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Duality and Integral Transform of a Class of Analytic Functions

Sarika Verma[1] · **Sushma Gupta**[1] · **Sukhjit Singh**[1]

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Abstract For $\alpha, \gamma \geq 0$ and $\beta < 1$, let $\mathcal{W}_\beta(\alpha, \gamma)$ denote the class of all normalized analytic functions f in the open unit disk $E = \{z : |z| < 1\}$ such that

$$\Re e^{i\phi}\left((1-\alpha+2\gamma)\frac{f(z)}{z} + (\alpha-2\gamma)f'(z) + \gamma z f''(z) - \beta\right) > 0, \quad z \in E$$

for some $\phi \in \mathbb{R}$. For $f \in \mathcal{W}_\beta(\alpha, \gamma)$, we consider the integral transform

$$V_\lambda(f)(z) := \int_0^1 \lambda(t)\frac{f(tz)}{t}\mathrm{d}t,$$

where λ is a non-negative real-valued integrable function satisfying the condition $\int_0^1 \lambda(t)\mathrm{d}t = 1$. In a very recent paper, Ali et al. (J Math Anal Appl 385:808–822, 2012) discussed the starlikeness of the integral transform $V_\lambda(f)$ when $f \in \mathcal{W}_\beta(\alpha, \gamma)$. The aim of present paper is to find conditions on $\lambda(t)$ such that $V_\lambda(f)$ is starlike of order δ $(0 \leq \delta \leq 1/2)$ when $f \in \mathcal{W}_\beta(\alpha, \gamma)$. As applications, we study various choices of $\lambda(t)$, related to classical integral transforms.

Keywords Starlike function · Hadamard product · Duality

Mathematics Subject Classification 30C45 · 30C80

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✉ Sarika Verma
sarika.16984@gmail.com

[1] Sant Longowal Institute of Engineering and Technology, Longowal 148106, Punjab, India

1 Introduction

Let $\mathcal{A}$ denotes the class of analytic functions f defined in the open unit disk $E = \{z : |z| < 1\}$ with the normalization $f(0) = f'(0) - 1 = 0$ and $\mathcal{A}_0 = \{g : g(z) = f(z)/z, \ f \in \mathcal{A}\}$. Let S be the subclass of $\mathcal{A}$ consisting of functions univalent in E. A function f in $\mathcal{A}$ is said to be starlike of order β if it satisfies

$$\Re\left(\frac{zf'(z)}{f(z)}\right) > \beta, \quad z \in E,$$

for some β $(0 \leq \beta < 1)$. We denote by $S^*(\beta)$, the subclass of S consisting of functions which are starlike of order β in E. Set $S^*(0) = S^*$. For any two functions $f(z) = z + a_2z^2 + a_3z^3 + \cdots$ and $g(z) = z + b_2z^2 + b_3z^3 + \cdots$ in $\mathcal{A}$, the Hadamard product (or convolution) of f and g is the function $f * g$ defined by

$$(f * g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n.$$

For $f \in \mathcal{A}$, Fournier and Ruscheweyh [4] introduced the operator

$$V_\lambda(f)(z) := \int_0^1 \lambda(t)\frac{f(tz)}{t}\mathrm{d}t, \tag{1.1}$$

where λ is a non-negative real-valued integrable function satisfying the condition $\int_0^1 \lambda(t)\mathrm{d}t = 1$. This operator contains some of the well-known operators (such as Alexander, Libera, Bernardi, and Komatu) as its special cases. This operator has been studied by a number of authors for various choices of $\lambda(t)$ [1,3,4,6,9,12]. Fournier and Ruscheweyh [4] applied the Duality theory [10,11] to prove the starlikeness of the linear integral transform $V_\lambda(f)$ over functions f in the class

$$\mathcal{P}(\beta) := \left\{f \in \mathcal{A} : \exists\phi \in \mathbb{R}|\Re e^{i\phi}\left(f'(z) - \beta\right) > 0, \ z \in E\right\}.$$

In 2001, Kim and Rønning [5] investigated starlikeness properties of the integral transform (1.1) for functions f in the class

$$\mathcal{P}_\gamma(\beta) := \left\{f \in \mathcal{A} : \exists\phi \in \mathbb{R}|\Re e^{i\phi}\left((1-\gamma)\frac{f(z)}{z} + \gamma f'(z) - \beta\right) > 0, \ z \in E\right\}.$$

Recently in 2008, Ponnusamy and Rønning [9] discussed the same problem for the functions in the class

$$\mathcal{R}_\gamma(\beta) := \left\{f \in \mathcal{A} : \exists\phi \in \mathbb{R}|\Re e^{i\phi}\left(f'(z) + \gamma z f''(z) - \beta\right) > 0, \ z \in E\right\}.$$

It is evident that $\mathcal{R}_\gamma(\beta)$ is closely related to the class $\mathcal{P}_\gamma(\beta)$. Clearly, $f \in \mathcal{R}_\gamma(\beta)$ if and only if zf' belongs to $\mathcal{P}_\gamma(\beta)$.

In a very recent paper, Ali et al. [1] discussed this problem for the functions f in the class

$$\mathcal{W}_\beta(\alpha,\gamma):=\left\{f \in \mathcal{A}: \exists\phi \in \mathbb{R} | \Re e^{i\phi}\left((1-\alpha+2\gamma)\frac{f(z)}{z}+(\alpha-2\gamma)f'(z)+\gamma z f''(z)-\beta\right)>0,\ z\in E\right\}. \tag{1.2}$$

Note that $\mathcal{W}_\beta(1,0) \equiv \mathcal{P}(\beta)$, $\mathcal{W}_\beta(\alpha,0) \equiv \mathcal{P}_\alpha(\beta)$ and $\mathcal{W}_\beta(1+2\gamma,\gamma) \equiv \mathcal{R}_\gamma(\beta)$.

In Sect. 3 of the paper, we shall mainly tackle the problem: For given $0 \le \delta \le 1/2$, to find conditions on β such that $V_\lambda(f) \in S^*(\delta)$ whenever $f \in \mathcal{W}_\beta(\alpha,\gamma)$. In Sect. 4, we find easier criteria of starlikeness of the integral operator $V_\lambda(f)$. While in the last section of the paper, we discussed applications of results obtained for various choices of $\lambda(t)$.

To prove our result, we shall need the duality theory for convolutions, so we include here some basic concepts and results from this theory. For a subset $\mathcal{B} \subset \mathcal{A}_0$ we define

$$\mathcal{B}^* = \{g \in \mathcal{A}_0 : (f * g)(z) \neq 0,\ z \in E,\ \text{for all } f \in \mathcal{B}\}.$$

The set $\mathcal{B}^*$ is called the dual of $\mathcal{B}$. Further, the second dual of $\mathcal{B}$ is defined as $\mathcal{B}^{**} = (\mathcal{B}^*)^*$. The basic reference to this theory is the book by Ruscheweyh [11] (see also [10]). We shall need the following fundamental result.

Theorem 1.1 (Duality Principle) *Let*

$$\mathcal{B} = \left\{\beta + (1-\beta)\left(\frac{1+xz}{1+yz}\right) : |x| = |y| = 1\right\},\ \beta \in \mathbb{R},\ \beta \neq 1.$$

We have

(1) $\mathcal{B}^{**} = \left\{g \in \mathcal{A}_0 : \exists\phi \in \mathbb{R} \text{ such that } \Re\left(e^{i\phi}(g(z)-\beta)\right) > 0,\ z \in E\right\}$.
(2) *If* Γ_1 *and* Γ_2 *are two continuous linear functionals on* $\mathcal{B}$ *with* $0 \notin \Gamma_2$, *then for every* $g \in \mathcal{B}^{**}$ *we can find* $v \in \mathcal{B}$ *such that*

$$\frac{\Gamma_1(g)}{\Gamma_2(g)} = \frac{\Gamma_1(v)}{\Gamma_2(v)}.$$

2 Preliminaries

We use the notations introduced in [1]. Let $\mu \ge 0$ and $\nu \ge 0$ satisfy

$$\mu + \nu = \alpha - \gamma \quad \text{and} \quad \mu\nu = \gamma. \tag{2.1}$$

For $\gamma = 0$, μ is also taken to be 0, in which case, $\nu = \alpha \ge 0$. Writing $\alpha = 1 + 2\gamma$ in (2.1), we get $\mu + \nu = 1 + \gamma = 1 + \mu\nu$, or $(\mu - 1)(1 - \nu) = 0$.

(i) When $\gamma > 0$, then writing $\mu = 1$ gives $\nu = \gamma$.
(ii) If $\gamma = 0$, then $\mu = 0$ and $\nu = \alpha = 1$.

In the particular case $\alpha = 1 + 2\gamma$, the values of μ and ν for $\gamma > 0$ will be taken as $\mu = 1$ and $\nu = \gamma$ respectively, while in the case when $\gamma = 0$, we have $\mu = 0$ and $\nu = 1 = \alpha$.

Define

$$\phi_{\mu,\nu}(z) = 1 + \sum_{n=1}^{\infty} \frac{(n\nu + 1)(n\mu + 1)}{n+1} z^n, \tag{2.2}$$

and

$$\psi_{\mu,\nu}(z) = \phi_{\mu,\nu}^{-1}(z) = 1 + \sum_{n=1}^{\infty} \frac{n+1}{(n\nu + 1)(n\mu + 1)} z^n = \int_0^1 \int_0^1 \frac{\mathrm{d}s\mathrm{d}t}{(1 - t^{\nu} s^{\mu} z)^2}. \tag{2.3}$$

Here $\phi_{\mu,\nu}^{-1}$ denotes the convolution inverse of $\phi_{\mu,\nu}$ such that $\phi_{\mu,\nu} * \phi_{\mu,\nu}^{-1} = z/(1-z)$. If we take $\gamma = 0$, then $\mu = 0$, $\nu = \alpha$ in (2.3), we have

$$\psi_{0,\alpha}(z) = 1 + \sum_{n=1}^{\infty} \frac{n+1}{n\alpha + 1} z^n = \int_0^1 \frac{\mathrm{d}t}{(1 - t^{\alpha} z)^2}.$$

If $\gamma > 0$, then $\nu > 0$, $\mu > 0$, and making the change of variables $u = t^{\nu}$, $v = s^{\mu}$ results in

$$\psi_{\mu,\nu}(z) = \frac{1}{\mu\nu} \int_0^1 \int_0^1 \frac{u^{1/\nu - 1} v^{1/\mu - 1}}{(1 - uvz)^2} \mathrm{d}u\mathrm{d}v.$$

Thus the function $\psi_{\mu,\nu}$ can be written as

$$\psi_{\mu,\nu}(z) = \begin{cases} \frac{1}{\mu\nu} \int_0^1 \int_0^1 \frac{u^{1/\nu - 1} v^{1/\mu - 1}}{(1 - uvz)^2} \mathrm{d}u\mathrm{d}v, & \gamma > 0; \\ \int_0^1 \frac{\mathrm{d}t}{(1 - t^{\alpha} z)^2}, & \gamma = 0, \alpha > 0. \end{cases} \tag{2.4}$$

Further let g be the solution of the initial-value problem

$$\frac{\mathrm{d}}{\mathrm{d}t} t^{1/\nu}(1 + g(t)) = \begin{cases} \frac{2}{\mu\nu} t^{1/\nu - 1} \int_0^1 \frac{1 - \delta(1 + st)}{(1 - \delta)(1 + st)^2} s^{1/\mu - 1} \mathrm{d}s, & \gamma > 0; \\ \frac{2}{\alpha} \frac{1 - \delta(1 + t)}{(1 - \delta)(1 + t)^2} t^{1/\alpha - 1}, & \gamma = 0, \alpha > 0. \end{cases} \tag{2.5}$$

satisfying $g(0) = 1$, where $\delta \in [0, 1/2]$. A simple calculation leads to the solution given by

$$g(t) = \frac{2}{\mu\nu} \int_0^1 \int_0^1 \frac{1 - \delta(1 + swt)}{(1 - \delta)(1 + swt)^2} s^{1/\mu - 1} w^{1/\nu - 1} \mathrm{d}s\mathrm{d}w - 1. \tag{2.6}$$

In particular

$$g_\alpha(t) = \frac{2}{\alpha} t^{-1/\alpha} \int_0^t u^{1/\alpha - 1} \frac{1 - \delta(1+u)}{(1-\delta)(1+u)^2} \mathrm{d}u - 1, \ \ \gamma = 0, \ \alpha > 0. \tag{2.7}$$

3 Main Results

Theorem 3.1 *Let $\mu \geq 0$, $\nu \geq 0$ satisfy (2.1), and $\beta < 1$ satisfy*

$$\frac{\beta}{1-\beta} = -\int_0^1 \lambda(t) g(t) \mathrm{d}t, \tag{3.1}$$

where g is the solution of the initial-value problem (2.5). Further let

$$\Lambda_\nu(t) = \int_t^1 \frac{\lambda(x)}{x^{1/\nu}} \mathrm{d}x, \quad \nu > 0, \tag{3.2}$$

$$\Pi_{\mu,\nu}(t) = \begin{cases} \int_t^1 \Lambda_\nu(x) x^{1/\nu - 1 - 1/\mu} \mathrm{d}x, & \gamma > 0; \\ \Lambda_\alpha(t), & \gamma = 0, (\mu = 0, \ \nu = \alpha > 0). \end{cases} \tag{3.3}$$

and assume that $t^{1/\nu} \Lambda_\nu(t) \to 0$, and $t^{1/\mu} \Pi_{\mu,\nu}(t) \to 0$ as $t \to 0^+$. Then for $\delta \in [0, 1/2]$, we have $V_\lambda(\mathcal{W}_\beta(\alpha, \gamma)) \subset S^*(\delta)$ if and only if $\mathcal{L}_{\Pi_{\mu,\nu}}(h_\delta) \geq 0$, where $\mathcal{L}_{\Pi_{\mu,\nu}}(h_\delta)$ and h_δ are defined by following equations:

$$\mathcal{L}_{\Pi_{\mu,\nu}}(h_\delta) = \begin{cases} \Re \int_0^1 \Pi_{\mu,\nu}(t) t^{1/\mu - 1} \left(\frac{h(tz)}{tz} - \frac{1-\delta(1+t)}{(1-\delta)(1+t)^2} \right) \mathrm{d}t, & \gamma > 0; \\ \Re \int_0^1 \Pi_{0,\alpha}(t) t^{1/\alpha - 1} \left(\frac{h(tz)}{tz} - \frac{1-\delta(1+t)}{(1-\delta)(1+t)^2} \right) \mathrm{d}t, & \gamma = 0. \end{cases} \tag{3.4}$$

and

$$h_\delta(z) = \frac{z\left(1 + \frac{\epsilon + 2\delta - 1}{2 - 2\delta} z\right)}{(1-z)^2}, \quad |\epsilon| = 1, \tag{3.5}$$

respectively. This conclusion does not hold for any smaller values of β.

Proof The case $\gamma = 0 (\mu = 0, \ \nu = \alpha)$ corresponds to the Theorem 1.2 in [2], so we will prove the result only when $\gamma > 0$.

Let

$$H(z) = (1 - \alpha + 2\gamma) \frac{f(z)}{z} + (\alpha - 2\gamma) f'(z) + \gamma z f''(z).$$

Since $\mu + \nu = \alpha - \gamma$ and $\mu\nu = \gamma$, then

$$\begin{aligned} H(z) &= (1 + \gamma - (\alpha - \gamma)) \frac{f(z)}{z} + (\alpha - \gamma - \gamma) f'(z) + \gamma z f''(z) \\ &= (1 + \mu\nu - \mu - \nu) \frac{f(z)}{z} + (\mu + \nu - \mu\nu) f'(z) + \mu\nu z f''(z). \end{aligned}$$

Writing $f(z)=z+\sum_{n=2}^{\infty}a_nz^n$, we obtain from (2.2)

$$H(z)=1+\sum_{n=1}^{\infty}a_{n+1}(n\nu+1)(n\mu+1)z^n=f'(z)*\phi_{\mu,\nu}(z), \tag{3.6}$$

and (2.3) gives that

$$f'(z)=H(z)*\psi_{\mu,\nu}(z). \tag{3.7}$$

Now, let $f\in\mathcal{W}_\beta(\alpha,\gamma)$. Then, in the view of the Theorem 1.1, we may restrict our attention to functions $f\in\mathcal{W}_\beta(\alpha,\gamma)$ for which

$$\begin{aligned}H(z)&=(1-\alpha+2\gamma)\frac{f(z)}{z}+(\alpha-2\gamma)f'(z)+\gamma zf''(z)\\&=\beta+(1-\beta)\left(\frac{1+xz}{1+yz}\right),\quad |x|=|y|=1.\end{aligned}$$

Thus (3.7) gives

$$f'(z)=\left((1-\beta)\frac{1+xz}{1+yz}+\beta\right)*\psi_{\mu,\nu}(z), \tag{3.8}$$

and therefore

$$\frac{f(z)}{z}=\frac{1}{z}\int_0^z\left((1-\beta)\frac{1+xw}{1+yw}+\beta\right)\mathrm{d}w*\psi(z), \tag{3.9}$$

here $\psi:=\psi_{\mu,\nu}$. □

Also, a well-known result from the theory of convolutions [11] (see also [10]) implies that

$$F\in S^*(\delta)\quad\Leftrightarrow\quad\frac{1}{z}(F*h_\delta)(z)\neq 0,\quad z\in E,$$

where

$$h_\delta(z)=\frac{z\left(1+\frac{\epsilon+2\delta-1}{2-2\delta}z\right)}{(1-z)^2},\quad |\epsilon|=1.$$

Hence $F\in S^*(\delta)$ if and only if

$$\begin{aligned}0\neq\frac{1}{z}(V_\lambda(f)(z)*h_\delta(z))&=\frac{1}{z}\left[\int_0^1\lambda(t)\frac{f(tz)}{t}\mathrm{d}t*h_\delta(z)\right]\\&=\int_0^1\frac{\lambda(t)}{1-tz}\mathrm{d}t*\frac{f(z)}{z}*\frac{h_\delta(z)}{z}\end{aligned}$$

Using (3.9), we have

$$
\begin{aligned}
0 \neq & \int_0^1 \frac{\lambda(t)}{1-tz}\mathrm{d}t * \left[\frac{1}{z}\int_0^z \left((1-\beta)\frac{1+xw}{1+yw}+\beta\right)\mathrm{d}w * \psi(z)\right] * \frac{h_\delta(z)}{z} \\
&= \int_0^1 \frac{\lambda(t)}{1-tz}\mathrm{d}t * \frac{h_\delta(z)}{z} * \left[\frac{1}{z}\int_0^z \left((1-\beta)\frac{1+xw}{1+yw}+\beta\right)\mathrm{d}w\right] * \psi(z) \\
&= \int_0^1 \lambda(t)\frac{h_\delta(tz)}{tz}\mathrm{d}t * (1-\beta)\left[\frac{1}{z}\int_0^z \left(\frac{1+xw}{1+yw}+\frac{\beta}{(1-\beta)}\right)\mathrm{d}w\right] * \psi(z) \\
&= (1-\beta)\left[\int_0^1 \lambda(t)\frac{h_\delta(tz)}{tz}\mathrm{d}t + \frac{\beta}{(1-\beta)}\right] * \frac{1}{z}\int_0^z \frac{1+xw}{1+yw}\mathrm{d}w * \psi(z) \\
&= (1-\beta)\left[\int_0^1 \lambda(t)\left(\frac{1}{z}\int_0^z \frac{h_\delta(tw)}{tw}\mathrm{d}w\right)\mathrm{d}t + \frac{\beta}{(1-\beta)}\right] * \frac{1+xz}{1+yz} * \psi(z).
\end{aligned}
$$

This holds if and only if [10, p. 23]

$$
\begin{aligned}
& \Re(1-\beta)\left[\int_0^1 \lambda(t)\left(\frac{1}{z}\int_0^z \frac{h_\delta(tw)}{tw}\mathrm{d}w\right)\mathrm{d}t + \frac{\beta}{(1-\beta)}\right] * \psi(z) \geq \frac{1}{2}, \\
\Leftrightarrow\ & \Re(1-\beta)\left[\int_0^1 \lambda(t)\left(\frac{1}{z}\int_0^z \frac{h_\delta(tw)}{tw}\mathrm{d}w\right)\mathrm{d}t + \frac{\beta}{(1-\beta)} - \frac{1}{2(1-\beta)}\right] * \psi(z) \geq 0, \\
\Leftrightarrow\ & \Re\left[\int_0^1 \lambda(t)\left(\frac{1}{z}\int_0^z \frac{h_\delta(tw)}{tw}\mathrm{d}w\right)\mathrm{d}t + \frac{\beta}{(1-\beta)} - \frac{1}{2(1-\beta)}\right] * \psi(z) \geq 0, \\
\Leftrightarrow\ & \Re\left[\int_0^1 \lambda(t)\left(\frac{1}{z}\int_0^z \frac{h_\delta(tw)}{tw}\mathrm{d}w\right)\mathrm{d}t - \frac{1}{2} + \frac{\beta}{2(1-\beta)}\right] * \psi(z) \geq 0, \\
\Leftrightarrow\ & \Re\left[\int_0^1 \lambda(t)\left(\frac{1}{z}\int_0^z \frac{h_\delta(tw)}{tw}\mathrm{d}w - \frac{1+g(t)}{2}\right)\mathrm{d}t\right] * \psi(z) \geq 0, \quad \text{(Using (3.1))} \\
\Leftrightarrow\ & \Re\left[\int_0^1 \lambda(t)\left(\frac{h_\delta(tz)}{tz} - \frac{1+g(t)}{2}\right)\mathrm{d}t\right] * \frac{1}{z}\int_0^z \psi(w)\mathrm{d}w \geq 0, \\
\Leftrightarrow\ & \Re\left[\int_0^1 \lambda(t)\left(\frac{h_\delta(tz)}{tz} - \frac{1+g(t)}{2}\right)\mathrm{d}t\right] * \sum_{n=0}^{\infty}\frac{z^n}{(n\nu+1)(n\mu+1)} \geq 0, \quad \text{(Using (2.3))} \\
\Leftrightarrow\ & \Re\int_0^1 \lambda(t)\left(\sum_{n=0}^{\infty}\frac{z^n}{(n\nu+1)(n\mu+1)} * \frac{h_\delta(tz)}{tz} - \frac{1+g(t)}{2}\right)\mathrm{d}t \geq 0, \\
\Leftrightarrow\ & \Re\int_0^1 \lambda(t)\left(\int_0^1\int_0^1 \frac{\mathrm{d}\eta\mathrm{d}\zeta}{(1-\eta^\nu\zeta^\mu z)} * \frac{h_\delta(tz)}{tz} - \frac{1+g(t)}{2}\right)\mathrm{d}t \geq 0, \\
\Leftrightarrow\ & \Re\int_0^1 \lambda(t)\left(\int_0^1\int_0^1 \frac{h_\delta(tz\eta^\nu\zeta^\mu)}{tz\eta^\nu\zeta^\mu}\mathrm{d}\eta\mathrm{d}\zeta - \frac{1+g(t)}{2}\right)\mathrm{d}t \geq 0,
\end{aligned}
$$

which can also be written as

$$
\Re\int_0^1 \lambda(t)\left(\int_0^1\int_0^1 \frac{1}{\mu\nu}\frac{h_\delta(tzuv)}{tzuv}u^{1/\nu-1}v^{1/\mu-1}\mathrm{d}v\mathrm{d}u - \frac{1+g(t)}{2}\right)\mathrm{d}t \geq 0.
$$

Writing $w = tu$, we get

$$\Re \int_0^1 \frac{\lambda(t)}{t^{1/\nu}} \left[\int_0^t \int_0^1 \frac{h_\delta(wzv)}{wzv} w^{1/\nu-1} v^{1/\mu-1} \mathrm{d}v \mathrm{d}w - \mu\nu t^{1/\nu} \frac{1+g(t)}{2} \right] \mathrm{d}t \geq 0.$$

An integration by parts with respect to t and (2.5) gives

$$\Re \int_0^1 \Lambda_\nu(t) \left[\int_0^1 \frac{h_\delta(tzv)}{tzv} t^{1/\nu-1} v^{1/\mu-1} \mathrm{d}v - t^{1/\nu-1} \int_0^1 \frac{1-\delta(1+st)}{(1-\delta)(1+st)^2} s^{1/\mu-1} \mathrm{d}s \right] \mathrm{d}t \geq 0.$$

Again writing $w = vt$ and $\eta = st$ reduces the above inequality to

$$\Re \int_0^1 \Lambda_\nu(t) t^{1/\nu-1/\mu-1} \left[\int_0^t \frac{h_\delta(wz)}{wz} w^{1/\mu-1} \mathrm{d}w - \int_0^t \frac{1-\delta(1+\eta)}{(1-\delta)(1+\eta)^2} \eta^{1/\mu-1} \mathrm{d}\eta \right] \mathrm{d}t \geq 0,$$

which after integration by parts with respect to t yields

$$\Re \int_0^1 \Pi_{\mu,\nu}(t) t^{1/\mu-1} \left(\frac{h(tz)}{tz} - \frac{1-\delta(1+t)}{(1-\delta)(1+t)^2} \right) \mathrm{d}t \geq 0.$$

Thus $F \in S^*(\delta)$ if and only if $\mathcal{L}_{\Pi_{\mu,\nu}}(h_\delta) \geq 0$.

Finally, to prove the sharpness, let $f \in \mathcal{W}_\beta(\alpha, \gamma)$ be of the form for which

$$(1-\alpha+2\gamma)\frac{f(z)}{z} + (\alpha - 2\gamma) f'(z) + \gamma z f''(z) = \beta + (1-\beta)\frac{1+z}{1-z}.$$

Using a series expansion we obtain that

$$f(z) = z + 2(1-\beta) \sum_{n=1}^{\infty} \frac{1}{(n\nu+1)(n\mu+1)} z^{n+1}.$$

Thus

$$F(z) = V_\lambda(f)(z) = \int_0^1 \lambda(t) \frac{f(tz)}{t} \mathrm{d}t = z + 2(1-\beta) \sum_{n=1}^{\infty} \frac{\tau_n}{(n\nu+1)(n\mu+1)} z^{n+1},$$

where $\tau_n = \int_0^1 \lambda(t) t^n \mathrm{d}t$. From (2.6), it is a simple exercise to write $g(t)$ in a series expansion as

$$g(t) = 1 + \frac{2}{1-\delta} \sum_{n=1}^{\infty} \frac{(-1)^n (n+1-\delta)}{(n\nu+1)(n\mu+1)} t^n. \tag{3.10}$$

Now, by (3.1) and (3.10), we have

$$\begin{aligned}\frac{\beta}{1-\beta} &= -\int_0^1 \lambda(t)g(t)\mathrm{d}t \\ &= -\int_0^1 \lambda(t)\left[1+\frac{2}{1-\delta}\sum_{n=1}^{\infty}\frac{(-1)^n(n+1-\delta)}{(n\nu+1)(n\mu+1)}t^n\right]\mathrm{d}t \\ &= -1-\frac{2}{1-\delta}\sum_{n=1}^{\infty}\frac{(-1)^n(n+1-\delta)}{(n\nu+1)(n\mu+1)}\int_0^1 \lambda(t)t^n\mathrm{d}t.\end{aligned}$$

Therefore

$$\frac{1}{1-\beta} = -\frac{2}{1-\delta}\sum_{n=1}^{\infty}\frac{(-1)^n(n+1-\delta)\tau_n}{(n\nu+1)(n\mu+1)}. \tag{3.11}$$

Finally, we see that

$$F'(z) = 1+2(1-\beta)\sum_{n=1}^{\infty}\frac{(n+1)\tau_n}{(n\nu+1)(n\mu+1)}z^n.$$

For $z=-1$, we have

$$\begin{aligned}F'(-1) &= 1+2(1-\beta)\sum_{n=1}^{\infty}\frac{(-1)^n(n+1)\tau_n}{(n\nu+1)(n\mu+1)} \\ &= 1+2(1-\beta)\sum_{n=1}^{\infty}\frac{(-1)^n(n+1-\delta)\tau_n}{(n\nu+1)(n\mu+1)}+2(1-\beta)\sum_{n=1}^{\infty}\frac{(-1)^n\delta\tau_n}{(n\nu+1)(n\mu+1)} \\ &= 1-(1-\delta)+\delta 2(1-\beta)\sum_{n=1}^{\infty}\frac{(-1)^n\tau_n}{(n\nu+1)(n\mu+1)} \\ &= -\delta\left(-1+2(1-\beta)\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\tau_n}{(n\nu+1)(n\mu+1)}\right) \\ &= -\delta F(-1).\end{aligned}$$

Thus $zF'(z)/F(z)$ at $z=-1$ equals δ. This implies that the result is sharp for the order of starlikeness.

4 Consequences of Theorem 3.1

Theorem 4.1 *Let* $0 \leq \delta \leq 1/2$. *Assume that both* $\Pi_{\mu,\nu}(t)$ *and* $\Lambda_\nu(t)$, *as given in Theorem 3.1, are integrable on [0,1] and positive on (0,1). Further assume that* $\mu \geq 1$, *and*

$$\frac{\Pi_{\mu,\nu}(t)}{(1+t)(1-t)^{1+2\delta}} \text{ is decreasing on (0, 1).} \tag{4.1}$$

If β satisfies (3.1), then we have $V_\lambda(\mathcal{W}_\beta(\alpha,\gamma)) \subset S^*(\delta)$, where $V_\lambda(f)$ is defined by (1.1).

Proof For $\mu \geq 1$, the function $t^{1/\mu-1}$ is decreasing on (0,1). Thus the condition (4.1) along with [8, Theorem 2.3] gives

$$\Re \int_0^1 \Pi_{\mu,\nu}(t) t^{1/\mu-1}\left(\frac{h(tz)}{tz} - \frac{1-\delta(1+t)}{(1-\delta)(1+t)^2}\right) \mathrm{d}t \geq 0.$$

The result now, follows from Theorem 3.1. □

Below, we obtain the conditions to ensure starlikeness of $V_\lambda(f)$. As defined in Theorem 3.1, for $\gamma > 0$,

$$\Pi_{\mu,\nu}(t) = \int_t^1 \Lambda_\nu(x) x^{1/\nu-1-1/\mu}\mathrm{d}x, \quad \text{and} \quad \Lambda_\nu(t) = \int_t^1 \frac{\lambda(x)}{x^{1/\nu}}\mathrm{d}x.$$

In order to apply Theorem 4.1, we have to prove that the function

$$p(t) = \frac{\Pi_{\mu,\nu}(t)}{(1+t)(1-t)^{1+2\delta}}$$

is decreasing in (0,1). Since $p(t) > 0$ and

$$\frac{p'(t)}{p(t)} = -\frac{\Lambda_\nu(t)}{t^{1-1/\mu-1/\nu}\Pi_{\mu,\nu}(t)} + \frac{2(t+\delta(1+t))}{1-t^2},$$

or equivalently,

$$\frac{p'(t)}{p(t)} = \frac{2(t+\delta(1+t))}{(1-t^2)\Pi_{\mu,\nu}(t)}\left\{\Pi_{\mu,\nu}(t) - \frac{(1-t^2)\Lambda_\nu(t)t^{1/\nu-1-1/\mu}}{2(t+\delta(1+t))}\right\},$$

so it remains to show that $q(t)$ is increasing over (0,1), where

$$q(t) := \Pi_{\mu,\nu}(t) - \frac{(1-t^2)\Lambda_\nu(t)t^{1/\nu-1-1/\mu}}{2(t+\delta(1+t))}.$$

Since $q(1) = 0$, this will imply that $q(t) \leq 0$, and thus $p(t)$ is decreasing on (0,1). Now

$$q'(t) = -\frac{t^{1/\nu-1-1/\mu}(1+t)}{2(t+\delta(1+t))^2}\left\{ -\lambda(t)t^{-1/\nu}(1-t)(t+\delta(1+t)) + \Lambda_\nu(t)\left(\frac{(1-t)}{t}(1/\nu-1-1/\mu)(t+\delta(1+t)) - (1-t-\delta(1+t))(1+2\delta)\right)\right\}. \tag{4.2}$$

So, $q'(t) \geq 0$ for $t \in (0,1)$ is equivalent to the inequality $r(t) \leq 0$, where $r(t)$ is equal to

$$-\lambda(t)t^{-1/\nu}(1-t)(t+\delta(1+t)) \\ +\Lambda_\nu(t)\left(\frac{(1-t)}{t}(1/\nu-1-1/\mu)(t+\delta(1+t))-(1-t-\delta(1+t))(1+2\delta)\right).$$

By using the idea similar to the one used to prove Theorem 3.1 in [3], we can write

$$r(t) = -A(t)X(t) + \frac{Y(t)}{t}\int_t^1 A(s)\mathrm{d}s,$$

where,

$$\begin{aligned} A(t) &= \lambda(t)t^{-1/\nu}, \\ X(t) &= (1-t)(t+\delta(1+t)), \\ Y(t) &= X(t)(1/\nu-1-1/\mu)+Z(t), \\ Z(t) &= -t(1-t-\delta(1+t))(1+2\delta). \end{aligned} \tag{4.3}$$

Clearly, $A(t) > 0$ and $X(t) > 0$ for all $t \in (0,1)$.

Case (i). If $Y(t) \leq 0$ on (0,1), then $r(t) \leq 0$ on (0,1) and thus the result follows.

Case (ii). When $Y(t) > 0$. We may write

$$r(t) = \frac{Y(t)}{t}B(t), \text{ where } B(t) = -A(t)\frac{tX(t)}{Y(t)} + \int_t^1 A(s)\mathrm{d}s, \text{ and } B(1) = 0.$$

Thus, to prove that $r(t) \leq 0$, it is enough to prove that $B(t)$ is an increasing function of t. Now

$$\begin{aligned} B'(t) &= -A(t)\left[\frac{A'(t)}{A(t)}\frac{tX(t)}{Y(t)} + \left(\frac{tX}{Y}\right)'(t)+1\right] \\ &= -t^{-1/\nu}\lambda(t)\left[\left(\frac{t\lambda'(t)}{\lambda(t)}-\frac{1}{\nu}\right)\frac{X(t)}{Y(t)} + \left(\frac{tX}{Y}\right)'(t)+1\right]. \end{aligned}$$

For $Y(t) > 0$, $B'(t) \geq 0$ is equivalent to

$$\frac{t\lambda'(t)}{\lambda(t)} \leq \frac{1}{\nu} - \left[1+\left(\frac{tX}{Y}\right)'(t)\right]\frac{Y(t)}{X(t)}. \tag{4.4}$$

Now, following three possibilities arise:

(a). If $Y(t) > 0$ throughout the interval (0,1), then (4.4) implies that $B'(t) \geq 0$ on (0,1). Thus, $B(t)$ is increasing in (0,1) which implies that, $B(t) \leq B(1) = 0$. Therefore, $r(t) \leq 0$ on (0,1).

(b). If $Y(t) > 0$ on some interval $(0, t_0)$ and $Y(t) \leq 0$ on $[t_0, 1)$ for some $t_0 \in (0, 1)$, then (4.4) implies that $B'(t) \geq 0$ on $(0, t_0)$. Thus, $B(t)$ is increasing in $(0, t_0)$ which implies that, $B(t) \leq B(t_0)$ for any t in $(0, t_0)$. Since $B(t_0) \to -\infty$, this implies that $B(t)$ is negative. Therefore, $r(t) \leq 0$ on $(0, t_0)$. In view of Case (i), $r(t) \leq 0$ whenever $Y(t) \leq 0$. Thus, $r(t) \leq 0$ on (0,1).

(c). If $Y(t) \leq 0$ on some interval $(0, t_0]$ and $Y(t) > 0$ on $(t_0, 1)$ for some $t_0 \in (0, 1)$, then (4.4) implies that $B'(t) \geq 0$ on $(t_0, 1)$. Thus, $B(t)$ is increasing in $(t_0, 1)$ which implies that, $B(t) \leq B(1) = 0$ for any t in $(t_0, 1)$. Therefore, $r(t) \leq 0$ on $(t_0, 1)$. In view of Case (i), $r(t) \leq 0$ whenever $Y(t) \leq 0$ which implies that, $r(t) \leq 0$ on (0,1).

Subcase (i). For $\delta = 0$, $X(t)$ and $Y(t)$ reduces to the simple form

$$X(t) = t(1-t) \ \text{ and } \ Y(t) = t(1-t)\left(\frac{1}{\nu} - 2 - \frac{1}{\mu}\right).$$

Clearly $Y(t) \leq 0$ on (0,1) if $\frac{1}{\nu} - 2 - \frac{1}{\mu} \leq 0$ or simply $\nu \geq \mu/(2\mu + 1)$ and so $r(t) \leq 0$ in this case. On the other hand, if $0 < \nu < \mu/(2\mu + 1)$ on (0,1), then $Y(t) > 0$ on (0,1) and thus (4.4) gives that

$$\frac{t\lambda'(t)}{\lambda(t)} \leq 1 + \frac{1}{\mu}$$

on (0,1) and hence $r(t) \leq 0$ throughout the interval (0,1).
In the case when $\gamma = 0$, we have $\mu = 0$, $\nu = \alpha > 0$. Let

$$k(t) := \Lambda_\alpha(t) t^{1/\alpha - 1}, \quad \text{where } \ \Lambda_\alpha(t) = \int_t^1 \frac{\lambda(x)}{x^{1/\alpha}} \mathrm{d}x.$$

To apply Theorem 2.3 in [9] along with Theorem 3.1, the function $P(t) = \frac{k(t)}{(1+t)(1-t)^{1+2\delta}}$ must be shown decreasing on the interval (0,1). Since, $P(t) > 0$ on (0,1) and

$$\frac{P'(t)}{P(t)} = \frac{2(t + \delta(1+t))}{(1-t^2)k(t)} \left\{ \frac{(1-t^2)k'(t)}{2(t+\delta(1+t))} + k(t) \right\},$$

thus, $P(t)$ is decreasing in (0,1) provided

$$Q(t) := k(t) + \frac{(1-t^2)k'(t)}{2(t+\delta(1+t))} \leq 0.$$

Since, $Q(1) = 0$, thus $Q(t) \leq 0$ will certainly hold if Q is increasing on $(0, 1)$. Now $Q'(t) = \dfrac{(1+t)}{2(t+\delta(1+t))^2} \left\{ (1-t)(t+\delta(1+t))k''(t) + [2\delta(t+\delta(1+t)) -(1-t)(1+\delta)]k'(t) \right\}$, where $(1-t)(t+\delta(1+t))k''(t)+[2\delta(t+\delta(1+t))-(1-t)(1+\delta)]k'(t)$ is equal to

$$t^{1/\alpha-2}\left\{t(1-t)(t+\delta(1+t))\Lambda_\alpha{}''(t)+\left[2\left(\frac{1}{\alpha}-1\right)(1-t)(t+\delta(1+t))\right.\right.$$
$$\left.+2t\delta(t+\delta(1+t))-t(1-t)(1+\delta)\right]$$
$$\Lambda_\alpha{}'(t)+\left[\left(\frac{1}{\alpha}-2\right)\frac{(1-t)}{t}(t+\delta(1+t))+2\delta(t+\delta(1+t))\right.$$
$$\left.\left.-(1-t)(1+\delta)\right]\left(\frac{1}{\alpha}-1\right)\Lambda_\alpha(t)\right\}.$$

Thus, $Q'(t)\geq 0$, for $t\in(0,1)$, is equivalent to the inequality

$$\left\{t(1-t)(t+\delta(1+t))\Lambda_\alpha{}''(t)+\left[2\left(\frac{1}{\alpha}-1\right)(1-t)(t+\delta(1+t))\right.\right.$$
$$\left.+2t\delta(t+\delta(1+t))-t(1-t)(1+\delta)\right]\Lambda_\alpha{}'(t)$$
$$+\left[\left(\frac{1}{\alpha}-2\right)\frac{(1-t)}{t}(t+\delta(1+t))+2\delta(t+\delta(1+t))\right.$$
$$\left.\left.-(1-t)(1+\delta)\right]\left(\frac{1}{\alpha}-1\right)\Lambda_\alpha(t)\right\}\geq 0.$$

The latter condition is equivalent to $\Delta(t)\geq 0$, where

$$\Delta(t)\equiv\left\{-t\lambda'(t)(1-t)(t+\delta(1+t))+\lambda(t)\left[\left(2-\frac{1}{\alpha}\right)(1-t)(t+\delta(1+t))\right.\right.$$
$$\left.-2t\delta(t+\delta(1+t))+t(1-t)(1+\delta)\right]$$
$$+\left[\left(\frac{1}{\alpha}-2\right)(1-t)(t+\delta(1+t))+2t\delta(t+\delta(1+t))\right.$$
$$\left.\left.-t(1-t)(1+\delta)\right]\left(\frac{1}{\alpha}-1\right)t^{1/\alpha-1}\Lambda_\alpha(t)\right\}.$$

A simple computation along with (4.3) shows that Δ can be rewritten as

$$-tX(t)\lambda'(t)+\left[\left(3-\frac{1}{\alpha}\right)X(t)-(X(t)+Z(t))\right]\lambda(t)$$
$$+\left[\left(\frac{1}{\alpha}-3\right)X(t)+(X(t)+Z(t))\right]\left(\frac{1}{\alpha}-1\right)t^{1/\alpha-1}\Lambda_\alpha(t). \tag{4.5}$$

Since $\Lambda_\alpha(t) \geq 0$ and setting

$$\left[\left(\frac{1}{\alpha}-3\right)X(t)+(X(t)+Z(t))\right]\left(\frac{1}{\alpha}-1\right)\geq 0,$$

$\Delta \geq 0$ follows from

$$-tX(t)\lambda'(t)+\left[\left(3-\frac{1}{\alpha}\right)X(t)-(X(t)+Z(t))\right]\lambda(t)\geq 0.$$

Since $X(t)$ is non-negative on (0,1), thus the inequality $\Delta \geq 0$ follows from

$$\frac{t\lambda'(t)}{\lambda(t)}\leq\left(3-\frac{1}{\alpha}\right)-\frac{X(t)+Z(t)}{X(t)} \text{ and } \left[\left(\frac{1}{\alpha}-3\right)X(t)+(X(t)+Z(t))\right]\left(\frac{1}{\alpha}-1\right) \geq 0. \tag{4.6}$$

For $\delta = 0$, (4.6) reduces to

$$\frac{t\lambda'(t)}{\lambda(t)}\leq 3-\frac{1}{\alpha} \text{ for } \left(\frac{1}{\alpha}-1\right)\left(\frac{1}{\alpha}-3\right)\geq 0 \text{ or equivalently for } \alpha\in(0,1/3]\cup[1,\infty).$$

These observations for $\delta = 0$ lead to the following result by, Ali et al. [1, Theorem 4.2].

Corollary 4.1 *Assume that both $\Pi_{\mu,\nu}(t)$ and $\Lambda_\nu(t)$, as defined in Theorem 3.1 are integrable on [0,1], and positive on (0,1). Let $\lambda(t)$ be a normalized non-negative real-valued integrable function on [0,1]. Under the same conditions as stated in Theorem 3.1, if λ satisfies*

$$\frac{t\lambda'(t)}{\lambda(t)}\leq\begin{cases}1+\frac{1}{\mu}, & \mu\geq 1(\gamma>0);\\ 3-\frac{1}{\alpha}, & \gamma=0, \alpha\in(0,1/3]\cup[1,\infty),\end{cases} \tag{4.7}$$

then $F(z) = V_\lambda(f)(z) \in S^*$. The conclusion does not hold for smaller values of β.

Subcase (ii). If $0 < \delta \leq 1/2$ with $\gamma > 0$, then (4.4) can be rewritten as

$$\left(\frac{1}{\nu}-\frac{t\lambda'(t)}{\lambda(t)}\right)X(t)Y(t)\geq Y^2(t)+Y(t)(tX'(t)+X(t))-Y'(t)tX(t).$$

Since $Y(t) = X(t)(1/\nu - 1 - 1/\mu) + Z(t)$, so the above inequality is equivalent to

$$\begin{aligned}&\left(\frac{1}{\nu}-1-\frac{1}{\mu}\right)[X(t)+Z(t)]X(t)-\left(1+\frac{1}{\mu}-\frac{t\lambda'(t)}{\lambda(t)}\right)\\&\quad\left[\left(\frac{1}{\nu}-1-\frac{1}{\mu}\right)X(t)+Z(t)\right]\\&\quad\leq Z'(t)(tX(t))-Z(t)(tX(t))'-Z^2(t).\end{aligned} \tag{4.8}$$

Define $D(t) = t(1+\delta) - (1-\delta)$. Rewriting the expressions for $X(t)$ and $Z(t)$ in terms of $D(t)$, we get

$$X(t) = (1-t)(D(t)+1) \ \text{ and } \ Z(t) = (1+2\delta)tD(t)$$

and so a simple computation gives that

$$Z'(t)(tX(t)) - Z(t)(tX)'(t) - Z^2(t) = 2\delta(1+2\delta)t^2(1-D^2(t)). \tag{4.9}$$

Since $D^2(t) \leq 1$ for $t \in [0, 1]$ thus (4.9) is non-negative in (0,1). Since $X(t)+Z(t)$ and $X(t)$ are non-negative on (0,1), so if $(1/\nu - 1 - 1/\mu) \leq 0$ or simply $\nu \geq \mu/(\mu+1)$, then the inequality (4.8) holds on the interval where $Y(t) > 0$ and hence, $r(t) \leq 0$ on (0,1).

While on the other hand, for $0 < \delta \leq 1/2$ with $\gamma = 0$, from (4.6) we have

$$\frac{t\lambda'(t)}{\lambda(t)} \leq \left(3 - \frac{1}{\alpha}\right) - \frac{X(t)+Z(t)}{X(t)} \text{ and } \left[\left(\frac{1}{\alpha} - 3\right) X(t) + (X(t)+Z(t))\right]\left(\frac{1}{\alpha} - 1\right) \geq 0.$$

Since $X(t)$ and $X(t) + Z(t)$ are non-negative on (0,1), thus equivalently,

$$\frac{t\lambda'(t)}{\lambda(t)} \leq 3 - \frac{1}{\alpha}, \quad \text{for } \alpha \in (0, 1/3].$$

Hence, for $0 < \delta \leq 1/2$ with $\gamma = 0$, we have $\Delta \geq 0$ throughout the interval (0,1).

Thus, we see that above Corollary continues to hold for $\delta \in (0, 1/2]$ but with some restrictions. More precisely, we have

Theorem 4.2 *Let $\lambda(t)$ be a non-negative real-valued integrable function on [0,1]. Assume that both $\Pi_{\mu,\nu}(t)$ and $\Lambda_\nu(t)$ are integrable on [0,1], and positive on (0,1). Let λ satisfying the condition*

$$\frac{t\lambda'(t)}{\lambda(t)} \leq \begin{cases} 1 + \frac{1}{\mu}, & \mu \geq 1 (\gamma > 0); \\ 3 - \frac{1}{\alpha}, & \gamma = 0, \alpha \in (0, 1/3]. \end{cases} \tag{4.10}$$

Let $f \in \mathcal{W}_\beta(\alpha, \gamma)$ with $\nu \geq \mu/(\mu+1)$, and $\beta < 1$ with

$$\frac{\beta}{1-\beta} = -\int_0^1 \lambda(t) g(t) \mathrm{d}t, \tag{4.11}$$

where $g(t)$ is defined by (2.6) with $\delta \in (0, 1/2]$.Then $F(z) = V_\lambda(f)(z) \in S^*(\delta)$. The conclusion does not hold for smaller values of β.

Remark 4.1

1. For $\alpha = 1 + 2\gamma$ with $\gamma > 0$ and $\mu = 1$, Theorem 4.2 yields Theorem 3.1 in [3] with $0 < \delta \leq 1/2$.
2. With $\delta = 0$, our Corollary 4.1 coincides with the Theorem 4.2 in [1].

5 Applications

In this section, we present a number of applications of Theorem 4.2 for various well-known integral operators. Let $(a)_n$ denote the Pochhammer symbol, defined in terms of the Gamma function, by

$$(a)_n := \frac{\Gamma(a+n)}{\Gamma(a)} = \begin{cases} 1, & n = 0, \\ a(a+1)...(a+n-1), & \text{n} \in \mathbb{N}. \end{cases}$$

Define the Gaussian hypergeometric function by

$${}_2F_1(a, b; c; z) = F(a, b; c; z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n (1)_n} z^n, \quad |z| < 1,$$

where a, b and c are complex numbers with $c \neq 0, -1, -2, \ldots$. Note that the series ${}_2F_1$ converges absolutely for $z \in E$. Now let Φ be defined by $\Phi(1-t) = 1 + \sum_{n=1}^{\infty} b_n (1-t)^n$, $b_n \geq 0$ for $n \geq 1$, and

$$\lambda(t) = K t^{b-1} (1-t)^{c-a-b} \Phi(1-t), \tag{5.1}$$

where K is a constant chosen such that $\int_0^1 \lambda(t) \mathrm{d}t = 1$. The following result holds in this instance.

Theorem 5.1 *Let a, b, c, $\alpha > 0$, $\nu \geq \mu/(\mu+1)$ and $\beta < 1$ satisfy*

$$\frac{\beta}{1-\beta} = -K \int_0^1 t^{b-1} (1-t)^{c-a-b} \Phi(1-t) g(t) \mathrm{d}t,$$

where K is a constant such that $K \int_0^1 t^{b-1} (1-t)^{c-a-b} \Phi(1-t) \mathrm{d}t = 1$ and g is given by (2.6). Then for $\delta \in [0, 1/2]$, we have $V_\lambda(\mathcal{W}_\beta(\alpha, \gamma)) \subset S^*(\delta)$ provided the following condition hold

$$c \geq a + b \;\; and \;\; b \leq \begin{cases} 2 + \frac{1}{\mu}, & \gamma > 0 (\mu \geq 1); \\ 4 - \frac{1}{\alpha}, & \gamma > 0, \alpha \in (1/4, 1/3], \end{cases} \tag{5.2}$$

where

$$V_\lambda(f)(z) = K \int_0^1 t^{b-1} (1-t)^{c-a-b} \Phi(1-t) \frac{f(tz)}{t} \mathrm{d}t.$$

The value of β is sharp.

Proof Using (5.1), we have

$$\frac{t\lambda'(t)}{\lambda(t)} = (b-1) - \frac{(c-a-b)t}{1-t} - \frac{t\Phi'(1-t)}{\Phi(1-t)}.$$

The condition (4.10) is satisfied when

$$(b-1)-\frac{(c-a-b)t}{1-t}-\frac{t\Phi'(1-t)}{\Phi(1-t)}\leq\begin{cases}1+\frac{1}{\mu}, & \mu\geq 1(\gamma>0);\\ 3-\frac{1}{\alpha}, & \gamma=0,\alpha\in(0,1/3].\end{cases}$$

Since $\Phi(1-t)=1+\sum_{n=1}^{\infty}b_n(1-t)^n$, $b_n\geq 0$ for $n\geq 1$, so the functions $\Phi(1-t)$ and $\Phi'(1-t)$ are non-negative in (0,1). Therefore, a simple computation of $(b-1)-\frac{(c-a-b)t}{1-t}$ with $c-a-b\geq 0$, implies that the condition (4.10) is satisfied whenever b satisfies (5.2). Hence the result follows by applying Theorem 4.2. ◻

Writing $\gamma=0,\alpha>0$ in Theorem 5.1 leads to the following corollary:

Corollary 5.1 *Let a, b, c, $\alpha>0$, and $\beta<1$ satisfy*

$$\frac{\beta}{1-\beta}=-K\int_0^1 t^{b-1}(1-t)^{c-a-b}\Phi(1-t)g_\alpha(t)\mathrm{d}t,$$

where K is a constant such that $K\int_0^1 t^{b-1}(1-t)^{c-a-b}\Phi(1-t)\mathrm{d}t=1$ and g_α is given by (2.7). If $f\in\mathcal{W}_\beta(\alpha,0)\equiv\mathcal{P}_\alpha(\beta)$, then the function

$$V_\lambda(f)(z)=K\int_0^1 t^{b-1}(1-t)^{c-a-b}\Phi(1-t)\frac{f(tz)}{t}\mathrm{d}t$$

belongs to $S^*(\delta)$ with $\delta\in(0,1/2]$ whenever a, b, c are related by $c\geq a+b$ and $b\leq 4-\frac{1}{\alpha}$, $\alpha\in(1/4,1/3]$, for all $t\in(0,1)$. The value of β is sharp.

Writing $\alpha=1+2\gamma,\gamma>0$ and $\mu=1$ in Theorem 5.1 gives the following corollary, which is an improvement of the Theorem 4.3 in [3]:

Corollary 5.2 *Let a, b, $c>0$, $\gamma\geq 1/2$ and $\beta<1$ satisfy*

$$\frac{\beta}{1-\beta}=-K\int_0^1 t^{b-1}(1-t)^{c-a-b}\Phi(1-t)g_\gamma(t)\mathrm{d}t,$$

where K is constant such that $K\int_0^1 t^{b-1}(1-t)^{c-a-b}\Phi(1-t)\mathrm{d}t=1$ and g_γ is given by (2.7). If $f\in\mathcal{W}_\beta(1+2\gamma,\gamma)$, then the function

$$V_\lambda(f)(z)=K\int_0^1 t^{b-1}(1-t)^{c-a-b}\Phi(1-t)\frac{f(tz)}{t}\mathrm{d}t$$

belongs to $S^*(\delta)$ with $\delta\in(0,1/2]$ whenever a, b, c are related by $c\geq a+b$ and $0<b\leq 3$, for all $t\in(0,1)$ and $\gamma>1/2$. The value of β is sharp.

The following special case of Theorem 5.1 corresponds to Bernardi operator, which we state as a theorem.

Theorem 5.2 *Let $c > -1$, $\nu \geq \mu/(\mu+1)$ and $\beta < 1$ satisfy*

$$\frac{\beta}{1-\beta} = -(c+1)\int_0^1 t^c g(t)\mathrm{d}t,$$

where g in given by (2.6). If $f \in \mathcal{W}_\beta(\alpha, \gamma)$, then the Bernardi Transform

$$V_\lambda(f)(z) = (1+c)\int_0^1 t^{c-1} f(tz)\mathrm{d}t$$

belongs to $S^*(\delta)$ with $\delta \in (0, 1/2]$ if

$$c \leq \begin{cases} 1+\frac{1}{\mu}, & \mu \geq 1(\gamma > 0); \\ 3-\frac{1}{\alpha}, & \gamma = 0, \alpha \in (1/4, 1/3]. \end{cases}$$

The value of β is sharp.

Taking $\gamma = 0$, $\alpha > 0$ Theorem 5.2 reduces to the following corollary:

Corollary 5.3 *Let $-1 < c \leq 3 - 1/\alpha$, $\alpha \in (1/4, 1/3]$ and $\beta < 1$ satisfy*

$$\frac{\beta}{1-\beta} = -(c+1)\int_0^1 t^c g_\alpha(t)\mathrm{d}t,$$

where g_α is given by (2.7). If $f \in \mathcal{W}_\beta(\alpha, 0) \equiv \mathcal{P}_\alpha(\beta)$, then the function

$$V_\lambda(f)(z) = (1+c)\int_0^1 t^{c-1} f(tz)\mathrm{d}t$$

belongs to $S^*(\delta)$ with $\delta \in (0, 1/2]$. The value of β is sharp.

Remark 5.1 For $\alpha = 1 + 2\gamma$, $\gamma > 0$ and $\mu = 1$ in Theorem 5.2 yields Corollary 4.1 in [3].

To prove the next theorem, we define

$$\lambda(t) = \begin{cases} (a+1)(b+1)\frac{t^a(1-t^{b-a})}{b-a}, & b \neq a; \\ (a+1)^2 t^a \log(1/t), & b = a, \end{cases} \tag{5.3}$$

where $b > -1$ and $a > -1$.

Theorem 5.3 *Let $b > -1$, $a > -1$, $\nu \geq \mu/(\mu+1)$ and $\alpha > 0$. Let $\beta < 1$ satisfy*

$$\frac{\beta}{1-\beta} = -\int_0^1 \lambda(t) g(t)\mathrm{d}t,$$

where g is given by (2.6) and $\lambda(t)$ is defined by (5.3). If $f \in \mathcal{W}_\beta(\alpha, \gamma)$, then the convolution operator

$$G_f(a, b; z) = \begin{cases} \frac{(a+1)(b+1)}{b-a} \int_0^1 t^{a-1}(1 - t^{b-a}) f(tz)\mathrm{d}t, \; b \neq a; \\ (a+1)^2 \int_0^1 t^{a-1} \log(1/t) f(tz)\mathrm{d}t, \qquad b = a. \end{cases}$$

belongs to $S^*(\delta)$ with $\delta \in (0, 1/2]$ if

$$a \leq \begin{cases} 1 + \frac{1}{\mu}, & \mu \geq 1(\gamma > 0); \\ 3 - \frac{1}{\alpha}, & \gamma = 0, \alpha \in (1/4, 1/3]. \end{cases} \tag{5.4}$$

The value of β is sharp.

Proof We omitted the proof as it follows on the same lines as discussed in Theorem 5.3 [1].

Remark 5.2

1. For $\alpha = 1 + 2\gamma$, $\gamma > 0$ and $\mu = 1$ in Theorem 5.3 yields Theorem 4.1 in [3].
2. For $\gamma = 0$, Theorem 5.3 gives a result similar to Theorem 2.1 [2].

Now, we define

$$\lambda(t) = \frac{(1+a)^p}{\Gamma(p)} t^a \left(\log(1/t)\right)^{p-1}, \;\; a > -1, \;\; p \geq 0.$$

In this case, V_λ reduces to the Komatu operator

$$V_\lambda(f)(z) = \frac{(1+a)^p}{\Gamma(p)} \int_0^1 \left(\log\left(\frac{1}{t}\right)\right)^{p-1} t^{a-1} f(tz)\mathrm{d}t, \;\; a > -1, \;\; p \geq 0.$$

For $p = 1$ Komatu operator gives the Bernardi integral operator. For this λ, the following result holds.

Theorem 5.4 *Let $-1 < a$, $\alpha > 0$, $p \geq 1$, $\nu \geq \mu/(\mu + 1)$ and $\beta < 1$ satisfy*

$$\frac{\beta}{1-\beta} = -\frac{(1+a)^p}{\Gamma(p)} \int_0^1 t^a \left(\log(1/t)\right)^{p-1} g(t)\mathrm{d}t,$$

where g is given by (2.6). If $f \in \mathcal{W}_\beta(\alpha, \gamma)$, then the function

$$\Phi_p(a; z) * f(z) = \frac{(1+a)^p}{\Gamma(p)} \int_0^1 \left(\log\left(\frac{1}{t}\right)\right)^{p-1} t^{a-1} f(tz)\mathrm{d}t$$

belongs to $S^*(\delta)$ with $\delta \in (0, 1/2]$ if

$$a \leq \begin{cases} 1 + \frac{1}{\mu}, & \mu \geq 1(\gamma > 0); \\ 3 - \frac{1}{\alpha}, & \gamma = 0, \alpha \in (1/4, 1/3]. \end{cases} \tag{5.5}$$

The value of β is sharp.

Proof Since

$$\frac{t\lambda'(t)}{\lambda(t)} = a - \frac{p-1}{\log(1/t)},$$

therefore, using the fact that $\log(1/t) > 0$ for $t \in (0, 1)$, and $p \geq 1$, condition (4.10) is satisfied whenever a satisfies (5.5).

Remark 5.3 Setting $\alpha = 1+2\gamma, \gamma > 0$ and $\mu = 1$ in Theorem 5.4, we get Theorem 4.2 in [3].

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