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# Random Coincidence Points of Expansive Type Completely Random Operators

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**Abstract** In this paper, we present some results on the existence of random coincidence points of expansive type completely random operators. Some applications to random fixed point theorems and random equations are given.

**Keywords** Random operators · Completely random operators · Random fixed points · Random coincidence points

**Mathematics Subject Classification** 60B11 · 60G57 · 60K37 · 37L55 · 47H10

## 1 Introduction

Let $(\Omega, \mathcal{F}, P)$ be a probability space, $X, Y$ be separable metric spaces and $f : \Omega \times X \to Y$ be a random operator in the sense that for each fixed $x$ in $X$, the mapping $f(., x) : \omega \mapsto f(\omega, x)$ is measurable. The random operator $f$ is said to be continuous if for each $\omega$ in $\Omega$, the mapping $f(\omega, .) : x \mapsto f(\omega, x)$ is continuous. An $X$-valued random variable $\xi$ is said to be a random fixed point of the random operator $f : \Omega \times X \to X$ if $f(\omega, \xi(\omega)) = \xi(\omega)$ a.s. and an $X$-valued random variable $\xi$ is said to be a random coincidence point of the random operators $f, g : \Omega \times X \to X$ if $f(\omega, \xi(\omega)) = g(\omega, \xi(\omega))$ a.s.

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The theory of random fixed points and random coincidence points is an important topic of the stochastic analysis and has been investigated by various authors (see, e.g. [2–5,14–18]).

In this paper, we are concerned with mapping $\Phi : L_0^X(\Omega) \to L_0^Y(\Omega)$. Since a random operator $f$ can be viewed as an action which transforms each deterministic input $x$ in $X$ into a random output $f(x)$ in $L_0^Y(\omega)$ while $\Phi : L_0^X(\Omega) \to L_0^Y(\Omega)$ can be viewed as an action which transforms each random input $u$ in $L_0^X(\Omega)$ into a random output $\Phi u$, we call $\Phi$ a completely random operator. In the Sect. 2, we present some properties of completely random operators. Section 3 deals with the notion of random coincidence points of completely random operators and gives some conditions ensuring the existence of a random coincidence point of expansive type completely random operators. It should be noted that the existence of a random coincidence point of completely random operators does not follow from the existence of corresponding deterministic coincidence point theorem as in the case of the random operator. In the Sect. 4, some applications to random fixed point theorems and random equations are presented.

## 2 Some Properties of Completely Random Operators

Let $(\Omega, \mathcal{F}, P)$ be a complete probability space and $X$ be a separable Banach space. A mapping $\xi : \Omega \to X$ is called an $X$-valued random variable if $\xi$ is $(\mathcal{F}, \mathcal{B}(X))$-measurable, where $\mathcal{B}(X)$ denotes the Borel $\sigma$-algebra of $X$. The set of all (equivalent classes) $X$-valued random variables is denoted by $L_0^X(\Omega)$ and it is equipped with the topology of convergence in probability. For each $p > 0$, the set of $X$-valued random variables $\xi$ such that $E\|\xi\|^p < \infty$ is denoted by $L_p^X(\Omega)$.

At first, recall that (see, e.g. [22])

**Definition 2.1** Let $X, Y$ be two separable Banach spaces.

(1) A mapping $f : \Omega \times X \to Y$ is said to be a random operator if for each fixed $x$ in $X$, the mapping $\omega \mapsto f(\omega, x)$ is measurable.
(2) The random operator $f : \Omega \times X \to Y$ is said to be continuous if for each $\omega$ in $\Omega$ the mapping $x \mapsto f(\omega, x)$ is continuous.
(3) Let $f, g : \Omega \times X \to Y$ be two random operators. The random operator $g$ is said to be a modification of $f$ if for each $x$ in $X$, we have $f(\omega, x) = g(\omega, x)$ a.s. Noting that the exceptional set can depend on $x$.

The following is the notion of the completely random operator.

**Definition 2.2** Let $X, Y$ be two separable Banach spaces.

(1) A mapping $\Phi : L_0^X(\Omega) \to L_0^Y(\Omega)$ is called a completely random operator.
(2) The completely random operator $\Phi$ is said to be continuous if for each sequence $(u_n)$ in $L_0^X(\Omega)$ such that $\lim u_n = u$ a.s., we have $\lim \Phi u_n = \Phi u$ a.s.
(3) The completely random operator $\Phi$ is said to be continuous in probability if for each sequence $(u_n)$ in $L_0^X(\Omega)$ such that $\lim u_n = u$ in probability, we have $\lim \Phi u_n = \Phi u$ in probability.

(4) The completely random operator $\Phi$ is said to be an extension of a random operator $f : \Omega \times X \to Y$ if for each $x$ in $X$

$$\Phi x(\omega) = f(\omega, x) \text{ a.s.}$$

where for each $x$ in $X$, $x$ denotes the random variable $u$ in $L_0^X(\Omega)$ given by $u(\omega) = x$ a.s.

For later using, we list some following results.

**Theorem 2.3** [24, Theorem 2.3] *Let $f : \Omega \times X \to Y$ be a random operator admitting a continuous modification. Then, there exists a continuous completely random operator $\Phi : L_0^X(\Omega) \to L_0^Y(\Omega)$ such that $\Phi$ is an extension of $f$.*

**Proposition 2.4** [24, Proposition 2.4] *Let $\Phi : L_0^X(\Omega) \to L_0^Y(\Omega)$ be a completely random operator. Then, the continuity of $\Phi$ implies the continuity in probability of $\Phi$.*

## 3 Random Coincidence Points of Completely Random Operators

Let $f, g : \Omega \times X \to X$ be random operators. Recall that (see, e.g. [1,3,18]), an $X$-valued random variable $\xi$ is said to be a random fixed point of the random operator $f$ if

$$f(\omega, \xi(\omega)) = \xi(\omega) \quad \text{a.s.}$$

An $X$-valued random variable $u^*$ is said to be a random coincidence point of two random operators $f, g$ if

$$f(\omega, u^*(\omega)) = g(\omega, u^*(\omega)) \quad \text{a.s.}$$

Assume that $f, g$ are continuous. Then, by Theorem 2.3 the mappings $\Phi, \Psi : L_0^X(\Omega) \to L_0^X(\Omega)$ defined respectively by

$$\begin{aligned} \Phi u(\omega) &= f(\omega, u(\omega)) \\ \Psi u(\omega) &= g(\omega, u(\omega)) \end{aligned}$$

are completely random operators extending $f$ and $g$, respectively. For each random fixed point $\xi$ of $f$, we get

$$\Phi\xi(\omega) = \xi(\omega) \quad \text{a.s.}$$

and for each random coincidence point $u^*$ of two random operators $f, g$, we have

$$\Phi u^*(\omega) = \Psi u^*(\omega) \quad \text{a.s.}$$

This leads us to the following definition.

**Definition 3.1** (1) Let $\Phi : L_0^X(\Omega) \to L_0^X(\Omega)$ be a completely random operator. An $X$-valued random variable $\xi$ in $L_0^X(\Omega)$ is called a random fixed point of $\Phi$ if

$$\Phi\xi = \xi.$$

(2) Let $\Phi_1, \Phi_2, ..., \Phi_n : L_0^X(\Omega) \to L_0^X(\Omega)$ be completely random operators. An $X$-valued random variable $u^*$ in $L_0^X(\Omega)$ is called a random coincidence point of $\Phi_1, \Phi_2, ..., \Phi_n$ if

$$\Phi_1 u^* = \Phi_2 u^* = ... = \Phi_n u^*. \tag{3.1}$$

In this section, we present some conditions ensuring the existence of a random coincidence point of completely random operators.

**Theorem 3.2** *Let $\Phi, \Psi, \Theta : L_0^X(\Omega) \to L_0^X(\Omega)$ be continuous in probability completely random operators, $\Phi, \Psi$ be surjective and $f : [0, \infty) \to [0, \infty)$ be a mapping such that for each $t > 0$,*

$$h(t) = \inf_{s \geq t} \frac{f(s)}{s} > 0. \tag{3.2}$$

*Assume that for any random variables $u, v$ in $L_0^X(\Omega)$ and $t > 0$, we have*

$$P(\|\Phi u - \Psi v\| > t) \geq P(\|\Theta u - \Theta v\| + f(\|\Theta u - \Theta v\|) > t). \tag{3.3}$$

*Then, $\Phi, \Theta$ have a random coincidence point and $\Psi, \Theta$ have a random coincidence point if there exist random variables $u_0, v_0$ in $L_0^X(\Omega)$ and $p > 0$ such that $\Phi v_0 = \Theta u_0$ and*

$$M = E\|\Theta v_0 - \Theta u_0\|^p < \infty. \tag{3.4}$$

*Proof* Suppose that $E\|\Theta v_0 - \Theta u_0\|^p < \infty$ for random variables $u_0, v_0$ in $L_0^X(\Omega)$ such that $\Phi v_0 = \Theta u_0$ and $p > 0$. Because $\Phi, \Psi$ are surjective, there exists a random variable $u_1$ in $L_0^X(\Omega)$ such that $\Phi u_1 = \Theta u_0, u_1 = v_0$. Again, there exists a random variable $u_2$ in $L_0^X(\Omega)$ such that $\Psi u_2 = \Theta u_1$. By induction, there exists a sequence $(u_n)$ in $L_0^X(\Omega)$ such that

$$\Phi u_1 = \Theta u_0, \Psi u_2 = \Theta u_1, ..., \Phi u_{2n+1} = \Theta u_{2n}, \Psi u_{2n+2} = \Theta u_{2n+1} \quad n = 1, 2, ... \tag{3.5}$$

We will show that $(\xi_n)$ given by $\xi_n = \Theta u_{n-1}$ $(n = 1, 2, ...)$ in (3.5) is a Cauchy sequence in $L_0^X(\Omega)$. Define the function $g(t), t > 0$ by

$$g(t) = 1 + \frac{f(t)}{t}.$$

So, we have

$$f(t) = (g(t) - 1)\, t.$$

Since $f(t) > 0 \quad \forall t > 0$, we get $g(t) > 1 \quad \forall t > 0$. For any random variables $u$, $v$ in $L_0^X(\Omega)$, we have

$$P(\|\Phi u - \Psi v\| > t) \geq P(\|\Theta u - \Theta v\| + f(\|\Theta u - \Theta v\|) > t).$$

Equivalently,

$$P(\|\Phi u - \Psi v\| > t) \geq P(g(\|\Theta u - \Theta v\|)\, \|\Theta u - \Theta v\| > t). \tag{3.6}$$

Fixed $t > 0$. For each $s \geq t > 0$, we have

$$g(s) = 1 + \frac{f(s)}{s} \geq 1 + h(t) = q(t).$$

Since $g(t) > 1$, we get

$$\{g(\|\Theta u - \Theta v\|)\|\Theta u - \Theta v\| > t\} \supset \{\|\Theta u - \Theta v\| > t\}.$$

Hence,

$$\begin{aligned}
P(\|\Phi u - \Psi v\| > q(t)t) &\geq P(g(\Theta\|u - \Theta v\|)\|\Theta u - \Theta v\| > q(t)t)\\
&\geq P(g(\|\Theta u-\Theta v\|)\|\Theta u-\Theta v\| > q(t)t, \|\Theta u-\Theta v\| > t)\\
&\geq P(q(t)\|\Theta u - \Theta v\| > q(t)t, \|\Theta u - \Theta v\| > t)\\
&= P(\|\Theta u - \Theta v\| > t).
\end{aligned}$$

Put $q = q(t)$, noting that $q > 1$ since $h(t) > 0$.

From this, for each $n$, we obtain

$$\begin{aligned}
P(\|\xi_{2n+1} - \xi_{2n}\| > qt) &= P(\|\Phi u_{2n+1} - \Psi u_{2n}\| > qt)\\
&\geq P(\|\Theta u_{2n+1} - \Theta u_{2n}\| > t)\\
&= P(\|\xi_{2n+2} - \xi_{2n+1}\| > t),
\end{aligned}$$

and

$$\begin{aligned}
P(\|\xi_{2n} - \xi_{2n-1}\| > qt) &= P(\|\Psi u_{2n} - \Phi u_{2n-1}\| > qt)\\
&\geq P(\|\Theta u_{2n} - \Theta u_{2n-1}\| > t)\\
&= P(\|\xi_{2n+1} - \xi_{2n}\| > t).
\end{aligned}$$

By induction and Chebyshev inequality, we get

$$\begin{aligned}
P(\|\xi_{n+1} - \xi_n\| > t) &\leq P(\|\xi_n - \xi_{n-1}\| > qt)\\
&\leq \dots\\
&\leq P(\|\xi_2 - \xi_1\| > q^{n-1}t)\\
&= P(\|\Theta u_1 - \Theta u_0\| > q^{n-1}t)\\
&= P(\|\Theta v_0 - \Theta u_0\| > q^{n-1}t)\\
&\leq E\|\Theta v_0 - \Theta u_0\|^p \tfrac{1}{(q^{n-1})^p t^p} = M \tfrac{1}{(q^{n-1})^p t^p}.
\end{aligned}$$

Let $r$ be a number in $(1, q)$. Then, $r > 1$ and $(r-1)(\frac{1}{r} + \frac{1}{r^2} + \ldots + \frac{1}{r^m}) + \frac{1}{r^m} = 1 \quad \forall m \geq 1$. Thus, for any $t > 0$, $n \geq 2$ and $m$ in $N$, we have

$$
\begin{aligned}
P(\|\xi_{n+m} - \xi_n\| > t) &\leq P\left(\|\xi_{n+m} - \xi_n\| > \left(1 - \frac{1}{r^m}\right) t\right) \\
&\leq P\big(\|\xi_{n+m} - \xi_{n+m-1}\| > t(r-1)/_{r^m}\big) \\
&\quad + \ldots + P(\|\xi_{n+1} - \xi_n\| > t(r-1)/_{r}) \\
&\leq \frac{M}{[(r-1)t]^p}\Big[\frac{(r^m)^p}{(q^{n+m-2})^p} + \ldots + \frac{r^p}{(q^{n-1})^p}\Big] \\
&= \frac{M}{[(r-1)t]^p}\frac{r^p}{(q^{n-1})^p}\Big[\left(\frac{r}{q}\right)^{p(m-1)} + \ldots + \left(\frac{r}{q}\right)^p + 1\Big] \\
&= \frac{M}{[(r-1)t]^p}\frac{r^p}{(q^{n-1})^p}\frac{1 - \left(\frac{r}{q}\right)^{(m-1)p}}{1 - \left(\frac{r}{q}\right)^p} \\
&\quad < \frac{Mr^p}{[(r-1)t]^p\left[1 - \left(\frac{r}{q}\right)^p\right]}\frac{1}{(q^p)^{n-1}} \quad n \geq 2,
\end{aligned}
$$

which tends to 0 as $n \to \infty$. It implies that $(\xi_n)$ is a Cauchy sequence in $L_0^X(\Omega)$. Hence, there exists $\xi$ in $L_0^X(\Omega)$ such that p-lim $\xi_n = \xi$. Because $\Phi$ is surjective, there exists $u^*$ in $L_0^X(\Omega)$ such that $\Phi u^* = \xi$. So, we have

$$
\begin{aligned}
P\left(\|\xi - \xi_{2n}\| > qt\right) &= P\left(\|\Phi u^* - \xi_{2n}\| > qt\right) \\
&= P\left(\|\Phi u^* - \Psi u_{2n}\| > qt\right) \\
&\geq P\left(\|\Theta u_{2n} - \Theta u^*\| + f\left(\|\Theta u_{2n} - \Theta u^*\|\right) > qt\right) \\
&\geq P\left(\|\Theta u_{2n} - \Theta u^*\| > t\right) \\
&= P\left(\|\xi_{2n+1} - \Theta u^*\| > t\right).
\end{aligned}
$$

Let $n \to \infty$, we receive $P\left(\|\xi - \Theta u^*\| > t\right) = 0$ implying $\Theta u^* = \xi \;\, a.s.$ Then, $\Phi, \Theta$ have a random coincidence point $u^*$.

By the same argument, $\Psi, \Theta$ have a random coincidence point $v^*$. □

**Corollary 3.3** *Let $\Phi, \Theta : L_0^X(\Omega) \to L_0^X(\Omega)$ be continuous in probability completely random operators, $\Phi$ be surjective and $f : [0, \infty) \to [0, \infty)$ be a mapping such that for each $t > 0$,*

$$h(t) = \inf_{s \geq t} \frac{f(s)}{s} > 0. \tag{3.7}$$

*Assume that for each pair $u, v$ in $L_0^X(\Omega)$ and $t > 0$, we have*

$$P\left(\|\Phi u - \Phi v\| > t\right) \geq P\left(\|\Theta u - \Theta v\| + f\left(\|\Theta u - \Theta v\|\right) > t\right). \tag{3.8}$$

*Then* $\Phi, \Theta$ *have a random coincidence point if and only if there exist random variables* $u_0, v_0$ *in* $L_0^X(\Omega)$ *and* $p > 0$ *such that* $\Phi v_0 = \Theta u_0$

$$M = E\|\Theta v_0 - \Theta u_0\|^p < \infty. \tag{3.9}$$

*Proof* Put $\Psi v = \Phi v$, then all the conditions in the Theorem 3.2 are satisfied. □

**Corollary 3.4** *Let* $\Phi, \Theta$ *be completely random operators satisfying the conditions stated in the Corollary 3.3. Assume that there exists a number* $q > 1$ *such that*

$$P\left(\|\Phi u - \Phi v\| > t\right) \geq P\left(\|\Theta u - \Theta v\| > t/q\right) \tag{3.10}$$

*for all random variables* $u, v$ *in* $L_0^X(\Omega)$ *and* $t > 0$. *Then* $\Phi, \Theta$ *have a random coincidence point if and only if there exist random variables* $u_0, v_0$ *in* $L_0^X(\Omega)$ *and* $p > 0$ *such that* $\Phi v_0 = \Theta u_0$ *and* (3.9) *holds.*

*Proof* Consider the function $f(t) = (q-1)t$ and $h(t) = q - 1 > 0$. Then $f(t)$ satisfies the conditions stated in the Corollary 3.3. □

**Remark** The following simple example shows that the random coincidence point of $\Phi$ and $\Theta$ in the Corollary 3.3 need not be unique.

*Example 3.5* Define two completely random operators $\Phi, \Theta : L_0^R(\Omega) \to L_0^R(\Omega)$ by

$$\Phi u = q|u| + \eta, \Theta u = |u|$$

where $\eta$ is a positive random variable, $q > 1$.

It is easy to check that $\Phi, \Theta$ satisfy all assumptions of Corollary 3.3 with $f(t) = (q-1)t$. On the other hand, $\Phi$ and $\Theta$ have two random coincidence points $u_1^* = \frac{1}{q-1}\eta, u_2^* = -\frac{1}{q-1}\eta$.

**Theorem 3.6** *Let* $\Phi, \Psi, \Theta : L_0^X(\Omega) \to L_0^X(\Omega)$ *be continuous in probability completely random operators,* $\Phi, \Psi$ *be surjective and* $f : [0, \infty) \to [0, \infty)$ *be a continuous, increasing function such that* $f(0) = 0$, $\lim_{t\to\infty} f(t) = \infty$ *and* $q > 1$. *Assume that for any random variables* $u, v$ *in* $L_0^X(\Omega)$ *and* $t > 0$, *we have*

$$P\left(\|\Phi u - \Psi v\| > f(t)\right) \geq P\left(\|\Theta u - \Theta v\| > f(t/q)\right). \tag{3.11}$$

*If there exist random variables* $u_0, v_0$ *in* $L_0^X(\Omega)$ *and* $p > 0$ *such that* $\Phi v_0 = \Theta u_0$ *and*

$$M = \sup_{t>0} t^p P\left(\|\Theta v_0 - \Theta u_0\| > f(t)\right) < \infty. \tag{3.12}$$

*Then,*

(1) *Assume that there exists a number* $c > 1/q$ *such that*

$$\sum_{n=1}^{\infty} f(c^n) < \infty. \tag{3.13}$$

*Then, the condition (3.12) is sufficient for* $\Phi, \Theta$ *have a random coincidence point and* $\Psi, \Theta$ *have a random coincidence point.*

(2) *Assume that for each* $t, s > 0$

$$f(t+s) \geq f(t) + f(s). \tag{3.14}$$

*Then, the condition (3.12) is also sufficient for* $\Phi, \Theta$ *have a random coincidence point and* $\Psi, \Theta$ *have a random coincidence point.*

*Proof* Let $g = f^{-1}$ be the inverse function of $f$. Then, $g : [0, \infty) \to [0, \infty)$ is increasing with $g(0) = 0$, $\lim_{t\to\infty} g(t) = \infty$. The condition (3.11) is equivalent to the following

$$P\left(g\left(\|\Phi u - \Psi v\|\right) > t\right) \geq P\left(g(\|\Theta u - \Theta v\|) > t/q\right). \tag{3.15}$$

Let $u_0$ be a random variable in $L_0^X(\Omega)$ such that (3.12) holds. Because $\Phi, \Psi$ are surjective, there exists a random variable $u_1$ in $L_0^X(\Omega)$ such that $\Phi u_1 = \Theta u_0$, $u_1 = v_0$. Again, there exists a random variable $u_2$ in $L_0^X(\Omega)$ such that $\Psi u_2 = \Theta u_1$. By induction, there exists a sequence $(u_n)$ in $L_0^X(\Omega)$ by

$$\Phi u_1 = \Theta u_0, \Psi u_2 = \Theta u_1, ..., \Phi u_{2n+1} = \Theta u_{2n}, \Psi u_{2n+2} = \Theta u_{2n+1} \quad n = 1, 2, ... \tag{3.16}$$

Put $\xi_n = \Theta u_{n-1}, \quad n = 1, 2, ....$ From (3.15), for each $n$, we obtain

$$\begin{aligned} P(g(\|\xi_{2n+1} - \xi_{2n}\|) > qt) &= P(g(\|\Phi u_{2n+1} - \Psi u_{2n}\|) > qt) \\ &\geq P(g(\|\Theta u_{2n+1} - \Theta u_{2n}\|) > t) \\ &= P(g(\|\xi_{2n+2} - \xi_{2n+1}\|) > t), \end{aligned}$$

and

$$\begin{aligned} P(g(\|\xi_{2n} - \xi_{2n-1}\|) > qt) &= P(g(\|\Psi u_{2n} - \Phi u_{2n-1}\|) > qt) \\ &\geq P(g(\|\Theta u_{2n} - \Theta u_{2n-1}\|) > t) \\ &= P(g(\|\xi_{2n+1} - \xi_{2n}\|) > t). \end{aligned}$$

By induction, we obtain for each $n$

$$\begin{aligned} P\left(g\left(\|\xi_{n+1} - \xi_n\|\right) > t\right) &\leq P\left(g(\|\xi_2 - \xi_1\|) > q^{n-1}t\right) \\ &= P\left(g(\|\Theta u_1 - \Theta u_0\|) > q^{n-1}t\right) \\ &= P\left(g(\|\Theta v_0 - \Theta u_0\|) > q^{n-1}t\right). \end{aligned}$$

Then,

$$P\left(g\left(\|\xi_{n+1} - \xi_n\|\right) > t\right) \leq P\left(g(\|\Theta v_0 - \Theta u_0\|) > q^{n-1}t\right). \tag{3.17}$$

(1) From (3.12), we have

$$P\left(g(\|\Phi u_0 - \Theta u_0\|) > s\right) = P\left(\|\Phi u_0 - \Theta u_0\| > f(s)\right) \leq \frac{M}{s^p}. \tag{3.18}$$

From (3.17) and (3.18), we get

$$P\left(g\left(\|\xi_{n+1} - \xi_n\|\right) > t\right) \leq \frac{M}{q^{(n-1)p} t^p}. \tag{3.19}$$

Taking $t = c^n$, from (3.19), we get

$$P\left(g\left(\|\xi_{n+1} - \xi_n\|\right) > c^n\right) \leq M \frac{1}{q^{(n-1)p} c^{np}}, \tag{3.20}$$

i.e.

$$P\left(\|\xi_{n+1} - \xi_n\| > f(c^n)\right) \leq M \frac{1}{q^{(n-1)p} c^{np}}. \tag{3.21}$$

Since

$$\sum_{n=1}^{\infty} P\left(\|\xi_{n+1} - \xi_n\| > f(c^n)\right) \leq M \sum_{n=1}^{\infty} \frac{1}{q^{(n-1)p} c^{np}} < \infty,$$

by the Borel-Cantelli Lemma, there is a set $D$ with probability one such that for each $\omega$ in $D$ there is $N(\omega)$

$$\|\xi_{n+1}(\omega) - \xi_n(\omega)\| \leq f(c^n) \quad \forall n > N(\omega).$$

By (3.13), we conclude that $\sum_{n=1}^{\infty} \|\xi_{n+1}(\omega) - \xi_n(\omega)\| < \infty$ for all $\omega$ in $D$, which implies that there exists $\lim \xi_n(\omega)$ for all $\omega$ in $D$. Consequently, the sequence $(\xi_n)$ converges a.s. to $\xi$ in $L_0^X(\Omega)$.
Because $\Phi$ is surjective, there exists $u^*$ in $L_0^X(\Omega)$ such that $\Phi u^* = \xi$. So, we have

$$\begin{aligned} P\left(\|\xi - \xi_{2n}\| > f(qt)\right) &= P\left(\|\xi_{2n} - \Phi u^*\| > f(qt)\right) \\ &= P\left(\|\Psi u_{2n} - \Phi u^*\| > f(qt)\right) \\ &\geq P\left(\|\Theta u_{2n} - \Theta u^*\| > f(t)\right) \\ &\geq P\left(\|\xi_{2n+1} - \Theta u^*\| > f(t)\right). \end{aligned}$$

Let $n \to \infty$, we receive $P\left(\|\xi - \Theta u^*\| > f(t)\right) = 0$ for all $t > 0$ implying $\Theta u^* = \xi$ a.s. Then, $\Phi$, $\Theta$ have a random coincidence point $u^*$.
By the same argument, $\Psi$, $\Theta$ have a random coincidence point $v^*$.

(2) It is easy to see that for each $t, s > 0$

$$g(s + t) \leq g(t) + g(s).$$

Hence, for $a \geq \sum_{i=1}^{m} s_i$, we have

$$\begin{aligned} P\left(g(\|\xi_{n+m}-\xi_n\|)>a\right) &\leq P\left(g\left(\sum_{i=1}^{m}\|\xi_{n+i}-\xi_{n+i-1}\|\right)>a\right) \\ &\leq P\left(\sum_{i=1}^{m} g(\|\xi_{n+i}-\xi_{n+i-1}\|)>\sum_{i=1}^{m} s_i\right) \\ &\leq \sum_{i=1}^{m} P\left(g(\|\xi_{n+i}-\xi_{n+i-1}\|)>s_i\right). \end{aligned}$$

From (3.12), we have

$$P\left(g\left(\|\xi_{n+i}-\xi_{n+i-1}\|\right)>s_i\right) \leq \frac{Mq^{(n+i-1)p}}{s_i^p}. \tag{3.22}$$

Put $r$ be a number in $(1, q)$ and $s_i = s(r-1)/r^i$. An argument similar to that in the forward proof yields

$$\lim_{n\to\infty} P(g(\|\xi_{n+m}-\xi_n\|)>s)=0 \quad \forall s>0,$$

so

$$\lim_{n\to\infty} P(\|\xi_{n+m}-\xi_n\|>f(s))=0 \quad \forall s>0.$$

Thus, we obtain

$$\lim_{n\to\infty} P(\|\xi_{n+m}-\xi_n\|>t)=0 \quad \forall t>0.$$

Consequently, the sequence $(\xi_n)$ converges in probability to $\xi$ in $L_0^X(\Omega)$. Because $\Phi$ is surjective, there exists $u^*$ in $L_0^X(\Omega)$ such that $\Phi u^* = \xi$. So, we have

$$\begin{aligned} P\left(\|\xi-\xi_{2n}\|>f(qt)\right) &= P\left(\|\xi_{2n}-\Phi u^*\|>f(qt)\right) \\ &= P\left(\|\Psi u_{2n}-\Phi u^*\|>f(qt)\right) \\ &\geq P\left(\|\Theta u_{2n}-\Theta u^*\|>f(t)\right) \\ &\geq P\left(\|\xi_{2n+1}-\Theta u^*\|>f(t)\right). \end{aligned}$$

Let $n \to \infty$, we receive $P\left(\|\xi-\Theta u^*\|>f(t)\right)=0$ for all $t>0$ implying $\Theta u^* = \xi$ a.s. Then, $\Phi, \Theta$ have a random coincidence point $u^*$.
By the same argument, $\Psi, \Theta$ have a random coincidence point $v^*$. □

**Corollary 3.7** *Let $\Phi, \Theta : L_0^X(\Omega) \to L_0^X(\Omega)$ be continuous in probability completely random operators, $\Phi$ be surjective and $f : [0,\infty) \to [0,\infty)$ be a continuous, increasing function such that $f(0)=0$, $\lim_{t\to\infty} f(t)=\infty$ and $q>1$. Assume that for any $u, v$ in $L_0^X(\Omega)$ and $t>0$, we have*

$$P\left(\|\Phi u-\Phi v\|>f(t)\right) \geq P\left(\|\Theta u-\Theta v\|>f(t/q)\right). \tag{3.23}$$

*If there exist random variables $u_0, v_0$ in $L_0^X(\Omega)$ and $p>0$ such that $\Phi v_0 = \Theta u_0$ and*

$$M=\sup_{t>0} t^p P\left(\|\Theta v_0-\Theta u_0\|>f(t)\right)<\infty. \tag{3.24}$$

*Then,*

(1) *Assume that there exists a number $c > 1/q$ such that*

$$\sum_{n=1}^{\infty} f(c^n) < \infty. \tag{3.25}$$

*Then, the condition (3.24) is sufficient for $\Phi, \Theta$ to have a random coincidence point.*

(2) *Assume that for each $t, s > 0$*

$$f(t+s) \geq f(t) + f(s). \tag{3.26}$$

*Then, the condition (3.24) is also sufficient for $\Phi, \Theta$ to have a random coincidence point.*

*Proof* It is easy to receive the corollary when we take $\Psi v = \Phi v$ in Theorem 3.6. □

## 4 Applications to Random Fixed Point Theorems and Random Equations

In this section, we present some applications to random fixed point theorems and random equations.

**Theorem 4.1** *Let $\Phi : L_0^X(\Omega) \to L_0^X(\Omega)$ be surjective, continuous in probability completely random operator and $f : [0, \infty) \to [0, \infty)$ be a continuous, increasing function such that $f(0) = 0$, $\lim_{t\to\infty} f(t) = \infty$ and $q > 1$. Assume that for each pair $u, v$ in $L_0^X(\Omega)$*

$$P\left(\|\Phi u - \Phi v\| > f(t)\right) \geq P\left(\|u - v\| > f(t/q)\right). \tag{4.1}$$

*If there exist random variables $v_0$ in $L_0^X(\Omega)$ and $p > 0$ such that*

$$M = \sup_{t>0} t^p P\left(\|\Phi v_0 - v_0\| > f(t)\right) < \infty. \tag{4.2}$$

*Then*

(1) *Assume that there exists a number $c > 1/q$ such that*

$$\sum_{n=1}^{\infty} f(c^n) < \infty. \tag{4.3}$$

*Then, the condition (4.2) is sufficient for $\Phi$ to have a unique random fixed point.*

(2) *Assume that for each $t, s > 0$*

$$f(t+s) \geq f(t) + f(s). \tag{4.4}$$

*Then, the condition (4.2) is also sufficient for $\Phi$ to have a unique random fixed point.*

*Proof* Consider the completely random operator $\Theta$ given by $\Theta u = u$. By Corollary 3.7, $\Phi$ and $\Theta$ have a random coincidence point $\xi$ which is exactly the random fixed point of $\Phi$.
Let $\xi, \eta$ be two random fixed points of $\Phi$. Then, for each $t > 0$, we have

$$P\left(\|\xi - \eta\| > f(qt)\right) = P\left(\|\Phi\xi - \Phi\eta\| > f(qt)\right) \geq P\left(\|\xi - \eta\| > f(t)\right).$$

By induction, it follows that

$$P\left(\|\xi - \eta\| > f(t)\right) \leq P\left(\|\xi - \eta\| > f(q^n t)\right) \quad \forall n.$$

Since $\lim_{n\to\infty} f(q^n t) = +\infty$, we conclude that $P\left(\|\xi - \eta\| > f(q^n t)\right) = 0$ for each $t > 0$. Hence, $g(\|\xi - \eta\|) = 0 \quad$ a.s., with $g$ is the inverse function of $f$. So, we have $\xi = \eta \quad$ a.s. as claimed. □

**Theorem 4.2** *Let $\Phi, \Theta : L_0^X(\Omega) \to L_0^X(\Omega)$ be continuous in probability completely random operators, $\Phi$ be surjective and $f : [0, \infty) \to [0, \infty)$ be a mapping such that for each $t > 0$,*

$$h(t) = \inf_{s \geq t} \frac{f(s)}{s} > 0. \tag{4.5}$$

*Assume that for each pair $u, v$ in $L_0^X(\Omega)$ and $t > 0$, we have*

$$P\left(\|\Phi u - \Phi v\| > t\right) \geq P\left(\|\Theta u - \Theta v\| + f\left(\|\Theta u - \Theta v\|\right) > t\right). \tag{4.6}$$

*If $\Phi, \Theta$ commute i.e. $\Phi\Theta u = \Theta\Phi u$ for any random variable $u$ in $L_0^X(\Omega)$ then $\Phi$ and $\Psi$ have a unique common random fixed point if there exist random variables $u_0, v_0$ in $L_0^X(\Omega)$ and $p > 0$ such that $\Phi v_0 = \Theta u_0$ and*

$$M = E\|\Theta v_0 - \Theta u_0\|^p < \infty. \tag{4.7}$$

*Proof* Suppose that (4.7) holds. By Corollary 3.3, there exists $u^*$ such that $\Phi u^* = \Theta u^* = \xi$. For $t > 0$, we have

$$\begin{aligned} P(\|\Phi\xi - \xi\| > qt) &= P(\|\Phi\xi - \Phi u^*\| > qt) \\ &\geq P(\|\Theta\xi - \Theta u^*\| > t) \\ &= P(\|\Theta\Phi u^* - \xi\| > t) \\ &= P(\|\Phi\Theta u^* - \xi\| > t) \\ &= P(\|\Phi\xi - \xi\| > t). \end{aligned}$$

By induction, it follows that $P(\|\Phi\xi - \xi\| > t) \leq P(\|\Phi\xi - \xi\| > q^n t)$ for any $n \in N$. Let $n \to \infty$, we have $P(\|\Phi\xi - \xi\| > t) = 0$ for any $t > 0$. Thus, $\Phi\xi = \xi$ i.e. $\xi$ is a random fixed point of $\Phi$. We have $\Theta\xi = \Theta\Phi u^* = \Phi\Theta u^* = \Phi\xi = \xi$. So $\xi$ is also a random fixed point of $\Theta$.

Let $\xi_1$ and $\xi_2$ be two common random fixed points of $\Phi$ and $\Theta$. For each $t > 0$, we have

$$\begin{aligned} P(\|\xi_1 - \xi_2\| > q^n t) &= P(\|\Phi\xi_1 - \Phi\xi_2\| > q^n t) \\ &\geq P(\|\Theta\xi_1 - \Theta\xi_2\| > q^{n-1} t) \\ &= P(\|\xi_1 - \xi_2\| > q^{n-1} t) \\ &\geq \ldots \\ &\geq P(\|\xi_1 - \xi_2\| > t). \end{aligned}$$

Let $n \to \infty$, we have $P(\|\xi_1 - \xi_2\| > t) = 0$ for all $t > 0$. Hence, $\xi_1 = \xi_2$. □

**Corollary 4.3** *Let $\Phi : L_0^X(\Omega) \to L_0^X(\Omega)$ be a surjective, continuous in probability and probabilistic $q$-expansive completely random operator in the sense that there exists a number $q > 1$ such that*

$$P(\|\Phi u - \Phi v\| > t) \geq P(\|u - v\| > t/q) \tag{4.8}$$

*for all random variables $u, v$ in $L_0^X(\Omega)$ and $t > 0$. Then, $\Phi$ has a unique random fixed point if there exists a random variable $v_0$ in $L_0^X(\Omega)$ and $p > 0$ such that*

$$E\|\Phi v_0 - v_0\|^p < \infty. \tag{4.9}$$

*Proof* Consider $\Theta : L_0^X(\Omega) \to L_0^X(\Omega)$ given by $\Theta u = u$, the function $f(t) = (1-q)t$ and $h(t) = 1-q > 0$. Then $\Phi$, $\Theta$ and $f(t)$ satisfy the conditions stated in the Theorem 4.2 and $\Phi$, $\Theta$ commute. Thus, $\Phi$ and $\Theta$ have a common random fixed point $\xi$ i.e. $\Phi$ has a random fixed point $\xi$.

**Theorem 4.4** *Let $\Phi, \Theta : L_0^X(\Omega) \to L_0^X(\Omega)$ be continuous in probability completely random operators, $\Phi$ be surjective and*

$$P(\|\Phi u - \Phi v\| > f(t)) \geq P(\|\Theta u - \Theta v\| > f(t/q)) \tag{4.10}$$

*for all $u, v$ in $L_0^X(\Omega)$, $t > 0$ and $f : [0, \infty) \to [0, \infty)$ be a continuous, increasing function such that $f(0) = 0$, $\lim_{t\to\infty} f(t) = \infty$ satisfying either (4.3) or (4.4) and $q > 1$. Consider random equation of the form*

$$\Phi u - \lambda\Theta u = \eta, \tag{4.11}$$

*where $\lambda$ is a real number and $\eta$ is a random variable in $L_0^X(\Omega)$.*

*Assume that*

$$0 < |\lambda| \leq \inf_{t>0} \frac{f\left(\frac{q}{q'} t\right)}{f(t)}, \tag{4.12}$$

*where $q' > 1$. Then the equation (4.11) has a unique random solution if there exists a random variable $v_0$ in $L_0^X(\Omega)$ and a number $p > 0$ such that*

$$M = \sup_{t>0} t^p P\left(\|\Phi v_0 - \lambda\Theta v_0 - \eta\| > |\lambda| f(t)\right) < \infty. \tag{4.13}$$

*Proof* Suppose that the condition (4.13) holds. Define a completely random operator $\Psi$ by

$$\Psi u = \frac{\Phi u - \eta}{\lambda}.$$

From (4.13) it follows that

$$M = \sup_{t>0} t^p P\left(\|\Psi v_0 - \Theta u_0\| > f(t)\right) < \infty. \tag{4.14}$$

Let $g = f^{-1}$ be the inverse function of $f$. Then, $g : [0, \infty) \to [0, \infty)$ is continuous, increasing with $g(0) = 0$, $\lim_{t\to\infty} g(t) = \infty$. For each $t > 0$, there exists $t'$ so that $f(t') = |\lambda| f(t)$ i.e. $t' = g(|\lambda| f(t))$. So, we have

$$\begin{aligned} P\left(\|\Psi u - \Psi v\| > f(t)\right) &= P\left(\|\Phi u - \Phi v\| > |\lambda| f(t)\right) \\ &= P\left(\|\Phi u - \Phi v\| > f\left(t'\right)\right) \\ &\geq P\left(\|\Theta u - \Theta v\| > f\left(t'/q\right)\right) \\ &= P\left(\|\Theta u - \Theta v\| > f\left(\frac{t}{q'}\frac{q't'}{qt}\right)\right). \end{aligned}$$

From (4.12), we receive $|\lambda| f(t) \leq f\left(\frac{q}{q'}t\right)$. Then, we deduce $g\left(|\lambda| f(t)\right) \leq \frac{q}{q'}t$. So, $t' \leq \frac{q}{q'}t$ and $\frac{q't'}{qt} \leq 1$. Hence,

$$P\left(\|\Theta u - \Theta v\| > f\left(\frac{t}{q'}\frac{q't'}{qt}\right)\right) \geq P\left(\|\Theta u - \Theta v\| > f\left(t/q',\right)\right)$$

which implies

$$P\left(\|\Psi u - \Psi v\| > f(t)\right) \geq P\left(\|\Theta u - \Theta v\| > f\left(t/q'\right)\right).$$

Consequently, $\Theta$ and $\Psi$ satisfy the conditions stated in the Corollary 3.7. Hence, $\Theta$ and $\Psi$ has a random coincidence point $\xi$ i.e. the equation (4.11) has a random solution $\xi$. □

**Corollary 4.5** *Let $\Phi : L_0^X(\Omega) \to L_0^X(\Omega)$ be a surjective, continuous in probability completely random operator satisfying the following condition*

$$P\left(\|\Phi u - \Phi v\| > f(t)\right) \geq P\left(\|u - v\| > f(t/q)\right) \tag{4.15}$$

*for all* $u, v$ *in* $L_0^X(\Omega)$*,* $t > 0$ *, where* $f : [0, \infty) \to [0, \infty)$ *is a continuous, increasing function such that* $f(0) = 0, \lim_{t\to\infty} f(t) = \infty$ *satisfying either* (4.3) *or* (4.4) *and* $q > 1$*. Consider random equation of the form*

$$\Phi u - \lambda u = \eta, \tag{4.16}$$

*where* $\lambda$ *is a real number and* $\eta$ *is a random variable in* $L_0^X(\Omega)$*.*

*Assume that*

$$0 < |\lambda| \le \inf_{t>0} \frac{f\left(\frac{q}{q'}t\right)}{f(t)}, \tag{4.17}$$

*where* $q' > 1$*. Then the equation* (4.16) *has a unique random solution if and only if there exists a random variable* $v_0$ *in* $L_0^X(\Omega)$ *and a number* $p > 0$ *such that*

$$M = \sup_{t>0} t^p P\left(\|\Phi v_0 - \lambda v_0 - \eta\| > |\lambda| f(t)\right) < \infty. \tag{4.18}$$

*Proof* Applying the Theorem 4.4 for the completely random operator $\Theta$ given by $\Theta u = u$. □

**Corollary 4.6** *Let* $\Phi, \Theta : L_0^X(\Omega) \to L_0^X(\Omega)$ *be continuous in probability completely random operators,* $\Phi$ *be surjective satisfying the following condition*

$$P\left(\|\Phi u - \Phi v\| > t\right) \ge P\left(\|\Theta u - \Theta v\| > t/q\right) \tag{4.19}$$

*for all* $u, v$ *in* $L_0^X(\Omega)$ *and a number* $q > 1$*. Consider the random equation*

$$\Phi u - \lambda \Theta u = \eta, \tag{4.20}$$

*where* $\lambda$ *is a real number and* $\eta$ *is a random variable in* $L_p^X(\Omega)$*,* $p > 0$*.*

*Assume that* $0 < |\lambda| < q$*. Then, the random equation* (4.20) *has a solution if there exists a random variable* $v_0$ *in* $L_0^X(\Omega)$ *such that*

$$E\|\Phi v_0 - \lambda \Theta v_0\|^p < \infty. \tag{4.21}$$

*Proof* Suppose that there exists a random variable $u_0$ in $L_0^X(\Omega)$ such that (4.10) holds. So, $\Phi$ and $\Theta$ satisfy (4.15) where $f(t) = t$. Take $|\lambda| < s < q$, then $q' = q/s > 1$ and

$$0 < |\lambda| < s = \frac{q}{q'} = \frac{f\left(\frac{q}{q'}t\right)}{f(t)}.$$

Moreover, for each $t > 0$

$$t^p P\left(\|\Phi v_0 - \lambda \Theta v_0 - \eta\| > |\lambda| t\right) \le \frac{E\|\Phi v_0 - \lambda \Theta v_0 - \eta\|^p}{|\lambda|^p} < \infty \tag{4.22}$$

since

$$E(\|\Phi u_0 - \lambda\Theta u_0 - \eta\|^p) \leq C_p E(\|\Phi u_0 - \lambda\Theta u_0\|^p) + C_p E\|\eta\|^p < \infty,$$

where $C_p$ is a constant. Hence, the condition (4.13) is satisfied. By Theorem 4.4, we conclude that the equation (4.20) has a random solution. □

Taking the completely random operator $\Theta$ given by $\Theta u = u$, we obtain

**Corollary 4.7** *Let $\Phi : L_0^X(\Omega) \to L_0^X(\Omega)$ be a surjective, continuous in probability completely random operator satisfying the following condition*

$$P(\|\Phi u - \Phi v\| > t) \geq P(\|u - v\| > t/q) \tag{4.23}$$

*for all $u, v$ in $L_0^X(\Omega)$ and a number $q > 1$. Consider the random equation*

$$\Phi u - \lambda u = \eta, \tag{4.24}$$

*where $\lambda$ is a real number satisfying $0 < |\lambda| < q$ and $\eta$ is a random variable in $L_p^X(\Omega)$, $p > 0$. Then, the random equation (4.24) has a unique random solution if there exists a random variable $v_0$ in $L_0^X(\Omega)$ such that*

$$E\|\Phi v_0 - \lambda v_0\|^p < \infty. \tag{4.25}$$

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