U$. □

Theorem 2.19 *Let P be a topological polygroup such that every open subset of P is a complete part, U be any neighborhood of e, and F be a any compact subset of P. Then, there exists a neighborhood V of e such that $x \circ V \circ x^{-1} \subseteq U$ for all $x \in F$.*

Proof Suppose that U be a neighborhood of e so by Theorem 2.16, there exists a symmetric neighborhood T of e such that $T \circ T \subseteq U$. Applying Theorem 2.16 for T, we conclude that there exists a symmetric neighborhood W of e such that $W \circ W \subseteq T$. So we have $W \circ W \circ W \subseteq T \circ T \subseteq U$. Since F is compact and $F \subseteq \cup_{x \in F} W \circ x$, it follows that there exist $x_1, \cdots, x_n \in F$ such that $F \subseteq \cup_{i=1}^{n} W \circ x_i$.

Let $V = \cap_{i=1}^{n} x_i^{-1} W \circ x_i$. We claim that $x_i^{-1} \circ V \circ x_i \subseteq W$, for $i = 1, \cdots, n$. Since W is a complete part and $w \in (x_i \circ x_i^{-1}) \circ w \circ (x_i^{-1} \circ x_i) \cap W$ for $i = 1, \cdots, n$ and $w \in W$, it follows that $(x_i \circ x_i^{-1}) \circ w \circ (x_i^{-1} \circ x_i) \subseteq W$. So for every $1 \leq k \leq n$ we have

$$\begin{aligned} x_k \circ V \circ x_k^{-1} &= x_k \circ \left(\bigcap_{i=1}^{n} (x_i^{-1} \circ W \circ x_i) \right) \circ x_k \\ &\subseteq x_k \circ x_k^{-1} \circ W \circ x_k \circ x_k^{-1} \subseteq W. \end{aligned}$$

Therefore, for every $x \in F$ there exist $w \in W$ and $1 \leq k \leq n$ such that $x \in w \circ x_k$, hence we have

$$\begin{aligned} x \circ V \circ x^{-1} &\subseteq (w \circ x_k) \circ V \circ (x_k^{-1} \circ w^{-1}) \subseteq w \circ (x_k \circ V \circ x_k^{-1}) \circ w^{-1} \\ &\subseteq w \circ W \circ w \subseteq W \circ W \circ W \subseteq U. \end{aligned}$$

□

Theorem 2.20 *Let P be a topological polygroup such that every open subset of P is a complete part, U be any neighborhood of e, and F be a any compact subset of P such that $F \subseteq U$. Then, there exists a neighborhood V of e such that $(F \circ V) \cup (V \circ F) \subseteq U$.*

Proof Suppose that F is a compact subset of P and U be a neighborhood of e such that $F \subseteq U$. Then, for every $x \in F$ there exist a neighborhood W_x of e such that $x \circ W_x \subseteq U$ and a neighborhood V_x of e such that $V_x \circ V_x \subseteq W_x$. Since F is compact and $F \subseteq \cup_{x \in F} x \circ V_x$, so there exist $x_1, \cdots, x_n \in F$ such that $F \subseteq \cup_{i=1}^{n} x_i \circ V_{x_i}$. Let $V_1 = \cap_{i=1}^{n} x_i \circ V_{x_i}$. Hence, we have

$$F \circ V_1 \subseteq \left(\bigcup_{i=1}^{n} x_i \circ V_{x_i} \right) \circ V_1 \subseteq \bigcup_{i=1}^{n} x_i \circ V_{x_i} \circ V_{x_i} \subseteq \bigcup_{i=1}^{n} x_i \circ W_{x_i} \subseteq U.$$

□

3 Subpolygroups of a Topological Polygroup

In this section we introduce the concept of subpolygroups of a topological polygroup. We consider the relative topology on a subpolygroup.

Theorem 3.1 *Let P be a topological polygroup. Then, every subpolygroup K of P with relative topology is a topological polygroup.*

Proof Since the restriction of mappings hyperoperation and inverse to K continues, the result holds. □

Lemma 3.2 *Let A and B be subsets of a topological polygroup P such that every open subset of P is a complete part. Then, the following assertions hold:*

(1) $\overline{A} \circ \overline{B} \subseteq \overline{A \circ B}$;
(2) $(\overline{A})^{-1} = \overline{(A^{-1})}$.

Proof (1) Suppose that $t \in \overline{A} \circ \overline{B}$. Then, $t \in x \circ y$ for some $x \in \overline{A}$ and $y \in \overline{B}$. We prove that each neighborhood U of t has a non-empty intersection with $A \circ B$. Since U is a complete part, it follows that $x \circ y \subseteq U$. Thus, there exist neighborhoods V and W containing x and y, respectively, such that $V \circ W \subseteq U$. From $x \in V \cap \overline{A}$ and $y \in W \cap \overline{B}$ we conclude that there exist $a \in V \cap A$ and $b \in W \cap B$. Now, we have $a \circ b \subseteq U \cap A \circ B$. Therefore, $t \in \overline{A \circ B}$.

(2) Suppose that $x \in \overline{A}^{-1}$. Then, $x^{-1} \in \overline{A}$. If $x \in U \in \tau$, then $x^{-1} \in U^{-1}$ so there exists $y \in A \cap U^{-1}$ thus $y^{-1} \in A^{-1} \cap U$. Hence, $x \in \overline{A^{-1}}$. Thus, $\overline{A}^{-1} \subseteq \overline{A^{-1}}$. Similarly, we can prove that $\overline{A^{-1}} \subseteq \overline{A}^{-1}$. Therefore, $(\overline{A})^{-1} = \overline{(A^{-1})}$. □

Theorem 3.3 *Let P be a topological polygroup such that every open subset of P is a complete part. Then, the following assertions hold:*

(1) *If K is a subsemihypergroup of P, then $\overline{K}$ is a subsemihypergroup of P.*
(2) *If K is a subpolygroup of P, then $\overline{K}$ is a subpolygroup of P.*

Proof (1) Suppose that K is a subsemihypergroup of P; then $K \circ K \subseteq K$. By Lemma 3.2, we have $\overline{K} \circ \overline{K} \subseteq \overline{K \circ K} \subseteq \overline{K}$; thus, $\overline{K}$ is a subsemihypergroup of P.

(2) Suppose that K is a subpolygroup of P; then $K^{-1} \subseteq K$. By Lemma 3.2, we have $\overline{K}^{-1} = \overline{K^{-1}} \subseteq \overline{K}$; thus, $\overline{K}$ is a subpolygroup of P. □

Theorem 3.4 *Let P be a topological polygroup such that every open subset of P is a complete part. Then, every subpolygroup K of P is open if and only if its interior is non-empty.*

Proof Suppose that x is an interior point of K. Then, there exists a neighborhood U of e such that $x \circ U \subseteq K$. Now, for every $y \in K$ we have

$$y \circ U \subseteq y \circ (x^{-1} \circ x) \circ K = (y \circ x^{-1}) \circ (x \circ K) = (y \circ x^{-1}) \circ K = K.$$

So y is an interior point of K. Hence, K is open. □

Theorem 3.5 *Let P be a topological polygroup such that every open subset of P is a complete part. Then, every open subpolygroup is closed.*

Proof Suppose that K is an open subpolygroup of P, then we have

$$P = \bigcup_{x \in P} x \circ K = K \cup \left(\bigcup_{x \notin K} x \circ K \right).$$

So $K^c = \bigcup_{x \notin K} x \circ K$. Now, since K is a complete part, it follows that $x \circ K$ is open. Thus, K^c is open and it conclude that K is closed. □

Theorem 3.6 *Let $\mathcal{A}$ be a family of neighborhoods of e in a topological polygroup P such that*

(1) *for each $U \in \mathcal{A}$, there is $V \in \mathcal{A}$ such that $V \circ V \subseteq U$;*
(2) *for each $U \in \mathcal{A}$, there is $V \in \mathcal{A}$ such that $V^{-1} \subseteq U$;*
(3) *for each $U, V \in \mathcal{A}$, there is $W \in \mathcal{A}$ such that $W \subseteq U \circ V$.*

Let $H = \cap\{U \mid U \in \mathcal{A}\}$. Then, H is a closed subpolygroup of P.

Proof Suppose that $x, y \in H$ and $U \in \mathcal{A}$. Then, by (1) there exists $V \in \mathcal{A}$ such that $V \circ V \subseteq U$. Thus, $x, y \in V$ so $x \circ y \subseteq V \circ V \subseteq U$. Hence, $x \circ y \subseteq H$. Similarly, we can prove that if $x \in H$, then $x^{-1} \in H$. Therefore, H is a subpolygroup of P. Now, we prove that H is closed. Let $x \in P \backslash H$. Then, $x \notin U$ for some $U \in \mathcal{A}$. So by (1), (2), and (3) there exist $V_1, V_2, V \in \mathcal{A}$ such that $V_1 \circ V_1 \subseteq U$, $V_2^{-1} \subseteq V_1$ and $V \subseteq V_1 \cap V_2$. Thus, $V \circ V^{-1} \subseteq U$. Hence, if $x \circ V \cap V \neq \emptyset$, then we have $x \in V \circ V^{-1} \subseteq U$ so $x \in H$, and it is a contradiction. Thus, $x \in x \circ V \subseteq P \backslash H$. Thus, $P \backslash H$ is open, that is, H is closed. □

Theorem 3.7 *Let U be a symmetric neighborhood of e in a topological polygroup P such that every open subset of P is a complete part. Then, the set $L = \bigcup_{n=1}^{\infty} U^n$ is an open and closed subpolygroup of P, where $U^2 = U \circ U$ and $U^n = U^{n-1} \circ U$ for every $n \in \mathbb{N}$.*

Proof If $x \in U^k$ and $y \in U^t$, then $x \circ y \subseteq U^{k+t}$ and $x^{-1} \in (U^{-1})^k = U^k$, for every $k, t \in \mathbb{N}$. Hence, L is a subpolygroup of P. By Theorem 3.4, L is open and closed. □

Theorem 3.8 *Let P be a topological polygroup such that every open subset of P is a complete part. Then, a subpolygroup H of P is closed if and only if there is an open subset U of P such that $U \cap H = U \cap \overline{H} \neq \emptyset$.*

Proof If H is closed subpolygroup of P, then it is sufficient to consider U as a neighborhood of e.

Conversely, suppose that there is an open subset U of P such that $U \cap H = U \cap \overline{H}$ and $U \cap H \neq \emptyset$. Let $x \in \overline{H}$ and $y \in U \cap H$. Then, $x \in x \circ y^{-1} \circ U$ and by Lemma 2.13 $x \circ y^{-1} \circ U$ is an open subset of P. So there exists $h \in H \cap x \circ y^{-1} \circ U$. Thus, $h \in x \circ y^{-1} \circ u$ for some $u \in U$, hence $u \in y \circ x^{-1} \circ h$. So $u \in U \cap \overline{H}$, since by Theorem 3.3, $\overline{H}$ is a subpolygroup of P. Thus, $u \in U \cap H$ hence $x \in h \circ u^{-1} \circ y \subseteq H$. Therefore, $\overline{H} = H$, that is, H is closed subpolygroup of P. □

Theorem 3.9 *Let P be a topological polygroup such that every open subset of P is a complete part and H is a non-closed subpolygroup of P. Then, $\overline{H} \cap H^c$ is dense in $\overline{H}$.*

Proof Suppose that H is a non-closed subpolygroup of P. Then, by the previous theorem, for every open subset U of P, $U \cap H = \emptyset$ or $\emptyset \neq U \cap H \subsetneq U \cap \overline{H}$. Let $x \in \overline{H}$ and U be a neighborhood of x. Then, $U \cap H \neq \emptyset$. So there exists $u \in U \cap \overline{H} \backslash U \cap H$. Thus, $u \in U \cap (\overline{H} \cap H^c)$. Therefore, $\overline{H} \cap H^c$ is dense in $\overline{H}$.

□

4 Isomorphism Theorems

In this section we state and prove the isomorphism theorems for topological polygroups.

Let $(P, \circ, e, ^{-1}, \tau)$ be a topological polygroup and N be a normal subpolygroup of P. Let π be the natural mapping $x \mapsto N \circ x$ of P onto P/N. Then, $(P/N, \overline{\tau})$ is a topological space, where $\overline{\tau}$ is the quotient topology induced by π. That is for every subset X of P we have $\{N \circ x \mid x \in X\}$ is an open subset of P/N if and only if $\pi^{-1}(\{N \circ x \mid x \in X\})$ is an open subset of P. In the following, the notation X/N is used for $\{N \circ x \mid x \in X\}$ for every subset X of P.

Theorem 4.1 *Let $(P, \circ, e, ^{-1}, \tau)$ be a topological polygroup such that every open subset of H is complete part. Then, $(P/\beta^*, \otimes, \overline{\tau})$ is a topological group, where β^* is the fundamental relation of P and $\beta^*(x) \otimes \beta^*(y) = \beta^*(z), z \in x \circ y$ for every $x, y \in P$.*

Proof It follows from Theorem 2.3. □

Definition 4.2 Let $< P_1, \circ_1, e_1, ^{-1}, \tau_1 >$ and $< P_2, \circ_2, e_2, ^{-1}, \tau_2 >$ be topological polygroups. A mapping φ from P_1 into P_2 is said to be a good topological homomorphism if for all $a, b \in P_1$,

(1) $\varphi(e_1) = e_2$;
(2) $\varphi(a \circ_1 b) = \varphi(a) \circ_2 \varphi(b)$;
(3) φ is continuous;
(4) φ is open.

Clearly, a good topological homomorphism φ is a topological isomorphism if φ is one to one and onto. We write $P_1 \cong P_2$ if P_1 is topologically isomorphic to P_2.

Because P_1 is a polygroup, $e_2 \in a \circ_1 a^{-1}$ for all $a \in P_1$; then we have $\varphi(e_1) \in \varphi(a) \circ_2 \varphi(a^{-1})$ or $e_2 \in \varphi(a) \circ_2 \varphi(a^{-1})$ which implies $\varphi(a^{-1}) \in \varphi(a)^{-1} \circ_2 e_2$; therefore, $\varphi(a^{-1}) = \varphi(a)^{-1}$ for all $a \in P_1$. Moreover, if φ is a strong topological homomorphism from P_1 into P_2, then the kernel of φ is the set $ker\varphi = \{x \in P_1 \mid \varphi(x) = e_2\}$. It is trivial that $ker\varphi$ is a subpolygroup of P_1 but in general is not normal in P_1.

As in polygroups, if φ is a good topological homomorphism from P_1 into P_2, then, φ it is injective if and only if $ker\varphi = \{e_1\}$.

Theorem 4.3 *Let $(P_1, \circ_1, ^{-1}, e_1, \tau_1)$ and $(P_2, \circ_2, ^{-1}, e_2, \tau_2)$ be two topological polygroups and $f : P_1 \longrightarrow P_2$ be a homomorphism. Then, f is continuous if and only if is continuous at e_1.*

Proof Obviously, if f is continuous, then f is continuous at e_1. Conversely, suppose that f is continuous at e_1 and $f(x) \in U_2$ for some $x \in P_1$ and open subset U_2 of P_2. Now, we have $f(e_1) \in f(x^{-1} \circ_1 x) = f(x)^{-1} \circ_2 f(x) \subseteq f(x)^{-1} \circ_2 U_2$, so there exists an open subset U_1 of P_1 containing e_1 such that $f(U_1) \subseteq f(x)^{-1} \circ_2 U_2$. Hence, by Lemma 2.12, we have $f(x \circ_1 U_1) = f(x) \circ_2 f(U_1) \subseteq U_2$. Therefore, f is continuous at x. □

Lemma 4.4 *Let P be a topological polygroup and N be a normal subpolygroup of P. Let π be the natural mapping $x \longmapsto N \circ x$ of P onto P/N. Then,*

(1) *$\pi^{-1}(\{N \circ x \mid x \in X\}) = N \circ X$ for every subset X of P;*
(2) *$\{N \circ x \mid x \in X\} = \{N \circ y \mid y \in N \circ X\}$ for every subset X of P;*
(3) *If every open subset of P is a complete part, then the natural mapping π is open.*

Proof (1) Obviously, we have that $N \circ X \subseteq \pi^{-1}(\{N \circ x \mid x \in X\})$ for every subset X of P. We prove the converse of inclusion. Suppose that $y \in \pi^{-1}(\{N \circ x \mid x \in X\})$. Then, $\pi(y) = N \circ y \in \{N \circ x \mid x \in X\}$. So $N \circ y = N \circ x$ for some $x \in X$. Thus, $y \circ x^{-1} \cap N \neq \emptyset$. Hence, there exists $n \in N$ such that $n \in x \circ y^{-1}$ and this implies $y \in n \circ x \subseteq N \circ X$. Therefore, the proof is complete.
(2) For every subset X of P we have $X \subseteq N \circ X$ so $\{N \circ x \mid x \in X\} \subseteq \{N \circ y \mid y \in N \circ X\}$. On the other hand, if $y \in N \circ X$, there exist $n \in N$ and $x \in X$ such that $y \in n \circ x$. Thus, $N \circ y = N \circ x$ and the proof is complete.
(3) If U is an open subset of P, then by (1) we have $\pi^{-1}(\pi(U)) = N \circ U$. Since U is a complete part, it follows that $N \circ U$ is open in P by Lemma 2.13. Therefore, π is open. □

Let N be a normal subpolygroup of topological polygroup P and every open subset of P be a complete part. Let A be an open subset of P/N. Then, by the previous lemma, $A = U/N$ for some open subset U of P.

Theorem 4.5 *Let N be a normal subpolygroup of topological polygroup P and every open subset of P be a complete part. Then, $< P/N, \odot, N, ^{-I} >$ is a topological polygroup, where $N \circ x \odot N \circ y = \{N \circ z \mid z \in x \circ y\}$ and $(N \circ x)^{-I} = N \circ x^{-1}$.*

Proof We prove that the hyperoperation $\odot$ and the unitary operation $^{-I}$ are continuous. Suppose $N \circ x, N \circ y \in P/N$, and $\mathcal{A}$ is an open subset of P/N such that $N \circ x \odot N \circ y \subseteq \mathcal{A}$. Then, $x \circ y \subseteq \pi^{-1}(\mathcal{A})$. Since $\pi^{-1}(\mathcal{A})$ is open in P, there exist open subsets V and W of P containing x and y, respectively, such that $V \circ W \subseteq \pi^{-1}(\mathcal{A})$. It follows that $\pi(V)$ and $\pi(W)$ are open in P/N containing $N \circ x$ and $N \circ y$, respectively, such that $\pi(V) \odot \pi(W) \subseteq \mathcal{A}$. Therefore, the hyperoperation $\odot$ is continuous.

Suppose that $(N \circ x)^{-I} = N \circ x^{-1} \in \mathcal{A}$. Then, $x^{-1} \in \pi^{-1}(\mathcal{A})$. Thus, there exists an open subset U in P such that $x^{-1} \in U^{-1} \subseteq \pi^{-1}(\mathcal{A})$ so $\pi(x^{-1}) = N \circ x^{-1} \in \pi(U^{-1}) \subseteq \mathcal{A}$ and $\pi(U^{-1})$ is open in P/N. □

The isomorphism theorems of polygroups are presented in [10]. In the following we prove them for topological polygroups.

Theorem 4.6 *Let $(P_1, \circ_1, e_1, ^{-1}, \tau_1)$ and $(P_2, \circ_2, e_2, ^{-1}, \tau_2)$ be topological polygroups such that every open subset of P_1 is a complete part. Let φ be an open and continuous good topological homomorphism from P_1 onto P_2 such that $N = ker\varphi$ is a normal subpolygroup of P_1. Then, P_1/N and P_2 are topologically isomorphic.*

Proof We define the mapping $\psi : P_2 \longrightarrow P_1/N$ by setting $\psi(x_2) = N \circ_1 x_1$ where, $\varphi(x_1) = x_2$, for all $x_2 \in P_2$. Since φ is onto, so $\varphi^{-1}(x_2) \neq \emptyset$. If $x_1, y_1 \in \varphi^{-1}(x_2)$, then $\varphi(x_1) = x_2 = \varphi(y_1)$. Thus, $e_2 \in \varphi(x_1 \circ_1 y_1^{-1})$, hence there exists $n \in x_1 \circ_1 y_1^{-1}$ such that $\varphi(n) = e_2$. Now, we have $N \circ_1 x_1 \subseteq N \circ_1 (n \circ_1 y_1) = N \circ_1 y_1 \subseteq N \circ_1 n^{-1} \circ_1 x_1 = N \circ_1 x_1$. Therefore, ψ is well defined. Obviously, ψ is onto and an algebraic homomorphism. If $\psi(x_2) = N \circ_1 x_1 = \psi(y_2) = N \circ_1 y_1$, then $x_1 \in n \circ_1 y_1$ for some $n \in N$. Thus, $x_2 = \varphi(x_1) \in \varphi(n) \circ_2 \varphi(y_1) = y_2$ hence, ψ is one-to-one. Therefore, ψ is an algebraic isomorphism.

Now, we show that ψ is open and continuous. Suppose that U_2 is an open subset of P_2. Then, $\psi(U_2) = \{N \circ_1 u_1 \mid u_1 \in \varphi^{-1}(U_2)\} = \varphi^{-1}(U_2)/N$. Since φ is continuous, it follows that $\varphi^{-1}(U_2)/N$ is open in P_1/N. Therefore, ψ is open.

If U_1/N is an open subset of P_1/N, then $\psi^{-1}(U_1/N)$ is open in P_2 since φ is open and we have

$$\psi^{-1}(U_1/N)=\{u_2 \mid \psi(u_2) \in U_1/N\}=\{u_2 \mid N \circ_1 u_1 \in U_1/N, \varphi(u_1)=u_2\}=\varphi(U_1).$$

Therefore, ψ is continuous, and the proof is complete. □

Theorem 4.7 *Let K and N be subpolygroups of a polygroup P with N normal and K open in P such that every open subset of P is a complete part. Then, $K/(N \cap K)$ and $(N \circ K)/N$ are topologically isomorphic.*

Proof Define $\varphi : K \longrightarrow P/N$ by $\varphi(k) = N \circ k$. Then, φ is a strong homomorphism and $ker\varphi = N \cap K$. Since $K \subseteq N \circ K$ and φ is the restriction of π on K, it follows that φ is open and continuous. It remains to show that $Im(\varphi) = N \circ K/N$. If $x \in N \circ K$, then $x \in n \circ k$, for some $n \in N$ and $k \in K$. Hence, $\varphi(k) = N \circ k = N \circ n \circ k = N \circ x$. So $N \circ K/N \subseteq Im(K) \subseteq N \circ K/N$. Therefore, by previous theorem, $K/(N \cap K) \cong (N \circ K)/N$. □

Theorem 4.8 *Let K and N be normal subpolygroups of a polygroup P such that every open subset of P is a complete part and $N \subseteq K$. Then, $(P/N)/(K/N)$ and P/K are topological isomorphic.*

Proof The mapping $\varphi : P/N \longrightarrow P/K$, where $\varphi(N \circ x) = K \circ x$ is a good homomorphism and we have $Ker\varphi = K/N$. If U is an open subset of P, then we have $\varphi(U/N) = U/K$. Therefore, φ is open and continuous. So by Theorem 4.6 we conclude that $(P/N)/(K/N)$ and P/K are topologically isomorphic. □

Theorem 4.9 *If N_1, N_2 are normal subpolygroups of P_1 and P_2, respectively, then $N_1 \times N_2$ is a normal subpolygroup of $P_1 \times P_2$ and $(P_1 \times P_2)/(N_1 \times N_2)$ and $P_1/N_1 \times P_2/N2$ are topological isomorphic.*

Proof It is straightforward. □

5 Conclusion

This paper deals with one of the newest argument from hyperstructure theory namely topological hypergroups. Applications of hypergroups have mainly appeared in special subclasses. One of the important subclasses is the class of polygroups. Indeed the structure of a polygroup is more near to the structure of a group. So, in the paper we studied the concept of topological polygroups. The concept of topological polygroups is a generalization of the concept of topological groups. It is important to mention that in this paper, the topological polygroups and topological hypergroups are different from topological hypergroups which were initiated by Dunkl [12] and Jewett [19].

Acknowledgments The authors are highly grateful to referees for their valuable comments and suggestions for improving the paper.

References

1. Aghabozorgi, H., Davvaz, B., Jafarpour, M.: Solvable polygroups and derived subpolygroups. Commun. Algebra **41**(8), 3098–3107 (2013)
2. Ameri, R.: Topological (transposition) hypergroups. Ital. J. Pure Appl. Math. **13**, 171–176 (2003)
3. Comer, S.D.: Polygroups derived from cogroups. J. Algebra **89**, 397–405 (1984)
4. Comer, S.D.: Extension of polygroups by polygroups and their representations using colour schemes, Lecture notes in Meth., No 1004, Universal Algebra and Lattice Theory, pp. 91–103 (1982)
5. Corsini, P.: Prolegomena of Hypergroup Theory. Aviani Editore, Tricesimo, (1993)
6. Corsini, P., Leoreanu, V.: Applications of Hyperstructures Theory. Advances in Mathematics. Kluwer Academic Publisher, Dordrecht (2003)
7. Cristea, I., Zhan, J.: Lower and upper fuzzy topological subhypergroups. Acta Math. Sin. (Engl. Ser.) **29**(2), 315–330 (2013)
8. Cristea, I., Hoskova, S.: Fuzzy pseudotopological hypergroupoids. Iran. J. Fuzzy Syst. **6**(4), 11–19 (2009)
9. Davvaz, B.: Isomorphism theorems of polygroups. Bull. Malays. Math. Sci. Soc. (2) **33**(3), 385–392 (2010)
10. Davvaz, B.: Polygroup Theory and Related Systems. World Scientific Publishing Co. Pte. Ltd., Hackensack (2013)
11. Dehkordi, S.O., Davvaz, B.: Γ -semihyperrings: approximations and rough ideals. Bull. Malays. Math. Sci. Soc. (2) **35**(4), 1035–1047 (2012)
12. Dunkl, C.F.: The measure algebra of a locally compact hyper group. Trans. Am. Math. Soc **179**, 331–348 (1973)
13. Freni, D.: Une note sur le coeur dun hypergroupe et sur la cloture transitive $\beta^* de \beta$. Riv. di Mat. Pura Appl. **8**, 153–156 (1991)
14. Heidari, D., Modarres, S.M.S., Davvaz, B.: Topological hypergroups in the sense of Marty. Commun. Algebra **42**(11), 4712–4721 (2014)
15. Hewitt, E., Ross, K.A.: Abstract Harmonic Analysis, vol. 1. Springer Verlag, Berlin (1963)
16. Hoskova-Mayerova, S.: Topological hypergroupoids. Comput. Math. Appl. **64**(9), 2845–2849 (2012)
17. Jafarabadi, H.M., Sarmin, N.H., Molaei, M.R.: Completely simple and regular semi hypergroups. Bull. Malays. Math. Sci. Soc. (2) **35**(2), 335–343 (2012)
18. Jafarpour, M., Aghabozorgi, H., Davvaz, B.: On nilpotent and solvable polygroups. Bull. Iran. Math. Soc. **39**(3), 487–499 (2013)
19. Jewett, R.I.: Spaces with an abstract convolution of measures. Adv. Math. **18**, 1–101 (1975)
20. Koskas, M.: Groupoides, demi-hypergroupes et hypergroupes. J. Math. Pures Appl. **49**(9), 155–192 (1970)
21. Marty, F.: Sur une generalization de la notion de groupe, 8^{iem} congres Math. Stockholm, Scandinaves (1934)

22. Ostadhadi-Dehkordi, S. and Heidari, M.: General Γ-hypermodules: Θ-relation, T-functor and fundamental modules, Bull. Malays. Math. Sci. Soc. **37**(3), 907–921 (2014)
23. Vougiouklis, T.: Hyperstructures and Their Representations, vol. 115. Hadronic Press Inc, Palm Harber (1994)