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Vertex Decomposability of 2-CM and Gorenstein Simplicial Complexes of Codimension 3

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Abstract Let Δ be a simplicial complex on vertex set $[n]$. It is shown that if Δ is complete intersection, Cohen–Macaulay of codimension 2, Gorenstein of codimension 3, or 2-Cohen–Macaulay of codimension 3, then Δ is vertex decomposable. As a consequence, we show that if Δ is a simplicial complex such that $I_\Delta = I_t(C_n)$, where $I_t(C_n)$ is the path ideal of length t of C_n, then Δ is vertex decomposable if and only if $t = n$, $t = n - 1$, or n is odd and $t = (n - 1)/2$.

Keywords Vertex decomposable · Simplicial complex · Monomial ideal · Weakly polymatroidal ideal

Mathematics Subject Classification 13F20 · 05E40 · 13F55

1 Introduction

Let Δ be a simplicial complex on vertex set $[n] = \{1, \ldots, n\}$, i.e., Δ is a collection of subsets of $[n]$ with the property that if $F \in \Delta$, then all subsets of F are also in Δ. An element of Δ is called a *face* of Δ, and the maximal faces of Δ under inclusion are called *facets*. We denote by $\mathscr{F}(\Delta)$ the set of facets of Δ. The *dimension* of a face F is

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defined as $\dim F = |F| - 1$, where $|F|$ is the number of vertices of F. The dimension of the simplicial complex Δ is the maximum dimension of its facets. A simplicial complex Δ is called *pure* if all facets of Δ have the same dimension. Otherwise it is called non-pure. We denote the simplicial complex Δ with facets $F_1, \ldots, F_t$ by $\Delta = \langle F_1, \ldots, F_t \rangle$. A simplex is a simplicial complex with only one facet.

For the simplicial complexes Δ_1 and Δ_2 defined on disjoint vertex sets, the join of Δ_1 and Δ_2 is $\Delta_1 * \Delta_2 = \{F \cup G \ : \ F \in \Delta_1, \ G \in \Delta_2\}$.

For the face F in Δ, the link, deletion, and $\star$ of F in Δ are, respectively, denoted by $\mathrm{link}_\Delta F$, $\Delta \setminus F$ and $\star_\Delta F$ and are defined by $\mathrm{link}_\Delta F = \{G \in \Delta \ : \ F \cap G = \varnothing, \ F \cup G \in \Delta\}$ and $\Delta \setminus F = \{G \in \Delta \ : \ F \nsubseteq G\}$ and $\star_\Delta F = \langle F \rangle * \mathrm{link}_\Delta F$.

Let $R = K[x_1, \ldots, x_n]$ be the polynomial ring in n indeterminates over a field K. To a given simplicial complex Δ on the vertex set $[n]$, the Stanley–Reisner ideal is the squarefree monomial ideal whose generators correspond to the non-faces of Δ. We say the simplicial complex Δ is complete intersection, Cohen–Macaulay or Gorenstein if $K[x_1, \ldots, x_n]/I_\Delta$ is complete intersection, Cohen–Macaulay, or Gorenstein, respectively.

The facet ideal of Δ is the squarefree monomial ideal generated by monomials $x_F = \prod_{i \in F} x_i$ where F is a facet of Δ and is denoted by $I(\Delta)$. The complement of a face F is $[n] \setminus F$ and is denoted by F^c. Also, the complement of the simplicial complex $\Delta = \langle F_1, \ldots, F_r \rangle$ is $\Delta^c = \langle F_1^c, \ldots, F_r^c \rangle$. The Alexander dual of Δ is defined by $\Delta^\vee = \{F^c \ : F \notin \Delta\}$. It is known that for the complex Δ, one has $I_{\Delta^\vee} = I(\Delta^c)$.

The simplicial complex Δ is (non-pure) shellable if its facets can be ordered $F_1, F_2, \ldots, F_r$ such that, for all $2 \leq i \leq r$, the subcomplex $\langle F_1, \ldots, F_{i-1} \rangle \cap \langle F_i \rangle$ is pure of dimension $\dim(F_i) - 1$.

Let $I \subset R$ be a monomial ideal. We denote by $G(I)$ the unique minimal system of monomial generators of I. We say that I has linear quotients with respect to the linear order $u_1, \ldots, u_r$ of $G(I)$ if for all $i = 2, \ldots, r$, the colon ideal $(u_1, \ldots, u_{i-1}) : (u_i)$ is generated by linear forms. It is well known that if I has linear quotients and generated in one degree, then I has a linear resolution, see [8]. In [10], the authors showed that the simplicial complex Δ is shellable if and only if $I_{\Delta^\vee}$ has linear quotients.

Billera and Provan [2] introduced the concept of pure vertex decomposable simplicial complexes. Then Björner and Wachs [4] extended the concept of vertex decomposability to non-pure complexes. An analogous extension of k-decomposability to non-pure complexes was given by Woodroofe [14]. Then Jonsson [11] extended Björner and Wachs's definition of shedding vertex in non-pure complexes to shedding face.

Definition 1.1 Let Δ be a simplicial complex on vertex set $[n]$. Then a face F is called a shedding face if every face G of $\star_\Delta F$ satisfies the following exchange property: for every $i \in F$, there is a $j \in [n] \setminus G$ such that $(G \cup \{j\}) \setminus \{i\}$ is a face of Δ.

Definition 1.2 [14] A simplicial complex Δ is recursively defined to be k-decomposable if either Δ is a simplex or else has a shedding face F with $\dim(F) \leq k$ such that both $\Delta \setminus F$ and $\mathrm{link}_\Delta F$ are k-decomposable.

Note that the complexes $\{\}$ and $\{\varnothing\}$ considered to be k-decomposable for all $k \geq -1$. 0-decomposable complexes are of special importance and called vertex decomposable.

It was shown by Billera and Provan [2] that a d-dimensional simplicial complex is d-decomposable if and only if it is shellable. This result was generalized to non-pure complexes by Woodroofe [14]. Also, since each k-decomposable complex is $(k+1)$-decomposable, therefore, we have the following implications:

vertex-decomposable $\Rightarrow$ 1-decomposable $\Rightarrow \cdots \Rightarrow$ d-decomposable $\Leftrightarrow$ shellable

This paper is organized as follows: In Sect. 2, we recall some definitions and some known results which will be needed later. The main results of the paper are in Sect. 3. First we show that each complete intersection simplicial complex and each Cohen–Macaulay simplicial complex of codimension 2 are vertex decomposable. In Theorem 3.5, Vertex decomposability of Gorenstein simplicial complexes of codimension 3 is shown. We also prove that any 2-CM simplicial complex of codimension 3 is vertex decomposable, see Theorem 3.13. Let C_n denote the n-cycle and $I_t(C_n)$ denote the path ideal of C_n of length t. We set $\Delta_t(C_n)$ for the simplicial complex whose Stanley–Reisner ideal is $I_t(C_n)$. In Sect. 4, as an application of our results, we show that $\Delta_t(C_n)$ is vertex decomposable if and only if $t = n$, $t = n-1$, or $t = (n-1)/2$, which extend the main result of [13].

2 Preliminaries

For a monomial $u = x_1^{a_1} \dots x_n^{a_n}$ in R, we denote the support of u by $\mathrm{supp}(u)$ and it is the set of those variables x_i that $a_i \neq 0$. Let m be another monomial in R. If for all $x_i \in \mathrm{supp}(u)$, $x_i^{a_i} \nmid m$ then we set $[u, m] = 1$, otherwise we set $[u, m] \neq 1$.

For a monomial ideal $I \subset R$, we set $I^u = (m_i \in G(I) : [u, m_i] \neq 1)$ and $I_u = (m_i \in G(I) : [u, m_i] = 1)$.

The concept of shedding monomial and k-decomposable monomial ideals was first introduced by Rahmati and Yassemi in [12].

Definition 2.1 Let I be a monomial ideal and $G(I) = \{m_1, \dots, m_r\}$. The monomial $u = x_1^{a_1} \dots x_n^{a_n}$ is called a shedding monomial of I if $I_u \neq 0$ and for each $m_i \in G(I_u)$ and each $x_l \in \mathrm{supp}(u)$ there exists $m_j \in G(I^u)$ such that $\langle m_j : m_i \rangle = \langle x_l \rangle$.

Definition 2.2 Let I be a monomial ideal and $G(I) = \{m_1, \dots, m_r\}$. Then I is a k-*decomposable* ideal if $r = 1$ or else has a shedding monomial u with $|\ \mathrm{supp}(u)\ | \leq k+1$ such that the ideals I^u and I_u are k-decomposable. Note that since $|\ G(I)\ |$ is finite, the recursion procedure will stop.

A 0-decomposable ideal is called *variable decomposable*. Also, a monomial ideal is decomposable if it is k-decomposable for some $k \geq 0$.

A monomial ideal $I \subset R = K[x_1, \dots, x_n]$ generated in a single degree is called polymatroidal if for any $u, v \in G(I)$ such that $\deg_{x_i}(u) > \deg_{x_i}(v)$ there exists an index j with $\deg_{x_j}(u) < \deg_{x_j}(v)$ such that $x_j(u/x_i) \in G(I)$. A squarefree polymatroidal ideal is called matroidal. Also, a monomial ideal I is called weakly polymatroidal if for every two monomials $u = x_1^{a_1} \dots x_n^{a_n} > v = x_1^{b_1} \dots x_n^{b_n}$ in $G(I)$ such that $a_1 = b_1, \dots, a_{t-1} = b_{t-1}$ and $a_t > b_t$, there exists $j > t$ such that $x_t(v/x_j) \in I$. It is clear from the definition that a polymatroidal ideal is weakly polymatroidal.

The following results from [12] are crucial in this paper.

Theorem 2.3 [12, Theorem 2.10] *Let Δ be a (not necessarily pure) d-dimensional simplicial complex on vertex set $[n]$. Then Δ is k-decomposable if and only if $I_{\Delta^\vee}$ is k-decomposable, where $k \leq d$.*

Proposition 2.4 [12, Lemma 3.8] *If I is an squarefree monomial ideal generated in degree 2 which has a linear resolution, then after suitable renumbering of the variables, I is weakly polymatroidal.*

Lemma 2.5 [12, Lemma 2.6] *Let $I \subset R$ be a monomial ideal with the minimal system of generators $G(I) = \{m_1, \ldots, m_r\}$ and u a monomial in R. Then the ideal I is k-decomposable if and only if uI is k-decomposable.*

Theorem 2.6 [12, Theorem 3.5] *Let $I \subset R$ be a weakly polymatroidal ideal. Then I is 0-decomposable.*

3 Some Vertex Decomposable Simplicial Complexes

First, we recall that a Noetherian local ring A is a complete intersection ring if its completion $\hat{A}$ is a residue class ring of a regular local ring R with respect to an ideal generated by an R-sequence. Note that a simplicial complex Δ is called complete intersection if R/I_Δ is a complete intersection ring, i.e., $I_\Delta = (u_1, \ldots, u_m)$ where $\gcd(u_i, u_j) = 1$ for all $i \neq j$. It is easy to see that in this case, $I_\Delta = \bigcap_{x_{i_j} \in \mathrm{supp}(u_j)} (x_{i_1}, \ldots, x_{i_m})$. On the other hand, we know that $I_\Delta = \bigcap_{F \in \mathscr{F}(\Delta)} P_{F^c}$, where $P_{F^c} = (x_i \ : \ i \in F^c)$. Therefore, we have the following:

Remark 3.1 Let Δ be a simplicial complex on vertex set $[n]$. Then Δ is complete intersection if and only if there are disjoint subsets $A_1, \ldots, A_m$ of $[n]$ such that $[n] = \bigcup_{i=1}^m A_i$ and F is a facet of Δ if and only if $F = [n] \setminus \{j_1, \ldots, j_m\}$, where $j_i \in A_i$.

A matroid complex Δ is a simplicial complex with the property that for all faces F and G in Δ with $|F| < |G|$, there exists $i \in G \setminus F$ such that $F \cup \{i\} \in \Delta$. Since link and deletion of any vertex of a matroid are again a matroid, induction on the number of vertices shows that any matroid complex is vertex decomposable. It is easy to see from Remark 3.1 that each complete intersection simplicial complex is a matroid. Hence, every complete intersection simplicial complex is vertex decomposable. However, in the following, we give a different proof of this fact.

Theorem 3.2 *Let Δ be a complete intersection simplicial complex on vertex set $[n]$. Then Δ is vertex decomposable.*

Proof Let $G(I_\Delta) = \{u_1, \ldots, u_m\}$. Since $u_1, \ldots, u_m$ is a regular sequence, we have $\gcd(u_i, u_j) = 1$ for all $i \neq j$. We set $P_{u_i} = (x_i \ : \ x_i \mid u_i)$ for all $i = 1, \ldots, m$. Then it is easy to see that $I_{\Delta^\vee} = \cap_{i=1}^m P_{u_i} = \prod_{i=1}^m P_{u_i}$. Hence, $I_{\Delta^\vee}$ is a transversal polymatroidal ideal and by Theorem 2.6, $I_{\Delta^\vee}$ is 0-decomposable. Thus, the assertion follows from Theorem 2.3. □

Theorem 3.3 *If* Δ *is a Cohen–Macaulay simplicial complex of codimension* 2*, then* Δ *is vertex decomposable.*

Proof Since Δ is Cohen–Macaulay simplicial complex of codimension 2, by a result of Eagon and Reiner [6], $I_{\Delta^\vee}$ is a squarefree monomial ideal which has 2-linear resolution. Hence, by Proposition 2.4 and Theorem 2.6, $I_{\Delta^\vee}$ is 0-decomposable. It follows from Theorem 2.3 that Δ is vertex decomposable. □

As an immediate consequence, we have the following:

Corollary 3.4 *Let* Δ *be a quasi-forest simplicial complex which is not a simplex. Then* $\Delta^\vee$ *is vertex decomposable.*

Proof It is proved in [15] that each quasi-forest is a flag complex. So I_Δ is generated by quadratic monomials and hence $\mathrm{height}(I_{\Delta^\vee}) = 2$. Since Δ is quasi-forest by [15, Corollary 5.5], we have $\mathrm{pd}(K[\Delta^\vee]) = 2$. Therefore, $\Delta^\vee$ is Cohen–Macaulay of codimension 2, and by Theorem 3.3, $\Delta^\vee$ is vertex decomposable. □

Next we consider Gorenstein simplicial complexes and prove the following:

Theorem 3.5 *Each Gorenstein simplicial complex of codimension* 3 *is vertex decomposable.*

Our proof is based on the following structure theorem that can be found in [3].

Theorem 3.6 *Let* Δ *be a Gorenstein simplicial complex of codimension* 3 *on vertex set* $[n]$*. Then* $\mid G(I_\Delta) \mid$ *is an odd number, say* $\mid G(I_\Delta) \mid = 2m+1 \leq n$*, and there exists a regular sequence of squarefree monomials* $u_1, \ldots, u_{2m+1}$ *in* $R = K[x_1, \ldots, x_n]$ *such that*

$$G(I_\Delta) = \{u_i u_{i+1}, \ldots, u_{i+m-1} : i = 1, \ldots, 2m+1\},$$

where $u_i = u_{i-2m-1}$ *whenever* $i > 2m+1$.

We will use the following remarks for our proof.

Remark 3.7 Let Δ be a Gorenstein simplicial complex of codimension 3 on vertex set $[n]$ with

$$G(I_\Delta) = \{u_i u_{i+1}, \ldots, u_{i+m-1} : i = 1, \ldots, 2m+1\},$$

where $u_i = u_{i-2m-1}$ whenever $i > 2m+1$. Then after relabeling of the variables, we may assume that $u_1 = \prod_{i=1}^{l_1} x_i$, $u_2 = \prod_{i=l_1+1}^{l_2} x_i, \ldots, u_{2m+1} = \prod_{i=l_{2m}+1}^{n} x_i$.

Remark 3.8 If Δ is a Gorenstein simplicial complex of codimension 3, then it is easy to see from Theorem 3.6 that $I_\Delta = \bigcap (x_{t_i}, x_{r_j}, x_{s_k})$ with $x_{t_i} \in \mathrm{supp}(u_i)$, $x_{r_j} \in \mathrm{supp}(u_j)$, $x_{s_k} \in \mathrm{supp}(u_k)$, where $1 \leq i < j < k \leq 2m+1$, and $j - i \leq m$, $k - j \leq m$, $k - i \geq m+1$. Thus, $I_{\Delta^\vee}$ is generated by the monomials $x_{t_i} x_{r_j} x_{s_k}$ with $x_{t_i} \in \mathrm{supp}(u_i)$, $x_{r_j} \in \mathrm{supp}(u_j)$, $x_{s_k} \in \mathrm{supp}(u_k)$, where $1 \leq i < j < k \leq 2m+1$ and $j - i \leq m$, $k - j \leq m$, $k - i \geq m+1$.

Example 3.9 Let Δ be a simplicial complex with

$$\begin{aligned}\mathscr{F}(\Delta) = \{&\{1, 2, 4, 5\}, \{1, 2, 4, 6\}, \{1, 2, 5, 6\}, \{1, 3, 4, 6\},\\ &\{1, 3, 4, 7\}, \{1, 3, 5, 6\}, \{1, 3, 5, 7\}, \{1, 4, 5, 7\}, \{2, 3, 5, 6\}, \{2, 3, 5, 7\},\\ &\{2, 3, 6, 7\}, \{2, 4, 5, 7\}, \{2, 4, 6, 7\}, \{3, 4, 6, 7\}\}.\end{aligned}$$

Then $I_\Delta = I_3(C_7) = (x_1x_2x_3, x_2x_3x_4, x_3x_4x_5, x_4x_5x_6, x_5x_6x_7, x_6x_7x_1, x_7x_1x_2)$, and $I_\Delta = \bigcap_{i,j,k}(x_i, x_j, x_k)$, where $j - i \leq 3, k - j \leq 3$ and $k - i \geq 4$. Therefore, by Remark 3.8, Δ is Gorenstein simplicial complex of codimension 3. Observe that 1 is a shedding vertex of Δ.

Lemma 3.10 *Let Δ be a Gorenstein simplicial complex of codimension 3, and $x_{t_i}x_{r_j}x_{s_k} \in G(I_{\Delta^\vee})$. If $k < k' \leq 2m+1$ or $1 \leq k' < i$, then for each $x_{s_{k'}} \in \operatorname{supp}(u_{k'})$, either $x_{t_i}x_{r_j}x_{s_{k'}}$ or $x_{r_j}x_{s_k}x_{s_{k'}}$ belongs to $G(I_{\Delta^\vee})$.*

Proof We set $v_1 = x_{t_i}x_{r_j}x_{s_{k'}}$ and $v_2 = x_{r_j}x_{s_k}x_{s_{k'}}$.

Case 1 Let $k < k' \leq 2m + 1$ and suppose on contrary v_1 and v_2 do not belong to $G(I_{\Delta^\vee})$. Since $x_{t_i}x_{r_j}x_{s_k} \in G(I_{\Delta^\vee})$, one has $j - i \leq m$ and $k' - i > k - i \geq m + 1$, hence $v_1 \notin G(I_{\Delta^\vee})$ if and only if

$$k' - j > m. \tag{1}$$

Again since $x_{t_i}x_{r_j}x_{s_k} \in G(I_{\Delta^\vee})$, we know that $k - j \leq m$ and $k' - k \leq m$. So $v_2 \notin G(I_{\Delta^\vee})$ if and only if

$$k' - j \leq m. \tag{2}$$

From 1 and 2, we get a contradiction.

Case 2 The same argument works also in the case $1 \leq k' < i$. □

Proposition 3.11 *Let Δ be a Gorenstein simplicial complex of codimension 3 on $[n]$, and $I = I_{\Delta^\vee}$. Then the following statements hold.*

(i) x_n is a shedding variable for I.
(ii) Let $l' + 1 \leq l \leq n - 1$, where l' is the smallest index such that there exists $x_ix_jx_{l'} \in G(I)$ with $i < j < l'$. Then x_l is a shedding variable for $I_{x_n,x_{n-1},\ldots,x_{l+1}}$.

Proof (i): Since Δ is a simplicial complex on $[n]$, $I_{x_n} \neq 0$. Suppose $x_{t_i}x_{r_j}x_{s_k} \in G(I_{x_n})$ be an arbitrary element with $s_k < n$. Let u_k be as in Theorem 3.6. If $k = 2m + 1$, then by Remark 3.8, $x_n \in \operatorname{supp}(u_k)$ and hence $x_{t_i}x_{r_j}x_n \in G(I)$. If $k < 2m + 1$, then by Lemma 3.10 either $x_{t_i}x_{r_j}x_n$ or $x_{r_j}x_{s_k}x_n$ belongs to $G(I)$. Hence, in any case one of the monomials, $x_{t_i}x_{r_j}x_n$ or $x_{r_j}x_{s_k}x_n$ belongs to $G(I^{x_n})$. This implies that x_n is a shedding variable for I.

(ii): By induction, we know that $I_{x_n,x_{n-1},\ldots,x_l} = (I_{x_n,x_{n-1},\ldots,x_{l+1}})_{x_l}$. If $x_{t_i}x_{r_j}x_{s_k} \in G(I_{x_n,x_{n-1},\ldots,x_l})$ with $s_k < l$, then as we showed in case (i), by Remark 3.8 and Lemma 3.10, either $x_{t_i}x_{r_j}x_l \in G(I^{x_l}_{x_n,x_{n-1},\ldots,x_{l+1}})$ or $x_{r_j}x_{s_k}x_l \in G(I^{x_l}_{x_n,x_{n-1},\ldots,x_{l+1}})$. This completes the proof. □

Proposition 3.12 *Let Δ be a Gorenstein simplicial complex of codimension 3. Let $1 \leq l \leq n$ and J_l be the monomial ideal which is generated by the set of those quadratic monomials $x_i x_j$, where $x_i x_j x_l \in G(I_{\Delta^\vee})$. Then J_l has linear quotients and in particular it is 0-decomposable.*

Proof We know by [9] Δ is shellable. Hence, $I_{\Delta^\vee}$ has linear quotients. Suppose that $G(I_{\Delta^\vee}) = \{v_1, v_2, \dots, v_t\}$ and it has linear quotients in the given order. Hence, for each v_c and v_d in $G(I_{\Delta^\vee})$ with $c < d$, there exists another monomial $v_{d'}$ with $d' < d$ such that $v_{d'} : v_d = x_{c'}$ for some c' and $x_{c'}$ divides $v_c : v_d$. We order the monomials in $G(J_l)$ by the induced order of $G(I_{\Delta^\vee})$ and claim that J_l has linear quotients in this order. Let w_p and w_q be arbitrary two elements in $G(J_l)$ with $p < q$. Thus $v_p = w_p x_l$ and $v_q = w_q x_1$ belong to $G(I_{\Delta^\vee})$. Therefore there exists another monomial $v_{k'}$ with $k' < q$ such that $v_{k'} : w_q x_l = x_s$ and x_s divides $w_p x_l : w_q x_l$. It is easy to see that $s \neq l$ and $x_l \mid v_{k'}$. Hence $w_{k'} = v_{k'}/x_l \in G(J_l)$, and $w_{k'} : w_q = x_s$ which divides $w_p : w_q$. This implies that J_l has linear quotients. Hence by Proposition 2.4 and Theorem 2.6, J_l is weakly polymatroidal and 0-decomposable. □

Proof of Theorem 3.5: By Theorem 2.3, Δ is 0-decomposable if and only if $I = I_{\Delta^\vee}$ is 0-decomposable. By Proposition 3.11, x_n is a shedding variable for I. Hence it is enough to show that I^{x_n} and I_{x_n} are 0-decomposable.

Since $I^{x_n} = \langle x_i x_j x_n : x_i x_j x_n \in G(I)\rangle = x_n \langle x_i x_j : x_i x_j x_n \in G(I)\rangle$, hence by Proposition 3.12 and Lemma 2.5, I^{x_n} is 0-decomposable. Now we show that I_{x_n} is 0-decomposable too. Again by using Proposition 3.11, we have x_{n-1} is a shedding monomial for I_{x_n}. But $I_{x_n}^{x_{n-1}} = \langle x_i x_j x_{n-1} : x_i x_j x_{n-1} \in G(I_{x_n})\rangle = x_{n-1}\langle x_i x_j : x_i x_j x_{n-1} \in G(I_{x_n})\rangle$. Then again by Proposition 3.12 and Lemma 2.5, $I_{x_n}^{x_{n-1}}$ is 0-decomposable. In order to show that $(I_{x_n})_{x_{n-1}} = I_{x_n, x_{n-1}}$ is 0-decomposable, we continue this procedure as follows: Let l' be the smallest integer that $x_i x_j x_{l'} \in G(I)$ with $i < j < l'$. Let $l' + 1 \leq l \leq n - 1$ be an integer. Then as we showed in the above, one can see that $I^{x_l}_{x_n, x_{n-1}, \dots, x_{l+1}}$ is 0-decomposable. Since $(I_{x_n, x_{n-1}, \dots, x_{l'+2}})_{x_{l'+1}} = \langle x_i x_j x_{l'} : x_i x_j x_{l'} \in G(I_{x_n, x_{n-1}, \dots, x_{l'+1}})\rangle = x_{l'}\langle x_i x_j : x_i x_j x_{l'} \in G(I_{x_n, x_{n-1}, \dots, x_{l'+1}})\rangle$. So by Proposition 3.12 and Lemma 2.5, $(I_{x_n, x_{n-1}, \dots, x_{l'+2}})_{x_{l'+1}}$ is 0-decomposable. Hence $I_{\Delta^\vee}$ is 0-decomposable and therefore Δ is vertex decomposable. □

Now we study the vertex decomposability property for another class of simplicial complexes, that is 2-CM simplicial complexes. According to [1], a Cohen–Macaulay simplicial complex Δ is 2-CM (doubly Cohen–Macaulay) if the deletion $\Delta \setminus \{k\}$ is Cohen–Macaulay of the same dimension as Δ, for each existing vertex $k \in \Delta$.

Theorem 3.13 *Let Δ be a 2-CM simplicial complex of codimension 3 on vertex set $[n]$. Then Δ is vertex decomposable.*

Proof We prove the theorem by induction on $|[n]|$ the number of vertices of Δ. If $|[n]| = 0$, then $\Delta = \{\}$ and it is vertex decomposable. Now Let $|[n]| > 0$ and $k \in [n]$ be a vertex of Δ. Then the simplicial complex $\text{link}_\Delta\{k\}$ is a complex on $|[n]| - 1$ vertices and its dimension is $\dim \Delta - 1$. It is known that $\text{link}_\Delta\{k\}$ is again 2-CM (see e.g. [1]) of codimension 3. Therefore, by induction hypothesis $\text{link}_\Delta\{k\}$ is vertex decomposable.

On the other hand, since Δ is a 2-CM, for each existing vertex $k \in \Delta$, $\Delta \setminus \{k\}$ is Cohen–Macaulay of codimension 2, and by Theorem 3.3, $\Delta \setminus \{k\}$ is vertex decomposable. It is easy to see that no face of $\mathrm{link}_{\Delta}\{k\}$ is a facet of $\Delta \setminus \{k\}$. Therefore any vertex k is a shedding vertex and Δ is vertex decomposable. □

Hochster's Tor formula provides that each Gorenstein simplicial complex is 2-CM (see [1]). Therefore, Theorem 3.5 is an immediate consequence of Theorem 3.13. But note that the proof of Theorem 3.5 is algebraic while the proof of Theorem 3.13 is combinatorial.

4 Path Ideals of Cycles

As an application of the above results, we show that simplicial complexes where associated to specific path ideals of an n-th cycle are vertex decomposable. Path ideal of a graph was first introduced by Conca and De Negri in [5]. Let G be a directed graph on vertex set $\{x_1, \ldots, x_n\}$. Fix an integer $2 \leq t \leq n$. A sequence $x_{i_1}, \ldots, x_{i_t}$ of distinct vertices of G is called a *path* of length t if there are $t-1$ distinct directed edges $e_1, \ldots, e_{t-1}$, where e_j is an edge from x_{i_j} to $x_{i_{j+1}}$. Then the *path ideal* of G of length t is the monomial ideal $I_t(G) = (\prod_{j=1}^{t} x_{i_j})$, where $x_{i_1}, \ldots, x_{i_t}$ is a path of length t in G. Let C_n denote the n-cycle with directed edges $e_1, \ldots, e_n$, where e_i is from x_i to x_{i+1} for $i = 1, \ldots, n-1$ and e_n is from x_n to x_1. Hence $I_t(C_n) = (u_1, \ldots, u_n)$, where $u_i = \prod_{v=0}^{t-1} x_{i+v}$ for all $i = 1, \ldots, n$, where $x_d = x_{d-n}$ whenever $d > n$. In [7, proposition4.1] it is shown that $R/I_2(C_n)$ is vertex decomposable/ shellable/ Cohen–Macaulay if and only if $n = 3$ or 5. Recently, Saeedi, Kiani and Terai in [13] showed that if $2 < t \leq n$, then $R/I_t(C_n)$ is sequentially Cohen–Macaulay if and only if $t = n, t = n-1$ or $t = (n-1)/2$. As a consequence of our result we can extend the main result of [13].

Theorem 4.1 *Let $3 \leq t \leq n$ and Δ be a simplicial complex on $[n]$ such that $I_\Delta = I_t(C_n)$. Then Δ is vertex decomposable if and only if $t = n, t = n-1$ or $t = (n-1)/2$.*

Proof If $t = n$, then Δ is complete intersection. If $t = n-1$, then Δ is Cohen–Macaulay of codimension 2, and if $t = (n-1)/2$, then Δ is Gorenstein of codimension 3. Hence in these three cases Δ is vertex decomposable. If t is not one of the above cases, then by [13], Δ is not sequentially Cohen–Macaulay and hence not vertex decomposable. □

For the simplicial complexes, one has the following implication:

$$\textbf{vertex decomposable} \Rightarrow \textbf{shellable} \Rightarrow \textbf{Cohen}-\textbf{Macaulay}.$$

Note that these implications are strict, but by the following corollary, for path ideals, the reverse implications are also valid.

Combining the main result of [13] with our result, we get the following:

Corollary 4.2 *Let $3 \leq t \leq n$ and Δ be a simplicial complex on $[n]$ such that $I_\Delta = I_t(C_n)$. Then the following conditions are equivalent:*

(i) Δ *is Cohen–Macaulay;*
(ii) Δ *is shellable;*
(iii) Δ *is vertex decomposable.*

Moreover, these equivalent condition hold if and only if $t = n, t = n - 1$ *or* $t = (n-1)/2$.

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