Bull. Malays. Math. Sci. Soc. (2015) 38:1689–1703
DOI 10.1007/s40840-014-0105-x

BULLETIN of the
MALAYSIAN MATHEMATICAL
SCIENCES SOCIETY
http://math.usm.my/bulletin

Gauss and Ricci Equations in Contact Manifolds with a Quarter-Symmetric Metric Connection

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Received: 27 June 2013 / Revised: 11 September 2014 / Published online: 17 December 2014

Abstract In the present paper, we study the extrinsic and intrinsic geometry of submanifolds of an almost contact metric manifold admitting a quarter-symmetric metric connection. We deduce Gauss, Codazzi and Ricci equations corresponding to the quarter-symmetric metric connection and show some applications of these equations. Finally, we give an example verifying the results.

Keywords Gauss Codazzi equation · Ricci equation · Contact manifold · Einstein manifold · Invariant submanifold · Totally umbilical · Quarter-symmetric metric connection

Mathematics Subject Classification 53C25 · 53C35 · 53D10

1 Introduction

The importance of the Gauss, Codazzi and Ricci equations in differential geometry is that if the ambient space has a constant sectional curvature, they play an analogous role to that of the compatibility equation in the local theory of surfaces. For a submanifold M of a Riemannian manifold $\bar{M}$, if the Riemannian curvature tensors are denoted by R and $\bar{R}$, respectively, then the usual Gauss, Codazzi and Ricci equations are given by the following:

Communicated by Young Jin Suh.

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$$g\big(\bar{R}(X,Y)Z,W\big) = g\big(R(X,Y)Z,W\big) - g\big(h(X,W),h(Y,Z)\big) + g\big(h(Y,W),h(X,Z)\big), \tag{1.1}$$

$$\big(\bar{R}(X,Y,Z)\big)^{\perp} = (\nabla_X h)(Y,Z) - (\nabla_Y h)(X,Z), \tag{1.2}$$

$$g\big(\bar{R}(X,Y)U,V\big) = g\big(R^{\perp}(X,Y)U,V\big) + g\big([A_V,A_U]X,Y\big), \tag{1.3}$$

for X, Y, Z, W tangent to M and U, V normal to M, where h is the second fundamental form, A is the associated shape operator of the immersion, and $R^{\perp}$ is the curvature tensor of the normal bundle. For an isometric immersion $i : M \to \bar{M}$ of Riemannian manifolds, the Gauss equation shows that the curvature tensor of $\bar{M}$, when evaluated on vector fields tangent to M, differs from the curvature tensor of M by a tensor involving only the second fundamental form of the immersion. Gauss–Codazzi–Ricci equations are very important instruments for describing a submanifold in a Riemannian space. By nature, these equations appear in the Cauchy problem of general relativity [20].

On the other hand, in 1975, Golab [13] introduced the notion of quarter-symmetric connection on a differentiable manifold. A linear connection ∇ is said to be quarter-symmetric if its torsion tensor T defined by $T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y]$, is of the form:

$$T(X,Y) = u(Y)\psi X - u(X)\psi Y, \tag{1.4}$$

where u is a $1-$form and ψ is a $(1,1)-$tensor field. When T vanishes, the connection ∇ is called symmetric; otherwise, it is called non-symmetric. ∇ is called a metric connection if there is a Riemannian metric g such that $\nabla g = 0$; otherwise, it is called non-metric. It is well known that a linear connection is both symmetric and metric if and only if it is the Riemannian (or, the Levi-Civita) connection. If in (1.4), ψ is an identity function, then it reduces to semi-symmetric metric connection. Hence, quarter-symmetric connection is a generalization of semi-symmetric connection. In the present paper, we deduce the Gauss, Codazzi and Ricci equations for submanifolds of an almost contact metric manifold admitting a quarter-symmetric metric connection.

In [8], De and Mondal proved the existence and uniqueness of quarter-symmetric metric connection in Riemannian manifolds. Many authors studied other geometric properties of almost Hermitian and almost contact manifolds with quarter-symmetric and semi-symmetric connections ([1,7,14,16,17,19]). Ozgur [16] proved several results including the equations of Gauss Codazzi and Ricci for submanifolds of a Riemannian manifold admitting a particular type of semi-symmetric non-metric connection. Later on, hypersurfaces and submanifolds of different ambient manifolds admitting quarter-symmetric metric connection have been studied by several authors ([9,11,15]). In this paper, we generalize all the results obtained in these previous studies, by considering submanifolds of any codimension of an almost contact metric manifold admitting a quarter-symmetric metric connection.

The present paper has been organized as follows:
After preliminaries, in sect. 3, we consider submanifolds of an almost contact metric manifold endowed with a quarter-symmetric metric connection, and we show that the connection induced on the submanifold is also a quarter-symmetric metric connection. Furthermore, we prove that the mean curvature with respect to both the connections coincides, and applying this result, we obtain a necessary condition of a submanifold to

be invariant. In sect. 4, we deduce the Gauss, Codazzi and Ricci equations corresponding to the quarter-symmetric metric connection and obtain some results applying these equations. Finally, in sect. 5, we provide an example verifying the obtained results.

2 Preliminaries

Let $\bar{M}$ be an $(n+p)-$dimensional (where $n+p$ is odd) differentiable manifold endowed with an almost contact metric structure (ϕ, ξ, η, g), where ϕ, ξ, η are tensor fields on $\bar{M}$ of types $(1,1)$, $(1,0)$, $(0,1)$, respectively, and g is a compatible metric with the almost contact structure, such that [2,3,5,21],

$$\phi^2 = -I + \eta \otimes \xi,\ \phi\xi = 0,\ \eta(\xi) = 1,\ \eta \circ \phi = 0, \quad (2.1)$$

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \quad (2.2)$$

$$g(\phi X, Y) + g(X, \phi Y) = 0, \quad (2.3)$$

for all vector fields $X, Y \in T\bar{M}$, where $T\bar{M}$ is the Lie algebra of vector fields of the manifold $\bar{M}$. The fundamental $2-$form Φ is defined by $\Phi(X,Y) = g(X, \phi Y)$. If $[\phi, \phi] + d\eta \otimes \xi = 0$, where $[\phi, \phi](X, Y) = \phi^2[X, Y] + [\phi X, \phi Y] - \phi[\phi X, Y] - \phi[X, \phi Y]$, then the almost contact structure is said to be *normal* [10]. If $\Phi = d\eta$, the almost contact structure becomes a contact structure. A normal contact metric manifold is called *Sasakian*. On a Sasakian manifold, we have the following [2,12]:

$$\left(\bar{\nabla}_X \phi\right) Y = g(X, Y)\xi - \eta(Y)X, \quad (2.4)$$

$$\bar{R}(X, Y)\xi = \eta(Y)X - \eta(X)Y. \quad (2.5)$$

An almost contact metric structure (ϕ, ξ, η, g) is cosymplectic if and only if ϕ is parallel. In a *cosymplectic* manifold, we have $\left(\bar{\nabla}_X \eta\right) Y = 0$.

A Riemannian manifold of dimension >2 is said to be *Einstein* if its Ricci tensor S satisfies $S(X, Y) = \mu g(X, Y)$, where μ is a constant [6].

Let M be a submanifold of an almost contact metric manifold $\bar{M}$ with a positive definite metric g. Let the induced metric on M also be denoted by g. The usual Gauss and Weingarten formulae are given, respectively, by [4,18]

$$\bar{\nabla}_X Y = \nabla_X Y + h(X, Y), \quad X, Y \in TM \quad (2.6)$$

$$\bar{\nabla}_X N = -A_N X + \nabla^{\perp}_X N, \quad N \in T^{\perp}M \quad (2.7)$$

where ∇ is the induced Riemannian connection on M, h is the second fundamental form of the immersion, A is the shape operator, and $\nabla^{\perp}$ is the normal connection on $T^{\perp}M$, the normal bundle of M. From (2.6) and (2.7), one gets

$$g(h(X, Y), N) = g(A_N X, Y). \quad (2.8)$$

The submanifold M of an almost contact manifold $\bar{M}$ is called *invariant* (resp. *anti-invariant*) if for each point $x \in M$, $\phi T_x M \subset T_x M$ (resp. $\phi T_x M \subset T^{\perp}_x M$. The submanifold is called *totally umbilical* if $h(X,Y) = g(X,Y)H$, for all $X, Y \in TM$, where H is the mean curvature vector of the submanifold, defined by $H = \frac{1}{n}\sum\{h(e_i, e_i)\}$, $\{e_i\}$, $i = 1, 2, ..., n$ being an orthonomal basis of TM and n the dimension of M. The submanifold is called *totally geodesic* if $h(X,Y) = 0$ for all $X, Y \in TM$. Let the codimension of M be p, and let $\{N_\alpha\}$, $\alpha = 1, 2, ..., p$ be an orthonormal basis of $T^\perp M$.

3 Basic Results

On a submanifold M of an almost contact metric manifold $\bar{M}$ with the quarter-symmetric metric connection ∇^*, we obtain the following results:

Theorem 3.1 *The connection induced on a submanifold of an almost contact metric manifold with a quarter-symmetric metric connection is also a quarter-symmetric metric connection.*

Proof We define the quarter-symmetric metric connection ∇^* on $\bar{M}$ by

$$\nabla^*_X Y = \bar{\nabla}_X Y - \eta(X)\phi Y. \tag{3.1}$$

If ∇' is the induced connection on M from the connection ∇^*, then we have

$$\nabla^*_X Y = \nabla'_X Y + m(X,Y), \tag{3.2}$$

where m is a tensor field of type $(1, 2)$ in $T^\perp M$, the normal part of M. We term $m(X,Y)$ the second fundamental form with respect to the quarter-symmetric connection.

For $X \in TM$ and $N \in T^\perp M$, we put

$$\phi X = PX + QX, \quad PX \in TM, \quad QX \in T^\perp M, \tag{3.3}$$

$$\phi N = tN + qN, \quad tN \in TM, \quad qN \in T^\perp M. \tag{3.4}$$

Using (3.3), from (3.1) and (3.2), we have

$$\nabla'_X Y + m(X,Y) = \nabla_X Y + h(X,Y) - \eta(X)PY - \eta(X)QY. \tag{3.5}$$

Now equating tangential and normal parts, we have

$$\nabla'_X Y = \nabla_X Y - \eta(X)PY, \tag{3.6}$$

and

$$m(X,Y) = h(X,Y) - \eta(X)QY. \tag{3.7}$$

From (3.6), the torsion tensor with respect to ∇' is given by

$$T'(X,Y) = \eta(Y)PX - \eta(X)PY.$$

Also using (3.2), we have

$$\big(\nabla'_X g\big)(Y, Z) = \big(\nabla^*_X g\big)(Y, Z). \tag{3.8}$$

Hence, the result. □

From (3.6), it follows that if the submanifold is anti-invariant, that is, $PX = 0$, then we have the following:

Theorem 3.2 *On an anti-invariant submanifold of an almost contact metric manifold with a quarter-symmetric metric connection, the induced quarter-symmetric connection and the induced Riemannian connection are equivalent.*

So we concentrate mostly on invariant submanifolds. Equation (3.2) is the Gauss formula for the quarter-symmetric metric connection. Also, from (3.1), we have

$$\begin{aligned}\nabla^*_X N &= \bar{\nabla}_X N - \eta(X)\phi N\\ &= -A_N X - \eta(X)tN + \nabla^{\perp}_X N - \eta(X)qN\\ &= D_X N - A'_N X,\end{aligned} \tag{3.9}$$

where $D_X N = \nabla^{\perp}_X N - \eta(X)qN$ is the normal connection, and $A'_N X = A_N X + \eta(X)tN$ is the shape operator corresponding to the quarter-symmetric metric connection. By simple calculations, we obtain

$$g(m(X, Y), N) = g(A'_N X, Y). \tag{3.10}$$

Equation (3.9) is the Weingarten formula with respect to the quarter-symmmetric metric connection.

Remark 3.1 Unlike the second fundamental form corresponding to the Levi-Civita connection, m is neither symmetric nor skew-symmetric, in general, which is evident from (3.7). Thus, the shape operator A'corresponding to the quarter-symmetric connection is also not symmetric. However, for invariant submanifolds both of them are symmetric.

We define the covariant derivative of m and η with respect to the quarter-symmetric metric connection as follows:

$$\big(\nabla^*_X m\big)(Y, Z) = D_X\big(m(Y, Z)\big) - m\big(\nabla'_X Y, Z\big) - m\big(Y, \nabla'_X Z\big), \tag{3.11}$$

$$\big(\nabla^*_X \eta\big)Y = X\big(\eta(Y)\big) - \eta\big(\nabla'_X Y\big). \tag{3.12}$$

Equation (3.11) may be called the van der Waerden–Bortolotti connection corresponding to the quarter-symmetric metric connection.

Now, we prove the following:

Theorem 3.3 *The mean curvature of the submanifold M with respect to the Riemannian connection coincides with that of M with respect to the quarter-symmetric metric connection.*

Proof Let $\{e_1, e_2, ..., e_n\}$ be an orthonormal basis of TM. We consider two cases:
Case I: $\xi \in TM$ and let $e_n = \xi$. Then from (3.7), we obtain

$$m(e_i, e_i) = h(e_i, e_i) - \eta(e_i)Q(e_i). \tag{3.13}$$

Since $\eta(e_i) = 0$, for $i = 1, 2, ..., n-1$, and $\phi(e_n) = 0$, summing up for $i = 1, 2, ..., n$ and dividing by n, we obtain the required result.
Case II: $\xi \notin TM$, then again from (3.7), we obtain

$$m(e_i, e_i) = h(e_i, e_i) - \eta(e_i)Q(e_i) \tag{3.14}$$

for each $i = 1, 2, ..., n$. From (3.14), we obtain $m(e_i, e_i) = h(e_i, e_i)$, since $\eta(e_i) = 0$, for all the $i = 1, 2, ..., n$. Summing up for $i = 1, 2, ..., n$ and dividing by n, we obtain the required result. □

The following corrolaries are the direct consequence of the above theorem.

Corollary 3.1 *Any submanifold of an almost contact manifold endowed with a quarter-symmetric metric connection is minimal with respect to the quarter-symmetric metric connection if and only if it is minimal with respect to the Riemannian connection.*

Corollary 3.2 *If a submanifold M of an almost contact manifold endowed with a quarter-symmetric metric connection is tangent to ξ and is totally umbilical with respect to both the connections, then M is invariant. Conversely, if M is invariant, then M is totally umbilical with respect to quarter-symmetric connection if and only if M is totally umbilical with respect to the Riemannian connection.*

Proof From (3.7), for all $X, Y \in TM$, we have,

$$\eta(X)QY = m(X, Y) - h(X, Y). \tag{3.15}$$

If M is totally umbilical with respect to both quarter-symmetric connection and Riemannian connection, then from Theorem 3.3, we have,

$$m(X, Y) = g(X, Y)H = h(X, Y).$$

Thus, from (3.15), we get for all $X, Y \in TM$,

$$\eta(X)QY = 0, \tag{3.16}$$

for any $X, Y \in TM$. Putting $X = \xi$ in (3.16), we obtain, $QY = 0$, for all $Y \in TM$, which implies that M is an invariant submanifold.

The converse part follows directly from (3.7). □

Remark 3.2 In this connection, we should note that, if any submanifold of a contact metric manifold is normal to ξ, then by the well-known result of Yano and Kon [22], the submanifold is always anti-invariant.

Theorem 3.4 *The covariant derivative of the fundamental* $2-$*form* Φ *with respect to the quarter-symmetric connection is equal to the covariant derivative of* Φ *with respect to the Riemannian connection.*

Proof We have $\Phi(X, Y) = g(X, \phi Y)$.

Therefore,

$$\begin{aligned}\big(\nabla^*_X \Phi\big)(Y, Z) &= X\Phi(Y, Z) - \Phi\big(\nabla^*_X Y, Z\big) - \Phi\big(Y, \nabla^*_X Z\big) \\ &= X\Phi(Y, Z) - \Phi\big(\bar{\nabla}_X Y, Z\big) + \eta(X)\Phi(Y, \phi Z) \\ &\quad - \Phi\big(Y, \bar{\nabla}_X Z\big) + \eta(X)\Phi(Y, \phi Z) \\ &= \big(\bar{\nabla}_X \Phi\big)(Y, Z), \text{ since, } \Phi(Y, \phi Z) = -\Phi(Y, \phi Z). \end{aligned} \tag{3.17}$$

Hence, the result. □

4 The Gauss, Codazzi-Mainardi and Ricci Equations

In this section, we find the relations between the curvature tensors corresponding to the Levi-civita connection and the quarter symmetric metric connection. We denote the Riemannian curvature tensors corresponding to the Levi-Civita connection and the quarter-symmetric connection by $\bar{R}$ and R^*, respectively, and that corresponding to the induced connections ∇ and $\nabla^{'}$ by R and $R^{'}$, respectively.

We have,

$$\begin{aligned}\nabla^*_X \nabla^*_Y Z &= \bar{\nabla}_X \bar{\nabla}_Y Z - \eta(X)\phi\big(\bar{\nabla}_Y Z\big) - \eta(Y)\bar{\nabla}_X \phi Z \\ &\quad + \eta(X)\eta(Y)\phi^2 Z - X(\eta(Y))\phi Z, \end{aligned} \tag{4.1}$$

$$\begin{aligned}\nabla^*_Y \nabla^*_X Z &= \bar{\nabla}_Y \bar{\nabla}_X Z - \eta(Y)\phi\big(\bar{\nabla}_X Z) - \eta(X)\bar{\nabla}_Y \phi Z \\ &\quad + \eta(X)\eta(Y)\phi^2 Z - Y\big(\eta(X)\big)\phi Z, \end{aligned} \tag{4.2}$$

and

$$\nabla^*_{[X,Y]} Z = \bar{\nabla}_{[X,Y]} Z - \eta\big([X, Y]\big)\phi Z. \tag{4.3}$$

Therefore, we have

$$\begin{aligned}R^*(X, Y)Z &= \bar{R}(X, Y)Z + \eta(X)\big(\bar{\nabla}_Y \phi\big)Z - \eta(Y)\big(\bar{\nabla}_X \phi\big)Z \\ &\quad - \big[(\bar{\nabla}_X \eta)Y - \big(\bar{\nabla}_Y \eta\big)X\big]\phi Z. \end{aligned} \tag{4.4}$$

Hence, we derive

$$\begin{aligned}R^*(X, Y, Z, W) &= g(R^*(X, Y)Z, W) \\ &= g\big(\bar{R}(X, Y)Z, W\big) + \eta(X)g\big((\bar{\nabla}_Y \phi)Z, W\big) - \eta(Y)g\big((\bar{\nabla}_X \phi)Z, W\big)\end{aligned}$$

$$
\begin{aligned}
&-\left[\left(\bar{\nabla}_X\eta\right)Y-\left(\bar{\nabla}_Y\eta\right)X\right]g(\phi Z,W)\\
&=\bar{R}(X,Y,Z,W)+\eta(X)g\big((\bar{\nabla}_Y\phi)Z,W\big)-\eta(Y)g\big((\bar{\nabla}_X\phi)Z,W\big)\\
&\quad-\left[\left(\bar{\nabla}_X\eta\right)Y-\left(\bar{\nabla}_Y\eta\right)X\right]g(\phi Z,W). \qquad (4.5)
\end{aligned}
$$

From (4.4) and (4.5), we can conclude the following:

Remark 4.1 (i) $R^*(X,Y,Z,W)\neq R^*(Z,W,X,Y)$

(ii) $R^*(X,Y,Z,W)\neq -R^*(X,Y,W,Z)$

(iii) The first Bianchi identity with respect to the quarter-symmetric connection is given by

$$R^*(X,Y)Z+R^*(Y,Z)X+R^*(Z,X)Y=k(X,Y)Z+k(Y,Z)X+k(Z,X)Y,$$

where $k(X,Y)Z$ is a (1,3)-tensor defined by
$k(X,Y)Z=\eta(Z)\left[\left(\bar{\nabla}_X\phi\right)Y-\left(\bar{\nabla}_Y\phi\right)X\right]-\left[\left(\bar{\nabla}_X\eta\right)Y-\left(\bar{\nabla}_Y\eta\right)X\right]\phi Z.$

Now, putting $Z=\xi$ in (4.4), we get

$$R^*(X,Y)\xi=\bar{R}(X,Y)\xi+\eta(X)\left(\bar{\nabla}_Y\phi\right)\xi-\eta(Y)\left(\bar{\nabla}_X\phi\right)\xi \qquad (4.6)$$

From (4.6), we obtain the following:

Remark 4.2 If the ambient manifold $\bar{M}$ is a Sasakian manifold, then we have

$$
\begin{aligned}
R^*(X,Y)\xi&=\bar{R}(X,Y)\xi+\eta(X)\left[\eta(Y)\xi-Y\right]-\eta(Y)\left[\eta(X)\xi-X\right]\\
&=2\bar{R}(X,Y)\xi.
\end{aligned}
$$

Remark 4.3 If the ambient manifold $\bar{M}$ is cosymplectic, then

$$R^*(X,Y)\xi=\bar{R}(X,Y)\xi.$$

Again, we have

$$
\begin{aligned}
\nabla^*_X\nabla^*_Y Z&=\nabla'_X\nabla'_Y Z+m\left(X,\nabla'_Y Z\right)\\
&\quad+D_X\left(m(Y,Z)\right)-A'_{m(Y,Z)}X. \qquad (4.7)\\
\nabla^*_Y\nabla^*_X Z&=\nabla'_Y\nabla'_X Z+m(Y,\nabla'_X Z)\\
&\quad+D_Y\left(m(X,Z)\right)-A_{m(X,Z)}Y. \qquad (4.8)\\
\nabla^*_{[X,Y]}Z&=\nabla'_{[X,Y]}Z+m\left([X,Y],Z\right). \qquad (4.9)
\end{aligned}
$$

By direct computations, we obtain

$$
\begin{aligned}
R^*(X,Y)Z&=R'(X,Y)Z+\left(\nabla'_X m\right)(Y,Z)-\left(\nabla'_Y m\right)(X,Z)\\
&\quad+A'_{m(X,Z)}Y-A'_{m(Y,Z)}X. \qquad (4.10)
\end{aligned}
$$

Hence, the Gauss equation for the quarter symmetric metric connection ∇^* is given by

$$\begin{aligned} R^*(X,Y,Z,W) &= g\big(R^*(X,Y)Z,W\big) \\ &= g\big(R'(X,Y)Z,W\big) + g\big(m(Y,W),m(X,Z)\big) \\ &\quad - g\big(m(Y,Z),m(X,W)\big) \\ &= R'(X,Y,Z,W) + g\big(m(Y,W),m(X,Z)\big) \\ &\quad - g\big(m(Y,Z),m(X,W)\big). \end{aligned} \tag{4.11}$$

Putting $X = W = e_i$, $Y = Z = e_j$, in (4.11), we obtain

$$\begin{aligned} R^*(e_i,e_j,e_j,e_i) = R'(e_i,e_j,e_j,e_i) &+ g\big(m(e_i,e_j),m(e_j,e_i)\big) \\ &- g\big(m(e_j,e_j),m(e_i,e_i)\big). \end{aligned} \tag{4.12}$$

Summing over i and j and using Theorem 3.3, we get

$$\tau^* = \tau' + ||m||^2 - n^2\|H\|^2, \tag{4.13}$$

where τ^* and τ' are the scalar curvatures corresponding to the quarter symmetric metric connection defined on $\bar{M}$ and the induced quarter-symmetric metric connection on M, respectively, and $||m||^2$ denotes the squared norm of the second fundamental form with respect to the quarter-symmetric connection. From (4.13), we can also write

$$\|H\|^2 \geq \frac{1}{n^2}\big(\tau' - \tau^*\big). \tag{4.14}$$

Hence, the following:

Theorem 4.1 *On a minimal submanifold of an almost contact metric manifold admitting a quarter-symmetric metric connection, the scalar curvature corresponding to the quarter-symmetric connection is never less than that of the induced quarter-symmetric connecton.*

Since $\{N_\alpha\}$, $\alpha = 1,2,...,p$ is a basis of $T^\perp M$, we can express $m(X,Y) = \sum_{\alpha=1}^{p} m_\alpha(X,Y)N_\alpha$, where each m_α is a (0, 2) tensor.

Hence, the Gauss equation (4.11) can be rewritten in the following form:

$$\begin{aligned} R^*(X,Y,Z,W) = R'(X,Y,Z,W) + \sum_{\alpha=1}^{p}\big[&m_\alpha(Y,W)m_\alpha(X,Z) \\ &- m_\alpha(Y,Z)m_\alpha(X,W)\big]. \end{aligned} \tag{4.15}$$

From (3.2) and (3.9), we can easily deduce that

$$m_\alpha(X,Y) = g\big(A'_{N_\alpha}X,Y\big). \tag{4.16}$$

Hence, by (4.15), the Gauss equation can also be represented in terms of the shape operator as

$$R^*(X, Y, Z, W) = R'(X, Y, Z, W) + \sum_{\alpha=1}^{p}\{g(A'_{N_\alpha}Y, W)g(A'_{N_\alpha}X, Z) - g(A'_{N_\alpha}Y, Z)g(A'_{N_\alpha}X, W)\}. \tag{4.17}$$

From (4.17), we get

$$R^*(X, Y, X, Y) = R'(X, Y, X, Y) + \sum_{\alpha=1}^{p}\{g(A'_{N_\alpha}Y, Y)g(A'_{N_\alpha}X, X) - g(A'_{N_\alpha}Y, X)g(A'_{N_\alpha}X, Y)\}. \tag{4.18}$$

Combining with Remark 3.1, we can state the following:

Theorem 4.2 *Let $\mathcal{P}$ be a $2-$dimensional invariant subspace of $T_x M$ and let $K^*(\mathcal{P})$ and $K'(\mathcal{P})$ be the sectional curvature of $\mathcal{P}$ in $\bar{M}$ and M, respectively, with respect to the quarter-symmetric metric connection. If $X$and Y form an orthonormal basis of $\mathcal{P}$, then*

$$K^*(\mathcal{P}) = K'(\mathcal{P}) + \sum_{\alpha=1}^{p}\{g(A'_{N_\alpha}Y, Y)g(A'_{N_\alpha}X, X) - g(A'_{N_\alpha}X, Y)^2\}.$$

Now, contracting equation (4.15) we have the expression of Ricci tensor corresponding to the quarter-symmetric connection as

$$\begin{aligned} S^*(Y, Z) &= S'(Y, Z) + \sum_{\alpha=1}^{p} R^*(N_\alpha, Y, Z, N_\alpha) \\ &\quad + \sum_{\alpha=1}^{p}\left[\sum_{i=1}^{n} g(A'_{N_\alpha}Y, e_i)g(A'_{N_\alpha}Z, e_i) - f_\alpha m_\alpha(Y, Z)\right] \text{(if } A \text{ is symmetric)} \\ &= S'(Y, Z) + \sum_{\alpha=1}^{p} R^*(N_\alpha, Y, Z, N_\alpha) \\ &\quad + \sum_{\alpha=1}^{p}\left[g(A'_{N_\alpha}A'_{N_\alpha}Y, Z) - f_\alpha m_\alpha(Y, Z)\right] \text{ (if } A \text{ is symmetric)} \\ &= S'(Y, Z) + \sum_{\alpha=1}^{p} R^*(N_\alpha, Y, Z, N_\alpha) \\ &\quad + \sum_{\alpha=1}^{p}\left[m_\alpha(A'_{N_\alpha}Y, Z) - f_\alpha m_\alpha(Y, Z)\right]. \end{aligned} \tag{4.19}$$

where f_α denote the trace of A'_{N_α}.

Suppose that the quarter-symmetric metric connection ∇^* is of constant sectional curvature. Then

$$R^*(X, Y, Z, W) = \lambda(g(Y, Z)g(X, W) - g(X, Z)g(Y, W)). \tag{4.20}$$

Therefore, from equation (4.19), we have

$$\begin{aligned} S'(Y, Z) = \sum_{\alpha=1}^{p} \left\{ f_\alpha m_\alpha(Y, Z) - m_\alpha(A'_{N_\alpha} Y, Z) \right\} \\ + \sum_{\alpha=1}^{p} R^*(N_\alpha, Y, Z, N_\alpha). \end{aligned}$$

Therefore,

$$S'(Y, Z) = \lambda(n-1)g(Y, Z) + \sum_{\alpha=1}^{p} \left\{ f_\alpha m_\alpha(Y, Z) - m_\alpha(A'_{N_\alpha} Y, Z) \right\}. \tag{4.21}$$

Thus, we can state the following:

Theorem 4.3 *Let M be an invariant submanifold of an almost contact metric manifold $\bar{M}$ of constant sectional curvature with a quarter-symmetric metric connection. Then*

(i) the Ricci tensor of M induced from the quarter-symmetric connection is symmetric.
(ii) The Ricci tensor of M induced from the quarter-symmetric connection is not parallel.

From Eq. (4.21), we can also conclude the following:

Theorem 4.4 *Let M be a totally umbilical submanifold of an almost contact metric manifold $\bar{M}$ of constant sectional curvature with a quarter-symmetric metric connection. Then the submanifold M is an Einstein manifold with respect to the quarter-symmetric metric connection.*

From (4.10), we obtain the normal part of $R^*(X, Y)Z$ as

$$R^{\perp}(X, Y)Z = (\nabla'_X m)(Y, Z) - (\nabla'_Y m)(X, Z), \tag{4.22}$$

which is the Codazzi equation corresponding to the quarter-symmetric metric connection ∇^*.

Let $\zeta_1, \zeta_2 \in T^{\perp}M$, then we have

$$\begin{aligned} R^*(X, Y)\zeta_1 = R'(X, Y)\zeta_1 - A'_{D_Y \zeta_1} X + A'_{D_X \zeta_1} Y + \nabla'_Y (A'_{\zeta_1} X) \\ + \nabla'_X (A'_{\zeta_1} Y) + m(Y, A'_{\zeta_1} X) - m(X, A'_{\zeta_1} Y) - A'_{\zeta_1}\big([X, Y]\big). \end{aligned} \tag{4.23}$$

So, we have

$$\begin{aligned} R^*(X, Y, \zeta_1, \zeta_2) &= g(R^*(X, Y)\zeta_1, \zeta_2) \\ &= g(R'(X, Y)\zeta_1, \zeta_2) + g(A'_{\zeta_1}X, A'_{\zeta_2}Y) - g(A'_{\zeta_1}Y, A'_{\zeta_2}X) \\ &= R'(X, Y, \zeta_1, \zeta_2) + g(A'_{\zeta_1}X, A'_{\zeta_2}Y) - g(A'_{\zeta_1}Y, A'_{\zeta_2}X), \quad (4.24) \end{aligned}$$

which is the Ricci equation corresponding to the quarter-symmetric metric connection.

Remark 4.4 If we consider the submanifold M to be invariant, then from Lemma 3.1, we have the shape operator to be symmetric. Thus, we can express the Ricci equation in the following form:

$$R^*(X, Y, \zeta_1, \zeta_2) = R'(X, Y, \zeta_1, \zeta_2) + g\big([A'_{\zeta_1}, A'_{\zeta_2}]X, Y\big). \quad (4.25)$$

5 Example

Example 5.1 Let us consider the 5-dimensional manifold $\bar{M} = \{(x, y, z, u, v) \in \mathbb{R}^5, (x, y, z, u, v) \neq (0, 0, 0, 0, 0)\}$, where (x, y, z, u, v) are the standard coordinates in $\mathbb{R}^5$. The vector fields

$$e_1 = 2\left(-\frac{\partial}{\partial x} + y\frac{\partial}{\partial z}\right), \; e_2 = 2\frac{\partial}{\partial y}, \; e_3 = 2\frac{\partial}{\partial z}, \; e_4 = 2\left(-\frac{\partial}{\partial u} + v\frac{\partial}{\partial z}\right), \; e_5 = 2\frac{\partial}{\partial v} \quad (5.1)$$

are linearly independent at each point of $\bar{M}$. Let g be the metirc defined by

$$g(e_i, e_j) = \begin{cases} 1, & \text{when } i = j \\ 0, & \text{when } i \neq j. \end{cases}$$

Let η be the 1-form defined by $\eta(Z) = g(Z, e_3)$, for all $Z \in T\bar{M}$, and let ϕ be the $(1, 1)$ tensor field defined by

$$\phi e_1 = e_2, \;\; \phi e_2 = -e_1, \;\; \phi e_3 = 0, \;\; \phi e_4 = e_5, \;\; \phi e_5 = -e_4. \quad (5.2)$$

Then, using the linearity of ϕ and g, we have

$$\eta(e_3) = 1, \;\; \phi^2 Z = -Z + \eta(Z)e_3$$

and

$$g(\phi Z, \phi W) = g(Z, W) - \eta(Z)\eta(W), \;\; \forall \; Z, W \in T\bar{M}.$$

Thus, for $e_3 = \xi$, $(\bar{M}, \phi, \xi, \eta, g)$ be an almost contact metric manifold. Let $\bar{\nabla}$ be the Levi-Civita connection with respect to the metric g. Then, we have

$$[e_1, e_2] = -2e_3 = [e_4, e_5] \;\text{ and }\; [e_i, e_j] = 0, \;\; \text{for all other } i, j.$$

Taking $e_3 = \xi$ and using Koszul's formula for the metric g, it can be easily calculated that

$$\begin{aligned}&\bar{\nabla}_{e_1} e_2 = -e_3,\ \ \bar{\nabla}_{e_1} e_3 = e_2,\ \ \bar{\nabla}_{e_2} e_1 = -e_3,\ \ \bar{\nabla}_{e_2} e_3 = -e_1,\ \ \bar{\nabla}_{e_3} e_1 = e_2,\\ &\bar{\nabla}_{e_3} e_2 = -e_1,\ \ \bar{\nabla}_{e_3} e_4 = e_5,\ \ \bar{\nabla}_{e_3} e_5 = -e_4,\ \ \bar{\nabla}_{e_4} e_3 = e_5,\ \ \bar{\nabla}_{e_5} e_3 = -e_4,\end{aligned} \tag{5.3}$$

and the rest of the terms are 0.

Since $\{e_1, e_2, e_3, e_4, e_5\}$ is a frame filed, then any vector field $X, Y \in T\bar{M}$ can be wrriten as

$$X = a_1 e_1 + b_1 e_2 + c_1 e_3 + d_1 e_4 + f_1 e_5,\ \ Y = a_2 e_1 + b_2 e_2 + c_c e_3 + d_2 e_4 + f_2 e_5$$

where $a_i, b_i, c_i, d_i, f_i \in \mathbb{R},\ \ i = 1, 2$ such that

$$a_1 a_2 + b_1 b_2 + c_1 c_2 + d_1 d_2 + f_1 f_2 \neq 0.$$

Hence, we derive

$$g(X, Y) = a_1 a_2 + b_1 b_2 + c_1 c_2 + d_1 d_2 + f_1 f_2. \tag{5.4}$$

Now, using (5.3), we get

$$\begin{aligned}\bar{\nabla}_X Y \ &= -(b_1 c_2 + b_2 c_1) e_1 + (a_1 c_2 + a_2 c_1) e_2 - (a_2 b_1 a_1 b_2) e_3\\ &\quad -(c_1 f_2 + c_2 f_1) e_4 + (c_1 d_2 + c_2 d_1) e_5.\end{aligned}$$

Thus, from (3.1), we obtain

$$\nabla^*_X Y = a_1 c_2 e_2 - b_1 c_2 e_1 - (a_2 b_1 + a_1 b_2) e_3 - c_2 f_1 e_4 + c_2 d_1 e_5. \tag{5.5}$$

Also from (5.4), it follows that $\nabla^* g = 0$. Thus, in an almost contact metric manifold, the quarter-symmetric metric connection is given by (5.5).

Now, let M be a subset of $\bar{M}$ and consider an isometric immersion $f : M \longrightarrow \bar{M}$ by

$$f(x, y, z) = (x, y, z, 0, 0).$$

It can be easily seen that M is a 3-dimensional submanifold of the 5 -dimensional almost contact metric manifold $\bar{M}$. Now, $M = \{(x, y, z) \in \mathbb{R}^3,\ (x, y, z) \neq 0\}$, where (x, y, z) are the standard coordinates in $\mathbb{R}^3$. The vector fields

$$e_1 = 2\left(-\frac{\partial}{\partial x} + y\frac{\partial}{\partial z}\right),\ \ e_2 = 2\frac{\partial}{\partial y},\ \ e_3 = 2\frac{\partial}{\partial z}$$

are linearly independent at each point of M. Let us denote the induced metric by the same symbol g such that

$$g(e_1, e_2) = g(e_1, e_3) = g(e_2, e_3) = 0$$

and

$$g(e_1, e_1) = g(e_2, e_2) = g(e_3, e_3) = 1.$$

Let ∇ be the Levi-Civita connection with respect to the metric g on M. Then, we have

$$[e_1, e_2] = -2e_3 \text{ and } [e_1, e_3] = 0 = [e_2, e_3].$$

Taking $e_3 = \xi$ and using Koszul's formula for the induced metric g, it can be easily calculated that

$$\nabla_{e_1} e_2 = -e_3, \quad \nabla_{e_1} e_3 = e_2, \quad \nabla_{e_1} e_1 = 0, \quad \nabla_{e_2} e_3 = -e_1, \quad \nabla_{e_2} e_2 = 0,$$

$$\nabla_{e_2} e_1 = -e_3, \quad \nabla_{e_3} e_3 = 0, \quad \nabla_{e_3} e_2 = -e_1, \quad \nabla_{e_3} e_1 = e_2.$$

Clearly, $\{e_4, e_5\}$ is the frame for the normal bundle $T^{\perp}M$. If we take $X, Y \in TM$, then we can express them as

$$X = a_1 e_1 = b_1 e_2 + c_1 e_3, \quad Y = a_2 e_1 = b_2 e_2 + c_2 e_3$$

and therefore

$$\nabla_X Y = -(b_1 c_2 + b_2 c_1) e_1 + (a_1 c_2 + a_2 c_1) e_2 - (a_1 b_2 + a_2 b_1) e_3 \tag{5.6}$$

which is the tangential part of $\bar{\nabla}_X Y$. The second fundamental form is given by

$$h(X, Y) = -(c_1 f_2 + c_2 f_1) e_4 + (c_1 d_2 + c_2 d_1) e_5. \tag{5.7}$$

Now, the tangential part of $\nabla^*_X Y$ is given by

$$\nabla'_X Y = -b_1 c_2 e_1 + a_1 c_2 e_2 - (a_1 b_2 + a_2 b_1) e_3 = \nabla_X Y - \eta(X) P Y. \tag{5.8}$$

And, the normal part of $\nabla^*_X Y$ will be

$$m(X, Y) = -c_2 f_1 e_4 + c_2 d_1 e_5 = h(X, Y) - \eta(X) Q Y. \tag{5.9}$$

It is easy to check that $\nabla'_X g = \nabla^*_X g$, for any $X \in TM$.

We can easily check that the submanifold M is minimal with respect to both the connections, the Levi-Civita as well as the quarter-symmetric connection.

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