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Rainbow C_4's and Directed C_4's: The Bipartite Case Study

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Abstract In this paper we obtain a new sufficient condition for the existence of directed cycles of length 4 in oriented bipartite graphs. As a corollary, a conjecture of Li is confirmed. As an application, a sufficient condition for the existence of rainbow cycles of length 4 in bipartite edge-colored graphs is obtained.

Keywords Rainbow cycle · Edge-colored graph · Directed cycle · Oriented bipartite graph

Mathematics Subject Classification 05C38 · 05C15 · 05C20

1 Introduction

For terminology and notation not defined here, we refer to [2]. Let $G = (V, E)$ be a simple graph. An *edge-coloring* of G is a mapping $C : E \to \mathbb{N}^+$, where $\mathbb{N}^+$ is the set of positive integers. We call G^c an *edge-colored graph* (or briefly, a *colored graph*) if G is assigned an edge-coloring C. Let v be a vertex of G^c. The *color degree* of v in

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G^c, denoted by $d^c_G(v)$ (or briefly, $d^c(v)$), is the number of colors of the edges incident to v. A *color neighborhood set* $N^c(v)$ of v is a subset of $N(v)$, the neighborhood of v, such that the colors of edges between v and $N^c(v)$ are pairwise distinct. Let H be a subgraph of G, then $C(H) = \{C(e) : e \in E(H)\}$ is called the *color set* of H.

A subgraph of a colored graph is called *rainbow* (sometimes called *heterochromatic* or *colorful*) if all edges of it have distinct colors. The existence of rainbow subgraphs has been studied for a long time. A problem of rainbow Hamilton cycles in colored complete graphs was mentioned by Erdös et al. [6], and later studied by Hahn and Thomassen [8], Frieze and Reed [7] and Albert et al. [1], respectively. Rainbow matchings were studied by Wang and Li [16], Lesaulnier et al. [11], and Kostochka and Yancey [10]. Chen and Li [4,5] studied the existence of long rainbow paths. A recent article on strong rainbow connection can be found in [15]. For a survey on the study of rainbow subgraphs in colored graphs, we refer to [9].

In particular, rainbow short cycles have received much attention. Broersma et al. [3] studied the existence of rainbow C_3's and C_4's under color neighborhood union condition. Later, Li and Wang [14] obtained two results on the existence of rainbow C_3's and C_4's under colored degree condition.

Theorem 1 (Li and Wang [14]) *Let G^c be a colored graph of order $n \geq 3$. If $d^c(v) \geq (4\sqrt{7}/7 - 1)n + 3 - 4\sqrt{7}/7$ for each $v \in V(G)$, then G^c has either a rainbow C_3 or a rainbow C_4.*

Theorem 2 (Li and Wang [14]) *Let G^c be a colored graph of order $n \geq 3$. If $d^c(v) \geq (\sqrt{7} + 1)n/6$ for each $v \in V(G)$, then G^c has a rainbow C_3.*

Li and Wang [14] conjectured that every colored graph G^c of order $n \geq 3$ has a rainbow C_3 if $d^c(v) \geq (n + 1)/2$ for each $v \in V(G)$. This conjecture was proved by Li [13] and stronger results were proved by Li et al. [12] with different methods as follows.

Theorem 3 (Li [13]) *Let G^c be a colored graph of order $n \geq 3$. If $d^c(v) \geq (n+1)/2$ for each $v \in V(G)$, then G^c has a rainbow C_3.*

Theorem 4 (Li et al. [12]) *Let G^c be a colored graph of order $n \geq 3$. If $\sum_{v \in V(G)} d^c(v) \geq n(n + 1)/2$, then G^c has a rainbow C_3.*

Theorem 5 (Li et al. [12]) *Let G^c be a colored graph of order $n \geq 3$. If $d^c(v) \geq n/2$ for each $v \in V(G)$, then G^c has a rainbow C_3 or $G \in \{K_{n/2,n/2}, K_4 - e, K_4\}$.*

The existence of rainbow C_4's in special colored graphs has also been studied. Wang et al. [17] obtained a result on the existence of rainbow C_4's in triangle-free colored graphs. Recently, Li [13] got a result on the existence of rainbow C_4's in balanced bipartite colored graphs.

Theorem 6 (Wang et al. [17]) *Let G^c be a triangle-free colored graph of order $n \geq 9$. If $d^c(v) \geq (3 - \sqrt{5})n/2 + 1$ for each $v \in V(G)$, then G^c has a rainbow C_4.*

Theorem 7 (Li [13]) *Let G^c be a balanced bipartite colored graph of order $2n$ with bipartition (A, B). If $d^c(v) > 3n/5 + 1$ for each $v \in A \cup B$, then G^c has a rainbow C_4.*

While in [13], Li made a tiny error in the proof of Theorem 7. Notice that $K_{3,3}$ is 3-edge-colorable, and a proper 3-edge-coloring of $K_{3,3}$ satisfies the condition of Theorem 7, but it has no rainbow C_4 since there are only 3 colors. We point out that, in order to correct it, the condition $d^c(u) > 3n/5 + 1$ should be changed into $d^c(u) > (3n + 8)/5$.

Now we turn to finite simple oriented graphs, i.e., finite graphs without multiple edges and loops in which each edge is replaced by exactly one arc. Let $D[A, B]$ be an oriented bipartite graph with bipartition (A, B). When there is no ambiguity, we use D instead of $D[A, B]$. For $A_1 \subseteq A$ and $B_1 \subseteq B$, we denote by $A_D(A_1, B_1)$ the set of arcs from A_1 to B_1 in $D[A, B]$.

The study of rainbow cycles in colored graphs is largely related to the study of oriented cycles in digraphs. For a wonderful example, see the introduction of [13]. In particular, motivated by the study of short rainbow cycles in colored graphs, Li [13] proposed the following nice conjecture and proved for balanced oriented bipartite graphs.

Conjecture 1 (Li [13]) Let D be an oriented bipartite graph with bipartition (A, B). If $d^+(u) > |B|/3$ for each $u \in A$ and $d^+(v) > |A|/3$ for each $v \in B$, then D has a directed C_4.

Theorem 8 (Li [13]) *Let D be a balanced oriented bipartite graph with bipartition (A, B), where $|A| = |B| = n$. If $d^+(v) > n/3$ for each $v \in A \cup B$, then D has a directed C_4.*

We state a construction from [17] to show that if Conjecture 1 holds, then it would be almost the best possible. Let m and n be two positive integers divisible by 3. Let $|M_0| = |M_1| = |M_2| = m/3$ and $|N_0| = |N_1| = |N_2| = n/3$. We construct an oriented bipartite graph with bipartition (M, N), where $M = M_0 \cup M_1 \cup M_2$ and $N = N_0 \cup N_1 \cup N_2$, by creating all possible arcs from M_i to N_i, and from N_i to M_{i+1}, $i = 0, 1, 2$ (modulo 3). In the rest parts, we use $D^*(m, n)$ to denote the construction above.

The first purpose of this paper is to confirm Conjecture 1. In fact, we prove a stronger result as follows.

Theorem 9 *Let D be an oriented bipartite graph with bipartition (A, B), where $|A| = m \geq 2$ and $|B| = n \geq 2$. If $d^+(u) \geq n/3$ for each $u \in A$ and $d^+(v) \geq m/3$ for each $v \in B$, then either D has a directed C_4 or $D = D^*(m, n)$.*

By using Theorem 9, we can extend Theorem 7 as follows.

Theorem 10 *Let G^c be a bipartite colored graph with bipartition (A, B). If $d^c(u) \geq (3|B| + 8)/5$ for each $u \in A$ and $d^c(v) \geq (3|A| + 8)/5$ for each $v \in B$, then G^c has a rainbow C_4.*

2 Proofs

2.1 Proof of Theorem 9

Let $\mathcal{D}(m, n)$ be the family of digraphs consisting of those oriented bipartite graphs with bipartition (A, B) which satisfy the condition of Theorem 9, where $m = |A|$ and $n = |B|$.

First, we claim that it is sufficient to prove for those m and n which are both multiples of 3. Suppose Theorem 9 holds for $\mathcal{D}(m, n)$ with $m \equiv n \equiv 0 \pmod 3$. For any $D \in \mathcal{D}(m', n')$, where m' and n' are not both multiples of 3, let $s_1 = 3\lceil \frac{m'}{3} \rceil - m'$, $A^* = \{u_1, \ldots, u_{s_1}\}$, and $A' = A \cup A^*$. Let $s_2 = 3\lceil \frac{n'}{3} \rceil - n'$, $B^* = \{v_1, \ldots, v_{s_2}\}$ and $B' = B \cup B^*$. Now we construct a new oriented bipartite graph D' with bipartition (A', B'), where $A(D') = A(D) \cup \{u'v, v'u : u \in A, u' \in A^*, v \in B, v' \in B^*\}$. Notice that $d^+_{D'}(u) \geq \lceil \frac{|B|}{3} \rceil = \frac{|B'|}{3}$ for each $u \in A$ and $d^+_{D'}(u') = n' > \frac{|B'|}{3}$ for each $u' \in A^*$. Similarly, $d^+_{D'}(v) \geq \lceil \frac{|A|}{3} \rceil = \frac{|A'|}{3}$ for each $v \in B$ and $d^+_{D'}(v') = m' > \frac{|A'|}{3}$ for each $v' \in B^*$. It follows that $D' \in \mathcal{D}(3\lceil \frac{m'}{3} \rceil, 3\lceil \frac{n'}{3} \rceil)$. Hence D' has a directed C_4 or $D' = D^*(3\lceil \frac{m'}{3} \rceil, 3\lceil \frac{n'}{3} \rceil)$. Since A^* and B^* are not both empty sets and the vertices in A^* and B^* only have outdegrees, $D' \neq D^*(3\lceil \frac{m'}{3} \rceil, 3\lceil \frac{n'}{3} \rceil)$, and moreover, the directed C_4 in D' is also in D. The proof of our claim is complete.

Now assume $D \in \mathcal{D}(m, n)$, where $m = 3m_1, n = 3n_1$, and m_1, n_1 are two positive integers. Let D_1 be a spanning subdigraph of D satisfying $d^+_{D_1}(u) = n_1$ for each $u \in A$ and $d^+_{D_1}(v) = m_1$ for each $v \in B$. Suppose D has no directed C_4, obviously, D_1 also has no directed C_4. Let u_0 be a vertex with maximum indegree k_1 among A, and v_0 be a vertex with maximum indegree k_2 among B. Let $B_1 = N^-_{D_1}(u_0)$, $B_2 = N^+_{D_1}(u_0)$, $A_3 = N^+_{D_1}(B_2)$, and $B_3 = N^+_{D_1}(A_3) - B_2$, where $|B_1| = k_1$, $|B_2| = n_1$. Since D_1 has no directed C_4, we have $N^+_{D_1}(A_3) \cap B_1 = \emptyset$. Since k_2 is the maximum indegree of all vertices in B, we get

$$|B_3|k_2 \geq |N^-_{D_1}(B_3)| = |A_{D_1}(A_3, B_3)|. \tag{1}$$

Since D_1 has no directed C_4, there is no arc from A_3 to B_1, which implies all arcs starting from A_3 have heads in $B_2 \cup B_3$. Hence $|A_{D_1}(A_3, B_3)| = |A_3|n_1 - |A_{D_1}(A_3, B_2)|$. Since

$$|A_{D_1}(A_3, B_2)| \leq |A_3||B_2| - |A_{D_1}(B_2, A_3)|, \tag{2}$$

we obtain

$$|A_{D_1}(A_3, B_3)| \geq |A_3|n_1 - (|A_3||B_2| - |A_{D_1}(B_2, A_3)|) = |A_{D_1}(B_2, A_3)| = m_1 n_1. \tag{3}$$

Together with (1) and (3), we obtain $|B_3| \geq \frac{m_1 n_1}{k_2}$. Therefore,

$$3n_1 = |B| \geq |B_1| + |B_2| + |B_3| \geq k_1 + n_1 + \frac{m_1 n_1}{k_2} \geq 3\sqrt[3]{\frac{k_1 m_1 n_1^2}{k_2}}. \quad (4)$$

It follows that $k_2 n_1 \geq k_1 m_1$. By symmetry, we also have $k_1 m_1 \geq k_2 n_1$. Thus, $k_1 m_1 = k_2 n_1$ and all the inequalities (1)–(4) are actually equalities. These facts imply $|B_1| = |B_2| = |B_3| = n_1 = k_1$, $m_1 = k_2$, $|A_{D_1}(B_2, A_3)| = |A_{D_1}(A_3, B_3)| = m_1 n_1$, and all vertices in A have indegree n_1 in D_1, all vertices in B have indegree m_1 in D_1.

Next we show $|A_3| = m_1$. First, choose a vertex $v' \in B_2$, $d^+(v') = m_1$, so $|A_3| \geq m_1$ by the definition of A_3. Assume that $|A_3| > m_1$. Since the inequality (2) becomes equality, the underlying graph of $D_1[A_3, B_2]$ is a complete bipartite graph. Hence there exists a vertex, say $u' \in A_3$, such that $u'v' \in D_1$. Since $u' \in A_3$, there exist a vertex $v'' \in B_2$, such that $v''u' \in D_1$ by the choice of A_3. Note that $d^+_{D[B_2,A_3]}(v') = d^+_{D[B_2,A_3]}(v'') = m_1$ and $u' \notin N^+(v')$. It follows that there exists a vertex, say $u'' \in A_3$, such that $v'u'' \in D_1$ and $v''u'' \notin D_1$. Since the underlying graph of $D_1[B_2, A_3]$ is a complete bipartite graph, $u''v'' \in D_1$. Now $C = u'v'u''v''u'$ is a directed C_4 in D_1, a contradiction. Hence $|A_3| = m_1$.

Now let $A_1 = N^+_{D_1}(B_3)$ and $A_2 = N^-_{D_1}(B_2)$. Note that $|\{vu : v \in B_2, u \in A_3\}| = m_1 n_1 = |A_{D_1}(B_2, A_3)|$. It follows that $A_{D_1}(B_2, A_3) = \{vu : v \in B_2, u \in A_3\}$. Similarly, $A_{D_1}(A_3, B_3) = \{uv : u \in A_3, v \in B_3\}$. So there is no arc with tail in A_3 and head in B_2, or arc with tail in B_3 and head in A_3, follows $A_1 \cap A_3 = A_2 \cap A_3 = \emptyset$. Since D_1 has no directed C_4, $A_1 \cap A_2 = \emptyset$.

Since $d^+_{D_1}(u) = d^-_{D_1}(u) = n_1$ for each $u \in A$ and $d^+_{D_1}(v) = d^-_{D_1}(v) = m_1$ for each $v \in B$, we obtain $|A_1| \geq m_1$ by its definition, and $n_1|A_2| \geq |A_{D_1}(A_2, B_2)| = \sum_{v \in B_2} d^-_{D_1}(v) = m_1 n_1$, follows that $|A_2| \geq m_1$. Since $|A_1| + |A_2| + |A_3| = 3m_1$ and A_1, A_2, A_3 are pairwise disjoint, $|A_1| = |A_2| = |A_3| = m_1$. Now apparently, $A_{D_1}(A_2, B_2) = \{uv : u \in A_2, v \in B_2\}$ and $A_{D_1}(B_3, A_1) = \{vu : v \in B_3, u \in A_1\}$. Since A_1 and A_2 are disjoint, $A_{D_1}(A_1, B_2) = \emptyset$. Furthermore, $A_{D_1}(A_1, B_3) = \emptyset$, hence $N^+(A_1) \subset B_1$. $\sum_{v \in A_1} d^+(v) = m_1 n_1$ implies $A_{D_1}(A_1, B_1) = \{uv : u \in A_1, v \in B_1\}$. Similarly, $A_{D_1}(B_1, A_2) = \{vu : v \in B_1, u \in A_2\}$. Therefore $D_1 = D^*(m, n)$. If there is any arc in D but not in D_1, then obviously, there would be a directed C_4 in D, a contradiction. Thus, $D = D_1 = D^*(m, n)$.

The proof is complete. □

2.2 Proof of Theorem 10

First note that the color degree condition implies that $|A| \geq 4$ and $|B| \geq 4$.

Suppose not. Let G^c be a colored graph which satisfies the condition of Theorem 10 but has no rainbow C_4. Set $|A| = n_1$ and $|B| = n_2$.

Choose an edge $e = xy \in E(G^c)$ such that $C(xy) = c_0$. Let $N^c(x) = \{y, y_1, y_2, \ldots, y_{r-1}\}$ and $N^c(y) = \{x, x_1, x_2, \ldots, x_{s-1}\}$. Since $d^c(x) \geq \frac{3n_2+8}{5}$ and $d^c(y) \geq \frac{3n_1+8}{5}$, we can set $r = \lceil \frac{3n_2+8}{5} \rceil$ and $s = \lceil \frac{3n_1+8}{5} \rceil$. Let $A_1 = \{x_1, x_2, \ldots, x_{s-1}\}$ and $B_1 = \{y_1, y_2, \ldots, y_{r-1}\}$. Note that $G^c[A_1, B_1]$ is also a bipartite colored graph.

The following claim can be deduced immediately from the definition of color neighborhood set and the assumption that G^c has no rainbow C_4.

Claim 1 For any edge x_iy_j and $C(xy_j) \neq C(yx_i)$, where $1 \leq i \leq s-1$ and $1 \leq j \leq r-1$, we have $C(x_iy_j) \in \{C(xy), C(xy_j), C(yx_i)\}$.

Now we construct an oriented bipartite graph $D = D[A_1, B_1]$ as follows. For any edge $x_iy_j \in E(G[A_1, B_1])$, such that $C(x_iy_j) \neq C(xy)$ and $C(xy_j) \neq C(yx_i)$, then $C(x_iy_j) = C(xy_j)$ or $C(x_iy_j) = C(yx_i)$ by Claim 1. If $C(x_iy_j) = C(xy_j)$, we define an arc x_iy_j in D, and if $C(x_iy_j) = C(yx_i)$, we define an arc y_jx_i in D. Let $G' = G[D]$ be the underlying graph of D.

In the following, for convenience, when we mention the color of an arc in D, we mean the color of the corresponding edge in $E(G')$.

Claim 2 There is no directed C_4 in D.

Proof Suppose $Q = x_iy_jx_py_qx_i$ is a directed C_4 in D. By the definition of D, we have $C(x_iy_j) = C(xy_j)$, $C(y_jx_p) = C(yx_p)$, $C(x_py_q) = C(xy_q)$, and $C(y_qx_i) = C(yx_i)$. The existence of arcs x_iy_j and y_jx_p implies $C(xy_j) \neq C(yx_i)$ and $C(xy_j) \neq C(yx_p)$, hence $C(x_iy_j) \neq C(y_qx_i)$ and $C(x_iy_j) \neq C(y_jx_p)$. We have $C(xy_j) \neq C(xy_q)$ from the definition of B_1, hence $C(x_iy_j) \neq C(x_py_q)$. So $C(x_iy_j)$ is different from the colors of all other three edges in the cycle Q in G^c. Similarly, we can prove that all edges in Q receive distinct colors in G^c, and therefore, Q is a rainbow C_4 in G^c, a contradiction. □

Claim 3 $D \neq D^*(|A_1|, |B_1|)$.

Proof Assume that $D = D^*(|A_1|, |B_1|)$. Without loss of generality, set $A_1 = X_1 \cup X_2 \cup X_3$ and $B_1 = Y_1 \cup Y_2 \cup Y_3$, where $|X_1| = |X_2| = |X_3|$ and $|Y_1| = |Y_2| = |Y_3|$. Since $s-1 \equiv 0 \pmod 3$ and $r-1 \equiv 0 \pmod 3$, we have $\lceil \frac{3n_i+3}{5} \rceil \equiv 0 \pmod 3$, $i = 1, 2$. It follows that $n_i \equiv 3$ or $4 \pmod 5$, $i = 1, 2$.

First, we claim that $D \neq D^*(3, 3)$. Suppose not. Then $\lceil \frac{3n_1+3}{5} \rceil = \lceil \frac{3n_2+3}{5} \rceil = 3$. Hence $n_1 = n_2 = 4$. In this case, we may suppose that $X_i = \{x_i\}$ and $Y_i = \{y_i\}$ for $i = 1, 2, 3$. The existence of arc x_1y_1 in D implies that $C(x_1y_1) = C(xy_1)$. Hence $d^c(y_1) \leq 3 < \frac{3|A|+8}{5}$, a contradiction.

Let v_0 be an arbitrary vertex of D. Without loss of generality, assume $v_0 \in Y_1$. If $n_1 = 5k + \lambda$, $\lambda = 3$ or 4, then $|X_1| = |X_2| = |X_3| = k + 1$. From the definition of D, we know that each edge in $\{uv_0 : u \in X_1\}$ has the same color $C(v_0x)$, and there are $k + 1$ different colors in $\{v_0u : u \in X_2\}$. Since $|A\backslash(A_1 \cup \{x\})| = 5k + \lambda - (3k + 3) - 1 = 2k + \lambda - 4$, there are at most $2k + \lambda - 4$ different colors in the edge set $\{v_0u : u \in A\backslash(A_1 \cup \{x\})\}$. These facts mean that there are at most $3k + \lambda - 2$ different colors in the edge set $\{v_0u : u \in A\backslash X_3\}$. Since $d^c(v) \geq 3k + \lambda$, there exists an edge $v_0u_0 \in \{v_0u : u \in X_3\}$, such that v_0u_0 has a new color and v_0u_0 is not in $E(G')$.

Next we show that any directed path of length 3 in $D = D^*(|A_1|, |B_1|)$ is rainbow. Without loss of generality, we choose a directed path $uv'u'v$, where $u' \in X_1$, $v \in Y_1$, $u \in X_3$, and $v' \in Y_3$. By the construction of D, $C(u'v) = C(xv)$, $C(v'u') = C(yu')$, and $C(uv') = C(xv')$. Since $u'v$ exists in D, $C(xv) \neq C(yu')$. It follows that $C(u'v) \neq C(v'u')$. Similarly, we have $C(v'u') \neq C(uv')$. Since $C(u'v) = C(xv)$ and $C(uv') = C(xv')$ and $C(xv) \neq C(xv')$ by the choice of $N^c(x)$, we have $C(u'v) \neq C(uv')$.

Now we fix a vertex $v_0 \in Y_1$. By the analysis above, there exists an edge $v_0u_0 \in \{v_0u \in E(G) : u \in X_3\}$, which is not in $E(G')$ and satisfies $C(v_0u_0) \notin \{C(uv_0) : u \in X_1\}$. Since $D = D^*(|A_1|, |B_1|) \neq D^*(3, 3)$, there are at least two arcs in $A_D(Y_3, X_1)$ with distinct colors, and we can choose one of them, say $v_0'u_0' \in A_D(Y_3, X_1)$, such that $C(v_0'u_0') \neq C(v_0u_0)$. By the analysis before, there is an edge $u_0v_0'' \in E(G^c)$, where $v_0'' \in Y_1$, such that it is not in $E(G')$ and satisfies $C(u_0v_0'') \neq C(u_0v_0')$. Now we will show that $v_0'' = v_0$. First, the deletion of u_0v_0 means $C(u_0v_0) = C(xy)$ or $C(xv_0) = C(yu_0)$. If $C(u_0v_0) = C(xy)$, then the existence of arcs $u_0'v_0$, $u_0'v_0'$, and u_0v_0' in D implies $C(xy) \neq C(u_0'v_0)$, $C(xy) \neq C(u_0'v_0')$, and $C(xy) \neq C(u_0v_0')$. Since colors of arcs in $\{u_0'v_0, v_0'u_0', u_0v_0'\}$ are pairwise distinct, $u_0'v_0'u_0v_0u_0'$ is a rainbow C_4 in G^c, a contradiction. Hence $C(xv_0) = C(yu_0)$. Similarly, we may obtain $C(xv_0'') = C(yu_0)$ from the deletion of u_0v_0'' when constructing D. It follows that $C(xv_0) = C(xv_0'')$. Now we can get $v_0 = v_0''$ directly from the definition of $N^c(x)$. From the analysis above, colors of arcs in $\{u_0'v_0, v_0'u_0', u_0v_0'\}$ are pairwise distinct, and $C(v_0u_0)$ is different from all of the three. Therefore, $u_0'v_0u_0v_0'u_0'$ is also a rainbow C_4 in G^c, a contradiction. □

By Claims 2 and 3, D has no directed C_4 and $D \neq D^*(|A_1|, |B_1|)$. By Theorem 9, there exists a vertex, without loss of generality, say $y_j \in B_1$, such that $d_D^+(y_j) < \frac{|A_1|}{3}$. By the construction of D, we know there are less than $\frac{|A_1|}{3} + 1$ different colors in $C(E_{G'}(y_j, A_1))$. For any edge e adjacent to y_j which is in $E(G[A_1, B_1])\backslash E(G')$, $C(e) = C(xy)$ or there exists an edge yx_i such that $C(yx_i) = C(xy_j)$ in G^c, and in this case, $e = x_iy_j$ and $C(e)$ may be missed in $C(E_{G'}(y_j, A_1))$. This implies that there are at most three colors in $C(xy_j) \cup C(E_G(y_j, A_1))\backslash C(E_{G'}(y_j, A_1))$. However, $C(xy_j)$ is also in $C(E_{G'}(y_j, A_1))$. Hence there are less than $\frac{|A_1|}{3} + 3$ different colors in $C(E_G(y_j, \{x\} \cup A_1))$. It follows that there are more than $d^c(y_j) - 3 - \frac{|A_1|}{3} \geq s - 3 - \frac{s-1}{3}$ different colors in the color set $\{y_jx' : x' \in A\backslash(A_1 \cup \{x\})\}$. Hence y_j has more than $s - 3 - \frac{s-1}{3}$ different neighbors in $A\backslash(A_1 \cup \{x\})$. Now we have

$$\begin{aligned} n_1 = |A| &\geq |A_1| + |\{x\}| + |N(y_j)\backslash(A_1 \cup \{x\})| \\ &> (s-1) + 1 + s - 3 - \frac{s-1}{3} = \frac{5s-8}{3} \geq n_1, \end{aligned}$$

a contradiction.

The proof is complete. □

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