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Maximal Thermo-geometric Parameter in a Nonlinear Heat Conduction Equation

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Abstract A nonlinear heat conduction equation is studied, and the maximal thermo-geometric parameter in the equation is analytically determined, above which thermal instability occurs. The first-order result yields an acceptable error, and the variational iteration method is recommended for a higher accurate prediction.

Keywords Heat transfer · Longitudinal fins · Variational iteration method · Thermo-geometric parameter

Mathematics Subject Classification 35K05 · 35Q79 · 80A20 · 34B15

1 Introduction

We consider a rectangular longitudinal one-dimensional fin, which is attached to a fixed base surface of temperature T_b and extends into a fluid of temperature T_b. The dimensionless governing equation is [1]

$$\frac{\mathrm{d}^2\theta}{\mathrm{d}x^2} + \beta\theta\frac{\mathrm{d}^2\theta}{\mathrm{d}x^2} + \beta\left(\frac{\mathrm{d}\theta}{\mathrm{d}x}\right)^2 - M^2(1+\beta\theta)^n = 0, \quad \theta > 0 \tag{1.1}$$

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with boundary conditions

$$\theta'(0) = 0 \quad \text{and} \quad \theta(1) = 1 \tag{1.2}$$

where θ is the dimensionless temperature, $\beta = \lambda(T_b - T_a)$ is the gradient of thermal conductivity, M is the thermo-geometric parameter. The exponent,n, represents laminar film boiling or condensation when $n = -1/4$, laminar natural convection when $n = 1/4$, turbulent natural convection when $n = 1/3$, nucleate boiling when $n = 2$, radiation when $n = 3$, and n vanishes for a constant heat transfer coefficient.

It is very important to study the effect of M on the heat transfer. It is obvious that the increase of M might result in negative θ at $x = 0$, contradicting the assumption. It is, therefore, important to identify the maximal value for the thermo-geometric parameter.

The detailed derivation of Eq. (1) was given in [1], and the thermal characteristics were elucidated in [2]. Harley and Moitsheki [1] gave a numerical investigation, and obtained the maximal values for various n. Some effective analytical methods were successfully applied to the problem [3–5]; there are alternative numerical/analytical methods, such as the three-point implicit block multistep method [6], the variational iteration method [7–12], reproducing kernel method [13, 14], and a complete review on various analytical methods is available in [15]. In this paper, we will suggest a simple analytical approach to identification of the maximal value of the thermo-geometric parameter.

2 Maximal Thermo-geometric Parameter

In this study, neither an exact solution nor an approximate solution is searched for, only the maximal M in Eq. (1.1) is considered. For this end, we choose a very simple trial function in the form

$$\theta(x) = a_0 + a_1 x + a_2 x^2 \tag{2.1}$$

By the boundary conditions, Eq. (1.2), we have

$$a_1 = 0 \tag{2.2}$$

$$a_0 + a_1 + a_2 = 1 \tag{2.3}$$

By Eqs. (1.1) and (1.2), we obtain

$$\theta''(0) = M^2(1 + \beta\theta(0))^{n-1} = M^2(1 + \beta a_0)^n \tag{2.4}$$

Eq. (2.4) means

$$2a_2 = M^2(1 + \beta a_0)^n \tag{2.5}$$

Submitting Eqs. (2.2) and (2.5) into Eq. (2.3) results in

$$a_0 + \frac{M^2}{2}(1+\beta a_0)^n = 1 \tag{2.6}$$

Setting $a_0 = \theta_{\min}(0) = 0$, we obtain maximal value for M, which is

$$M_{\max} = \sqrt{2} = 1.414 \tag{2.7}$$

Comparison of Eq. (2.7) with the numerical results given in [1] reveals that the maximal error is 16.5 % for $-4 < n < 3$. The accuracy is 3 and 5.4 % for $n = 2$ and $n = 3$, respectively.

When the thermo-geometric parameter reaches its maximal value, thermal instability occurs [2], so in practical applications, we should follow $M \ll M_{\max}$, and the 16.5 % error is acceptable.

If a higher accurate prediction is needed, the variational iteration algorithm [7, 15] is recommended.

According to the variational iteration method [7, 15], the following iteration formulation (variational iteration algorithm-II [15]) can be constructed

$$\theta_{p+1}(x) = \theta_0(x) + \int_0^x (x-s)\left\{\beta\theta_p(s)\frac{\mathrm{d}^2\theta_p(s)}{\mathrm{d}s^2} + \beta\left(\frac{\mathrm{d}\theta_p(s)}{\mathrm{d}s}\right)^2 - M^2\left[1+\beta\theta_p(s)\right]^n\right\}\mathrm{d}s \tag{2.8}$$

We begin with $\theta_0(x) = \theta(0) = a_0$, by Eq. (2.8), we have

$$\theta_1(x) = a_0 + \int_0^x (x-s)\left\{-M^2(1+\beta a_0)^n\right\}\mathrm{d}s = a_0 + \frac{1}{2}M^2(1+\beta a_0)^n x^2 \tag{2.9}$$

If the first-order approximate solution is enough, then by the boundary condition, $\theta(1) = 1$, the following result is obtained.

$$\theta_1(1) = a_0 + \frac{1}{2}M^2(1+\beta a_0)^n = 1 \tag{2.10}$$

which is exactly same with Eq. (2.6).

The solution process can continue without any difficulty by using some mathematical software, and a higher accurate result can be obtained.

3 Conclusion

In practical applications, we need neither an exact solution nor an approximate solution, but a criterion for some parameters in the studied equation, for example, the condition of resonance for a nonlinear oscillator. In this paper, we suggest a simple but effective approach to identification of the maximal thermo-geometric parameter in Eq. (1), the 16.5 % accuracy of the first-order prediction is acceptable considering it should follow $M \ll M_{\max}$ though a higher accuracy can be obtained by the variational iteration method.

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