Fixed Points of Weakly Compatible Mappings Satisfying Generalized φ-Weak Contractions

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Abstract In this paper, utilizing the notion of the common limit range property, we prove some new integral type common fixed point theorems for weakly compatible mappings satisfying a φ-weak contractive condition in metric spaces. Moreover, we extend our results to four finite families of self mappings, and furnish an illustrative example and an application to support our main theorem. Our results improve, extend, and generalize well-known results on the topic in the literature.

Keywords Metric space · Weakly compatible mappings · (CLR_S) property · (CLR_{ST}) property · Fixed point

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1 Introduction and Preliminaries

The celebrated Banach contraction mapping principle, also known as Banach fixed point theorem [9], plays an important role for solving existence problems in many branches of nonlinear analysis. For instance, it has been used to show the existence of solutions of nonlinear Volterra integral equations, nonlinear integro-differential equations in Banach spaces and to show the convergence of algorithms in computational mathematics. This famous theorem states that every contraction in a complete metric space has a unique fixed point and that point can be obtained as a limit of repeated iteration of the mapping at any point of X. It is evident that each contraction is a continuous function.

In 1997, Alber and Guerre-Delabriere [1] proposed the notion of weak contraction for single-valued mappings in Hilbert space. A self mapping T on a complete metric space (X, d) is a φ-weak contraction if there exists a function $\varphi : [0, +\infty) \rightarrow [0, +\infty)$ such that φ is positive on $(0, +\infty)$, $\varphi(0) = 0$, and

$$d(Tx, Ty) \leq d(x, y) - \varphi(d(x, y)) \tag{1.1}$$

for all $x, y \in X$. Subsequently, Rhoades [57] showed that the result which Alber and Guerre-Delabriere proved in [1] is also valid in the setting of complete metric spaces.

Theorem 1 [57, Theorem 2] *Let (X, d) be a non-empty complete metric space and let $T : X \rightarrow X$ be a φ-weak contraction on X. If φ is a continuous and non-decreasing function with $\varphi(t) > 0$ for all $t > 0$ and $\varphi(0) = 0$, then T has a unique fixed point.*

It is noticed that Alber and Guerre-Delabriere [1] assumed an additional condition on φ which is $\lim_{t \rightarrow +\infty} \varphi(t) = +\infty$; but Rhoades [57] obtained the result noted in Theorem 1 without using this particular assumption. If one takes $\varphi(t) = (1 - k)t$, where $k \in (0, 1)$, then (1.1) reduces to contraction, that is, every contraction is a φ-weak contraction. Following this trend, Karapınar in [40] proved the existence and uniqueness of a fixed point for cyclic mappings (see also, [42]) and in [41] obtained a fixed point for a φ-weak contraction. In fact, weak contractions are closely related to the mappings of Boyd and Wong's type [16], and Reich's type [55]. If φ is a lower semi-continuous function from the right, then $\psi(t) = t - \varphi(t)$ is an upper semi-continuous function from the right, and moreover, (1.1) turns into $d(Tx, Ty) \leq \psi(d(x, y))$. Therefore, the φ-weak contraction becomes a Boyd and Wong's type one. Similarly, if we define $\alpha(t) = \frac{1-\varphi(t)}{t}$ for $t > 0$ and $\alpha(0) = 0$, then (1.1) is replaced by $d(Tx, Ty) \leq \alpha(d(x, y))d(x, y)$. Thus, the φ-weak contraction becomes a Reich's type one. Following this direction of research, many authors have proved common fixed point theorems in metric spaces satisfying different contractive conditions, see [3,7,10,12,15,18–24,26,30,34,37,46–49,51–53,56,60,62–65,67,68,72–75,77].

In 2002, Branciari [17] proved a fixed point result for a single mapping satisfying an analog of Banach's contraction principle for an integral type inequality. The authors [5,25,45,58,59,70,71,78] proved several fixed point results involving more general integral type contractive conditions. Moreover, in [76], Suzuki showed that a Meir–Keeler contraction of integral type is still a Meir–Keeler contraction.

In 2009, Zhang and Song [79] proved a fixed point theorem for two mappings satisfying a generalized φ-weak contractive condition in a complete metric space. Later on, Razani and Yazdi [54] proved a common fixed point theorem for any even number of self mappings in a complete metric space and generalized the results of Zhang and Song [79].

In this paper, we prove an integral type common fixed point theorem for four mappings applying the common limit range property. As an application, we present fixed point theorems for six mappings and four finite families of mappings in metric spaces using the notion of the pairwise commuting mappings which is studied by Ali and Imdad [4]. We conclude with an example that supports the useability of our results and an application to some functional equations arising in dynamic programming.

The following definitions and results will be needed in the sequel.

Definition 1 Let $A, S : X \to X$ be two self mappings of a metric space (X, d). The mappings A and S are said to be:

(1) commuting if $ASx = SAx$, for all $x \in X$;
(2) weakly commuting if $d(ASx, SAx) \leq d(Ax, Sx)$, for all $x \in X$, see [61];
(3) compatible if $\lim_{n \to +\infty} d(ASx_n, SAx_n) = 0$ for each sequence $\{x_n\}$ in X such that $\lim_{n \to +\infty} Ax_n = \lim_{n \to +\infty} Sx_n$, see [38];
(4) non-compatible if there exists a sequence $\{x_n\}$ in X such that $\lim_{n \to +\infty} Ax_n = \lim_{n \to +\infty} Sx_n$ but $\lim_{n \to +\infty} d(ASx_n, SAx_n)$ is either nonzero or nonexistent, see [48];
(5) weakly compatible if they commute at their coincidence points, that is, $ASu = SAu$ whenever $Au = Su$, for some $u \in X$, see [39].

Definition 2 [2] A pair (A, S) of self mappings of a metric space (X, d) is said to satisfy the property (E.A) if there exists a sequence $\{x_n\}$ in X, for some $z \in X$ such that

$$\lim_{n \to +\infty} Ax_n = \lim_{n \to +\infty} Sx_n = z. \tag{1.2}$$

It can be noticed that any pair of non-compatible self mappings of a metric space (X, d) satisfies the property (E.A) but two mappings satisfying the property (E.A) need not be non-compatible (see [29, Example1]). On the other hand, the notions of weak compatibility and property (E.A) are independent to each other (see [50, Examples 2.1–2.2]). For more reading on the property (E.A), consider [27,28] and the references therein.

Definition 3 [44] Two pairs (A, S) and (B, T) of self mappings of a metric space (X, d) are said to satisfy the common property (E.A), if there exist two sequences $\{x_n\}$ and $\{y_n\}$ in X, and some $z \in X$ such that

$$\lim_{n \to +\infty} Ax_n = \lim_{n \to +\infty} Sx_n = \lim_{n \to +\infty} By_n = \lim_{n \to +\infty} Ty_n = z. \tag{1.3}$$

It is observed that the fixed point results always require the closedness of the underlying subspaces for the existence of common fixed points under the property (E.A) and common property (E.A). In 2011, Sintunavarat and Kumam [69] coined the idea

of "common limit range property". Recently, Imdad et al. [33] extended the notion of common limit range property to two pairs of self mappings which relaxes the requirement on closedness of the subspaces. Since then, a number of fixed point theorems have been established by several researchers in different settings under common limit range property. For detail description, we refer the reader to [8,32,35,36,43,66].

Definition 4 [69] A pair (A, S) of self mappings of a metric space (X, d) is said to satisfy the common limit range of S property, (CLR_S) property for short, if there exists a sequence $\{x_n\}$ in X such that

$$\lim_{n \to +\infty} Ax_n = \lim_{n \to +\infty} Sx_n = z,$$

where $z \in S(X)$.

Hence, it is assured that a pair (A, S) satisfying the property (E.A) along with closedness of the subspace $S(X)$ always enjoys the (CLR_S) property (see [33, Examples 2.16–2.17]).

Definition 5 Two pairs (A, S) and (B, T) of self mappings of a metric space (X, d) are said to satisfy the common limit range of S and T property, (CLR_{ST}) property for short, if there exist two sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$$\lim_{n \to +\infty} Ax_n = \lim_{n \to +\infty} Sx_n = \lim_{n \to +\infty} By_n = \lim_{n \to +\infty} Ty_n = z,$$

where $z \in S(X) \cap T(X)$.

Definition 6 [4] Two families of self mappings $\{A_i\}_{i=1}^{m}$ and $\{S_k\}_{k=1}^{n}$ are said to be pairwise commuting if

(1) $A_i A_j = A_j A_i$ for all $i, j \in \{1, 2, \ldots, m\}$,
(2) $S_k S_l = S_l S_k$ for all $k, l \in \{1, 2, \ldots, n\}$,
(3) $A_i S_k = S_k A_i$ for all $i \in \{1, 2, \ldots, m\}$ and $k \in \{1, 2, \ldots, n\}$.

2 Main Results

Let Φ denote the set of all non-decreasing functions $\varphi : [0, +\infty) \to [0, +\infty)$ that satisfy the following conditions:

(1) φ is lower semi-continuous on $[0, +\infty)$,
(2) $\varphi(0) = 0$,
(3) $\varphi(s) > 0$ for each $s > 0$.

We start with the following Lemma.

Lemma 1 *Let $A, B, S,$ and T be self mappings of a metric space (X, d). Suppose that the following hypotheses hold:*

(1) *the pair (A, S) satisfies the (CLR_S) property $\Big($or the pair (B, T) satisfies the (CLR_T) property$\Big)$,*

(2) $A(X) \subset T(X)$ $\big(or\ B(X) \subset S(X)\,\big)$,
(3) $T(X)$ $\big(or\ S(X)\,\big)$ *is a closed subset of X,*
(4) $\{By_n\}$ *converges for every sequence* $\{y_n\}$ *in X whenever* $\{Ty_n\}$ *converges* $\big(or$ $\{Ax_n\}$ *converges for every sequence* $\{x_n\}$ *in X whenever* $\{Sx_n\}$ *converges*),
(5) *there exists* $\varphi \in \Phi$ *such that*

$$\int_0^{d(Ax,By)} \phi(t)\mathrm{d}t \leq M(x, y) - \varphi(M(x, y)), \tag{2.1}$$

for all $x, y \in X$, *where*

$$M(x, y) = \int_0^{\max\{d(Ax,Sx),d(By,Ty),d(Sx,Ty),[d(Ax,Ty)+d(By,Sx)]/2\}} \phi(t)\mathrm{d}t$$

and $\phi : [0, +\infty) \to [0, +\infty)$ *is a Lebesgue-integrable mapping which is summable and such that*

$$\int_0^{\epsilon} \phi(t)\mathrm{d}t > 0, \tag{2.2}$$

for all $\epsilon > 0$.

Then the pairs (A, S) *and* (B, T) *share the* (CLR_{ST}) *property.*

Proof Since the pair (A, S) satisfies the (CLR_S) property, there exists a sequence $\{x_n\}$ in X such that

$$\lim_{n \to +\infty} Ax_n = \lim_{n \to +\infty} Sx_n = z,$$

where $z \in S(X)$. By (2), $A(X) \subset T(X)$ (wherein $T(X)$ is a closed subset of X), and for each $\{x_n\} \subset X$, there corresponds a sequence $\{y_n\} \subset X$ such that $Ax_n = Ty_n$. Therefore,

$$\lim_{n \to +\infty} Ty_n = \lim_{n \to +\infty} Ax_n = z,$$

where $z \in S(X) \cap T(X)$. Thus, we have $Ax_n \to z$, $Sx_n \to z$ and $Ty_n \to z$ as $n \to +\infty$. By (4), the sequence $\{By_n\}$ converges and in all we need to show that $By_n \to z$ as $n \to +\infty$. Putting $x = x_n$ and $y = y_n$ in condition (2.1), we get

$$\int_0^{d(Ax_n,By_n)} \phi(t)\mathrm{d}t \leq M(x_n, y_n) - \varphi(M(x_n, y_n)), \tag{2.3}$$

where

$$M(x_n, y_n) = \int_0^{\max\{d(Ax_n,Sx_n),d(By_n,Ty_n),d(Sx_n,Ty_n),[d(Ax_n,Ty_n)+d(By_n,Sx_n)]/2\}} \phi(t)\mathrm{d}t.$$

Let $By_n \to l\ (\neq z)$ for $t > 0$ as $n \to +\infty$. Then taking limit as $n \to +\infty$ (lower limit) in inequality (2.3), we have

$$\int_0^{d(z,l)} \phi(t)\mathrm{d}t \leq \lim_{n \to +\infty} M(x_n, y_n) - \varphi\left(\lim_{n \to +\infty} M(x_n, y_n)\right), \tag{2.4}$$

where

$$\lim_{n \to +\infty} M(x_n, y_n) = \int_0^{\max\{d(z,z),d(l,z),d(z,z),[d(z,z)+d(l,z)]/2\}} \phi(t)\mathrm{d}t$$
$$= \int_0^{d(z,l)} \phi(t)\mathrm{d}t.$$

Hence, inequality (2.4) implies

$$\int_0^{d(z,l)} \phi(t)\mathrm{d}t \le \int_0^{d(z,l)} \phi(t)\mathrm{d}t - \varphi\left(\int_0^{d(z,l)} \phi(t)\mathrm{d}t\right),$$

that is, $\varphi\left(\int_0^{d(z,l)} \phi(t)\mathrm{d}t\right) \le 0$. Thus, $\varphi\left(\int_0^{d(z,l)} \phi(t)\mathrm{d}t\right) = 0$ and by the property of the function φ, we have $d(z, l) = 0$ or equivalently $z = l$, which contradicts the hypothesis $l \ne z$. Hence, both the pairs (A, S) and (B, T) share the (CLR_{ST}) property. $\square$

In general, the converse of Lemma 1 is not true (see [33, Example 3.5]). Now, we are ready to state and prove the following theorem.

Theorem 2 *Let A, B, S, and T be self mappings of a metric space (X, d) satisfying the hypothesis (5) of Lemma 1. If the pairs (A, S) and (B, T) share the (CLR_{ST}) property, then (A, S) and (B, T) have a coincidence point each. Moreover, A, B, S, and T have a unique common fixed point provided both the pairs (A, S) and (B, T) are weakly compatible.*

Proof If the pairs (A, S) and (B, T) share the (CLR_{ST}) property, then there exist two sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$$\lim_{n \to +\infty} Ax_n = \lim_{n \to +\infty} Sx_n = \lim_{n \to +\infty} Ty_n = \lim_{n \to +\infty} By_n = z,$$

where $z \in S(X) \cap T(X)$. Since $z \in S(X)$, there exists a point $u \in X$ such that $Su = z$. Putting $x = u$ and $y = y_n$ in condition (2.1), we get

$$\int_0^{d(Au, By_n)} \phi(t)\mathrm{d}t \le M(u, y_n) - \varphi(M(u, y_n)), \qquad (2.5)$$

where

$$M(u, y_n) = \int_0^{\max\{d(Au,Su),d(By_n,Ty_n),d(Su,Ty_n),[d(Au,Ty_n)+d(By_n,Su)]/2\}} \phi(t)\mathrm{d}t.$$

Letting $n \to +\infty$ (taking the lower limit) in condition (2.5), we have

$$\int_0^{d(Au,z)} \phi(t)\mathrm{d}t \le \lim_{n \to +\infty} M(u, y_n) - \varphi\left(\lim_{n \to +\infty} M(u, y_n)\right), \qquad (2.6)$$

where

$$\lim_{n \to +\infty} M(u, y_n) = \int_0^{\max\{d(Au,z),d(z,z),d(z,z),[d(Au,z)+d(z,z)]/2\}} \phi(t)\mathrm{d}t$$

$$= \int_0^{\max\{d(Au,z),d(Au,z)/2\}} \phi(t)\mathrm{d}t$$

$$= \int_0^{d(Au,z)} \phi(t)\mathrm{d}t.$$

From (2.6), we obtain

$$\int_0^{d(Au,z)} \phi(t)\mathrm{d}t \le \int_0^{d(Au,z)} \phi(t)\mathrm{d}t - \varphi\left(\int_0^{d(Au,z)} \phi(t)\mathrm{d}t\right)$$

and it follows easily that $Au = z$. Therefore, $Au = Su = z$ which shows that u is a coincidence point of the pair (A, S).

As $z \in T(X)$, there exists a point $v \in X$ such that $Tv = z$. Putting $x = x_n$ and $y = v$ in condition (2.1), we have

$$\int_0^{d(Ax_n, Bv)} \phi(t)\mathrm{d}t \le M(x_n, v) - \varphi(M(x_n, v)), \tag{2.7}$$

where

$$M(x_n, v) = \int_0^{\max\{d(Ax_n,Sx_n),d(Bv,Tv),d(Sx_n,Tv),[d(Ax_n,Tv)+d(Bv,Sx_n)]/2\}} \phi(t)\mathrm{d}t.$$

Letting $n \to +\infty$ (taking the lower limit) in condition (2.7), we get

$$\int_0^{d(z, Bv)} \phi(t)\mathrm{d}t \le \lim_{n\to\infty} M(x_n, v) - \varphi\left(\lim_{n\to\infty} M(x_n, v)\right), \tag{2.8}$$

where

$$\lim_{n\to\infty} M(x_n, v) = \int_0^{\max\{d(z,z),d(Bv,z),d(z,z),[d(z,z)+d(Bv,z)]/2\}} \phi(t)\mathrm{d}t$$

$$= \int_0^{\max\{d(Bv,z),d(Bv,z)/2\}} \phi(t)\mathrm{d}t$$

$$= \int_0^{d(z,Bv)} \phi(t)\mathrm{d}t.$$

Hence, inequality (2.8) implies

$$\int_0^{d(z,Bv)} \phi(t)\mathrm{d}t \le \int_0^{d(z,Bv)} \phi(t)\mathrm{d}t - \varphi\left(\int_0^{d(z,Bv)} \phi(t)\mathrm{d}t\right)$$

and so $z = Bv$. Thus, $Bv = Tv = z$ which shows that v is a coincidence point of the pair (B, T).

Since the pairs (A, S) and (B, T) are weakly compatible, $Au = Su$ and $Bv = Tv$, therefore $Az = ASu = SAu = Sz$ and $Bz = BTv = TBv = Tz$. Putting $x = z$ and $y = v$ in condition (2.1), we have

$$\int_0^{d(Az,z)} \phi(t)\mathrm{d}t = \int_0^{d(Az,Bv)} \phi(t)\mathrm{d}t \leq M(z, v) - \varphi(M(z, v)), \qquad (2.9)$$

where

$$
\begin{aligned}
M(z, v) &= \int_0^{\max\{d(Az,Sz),d(Bv,Tv),d(Sz,Tv),[d(Az,Tv)+d(Bv,Sz)]/2\}} \phi(t)\mathrm{d}t \\
&= \int_0^{\max\{d(Az,Az),d(z,z),d(Az,z),[d(Az,z)+d(z,Az)]/2\}} \phi(t)\mathrm{d}t \\
&= \int_0^{d(Az,z)} \phi(t)\mathrm{d}t.
\end{aligned}
$$

From (2.9), we get

$$\int_0^{d(Az,z)} \phi(t)\mathrm{d}t \leq \int_0^{d(Az,z)} \phi(t)\mathrm{d}t - \varphi\left(\int_0^{d(Az,z)} \phi(t)\mathrm{d}t\right).$$

It follows that $z = Az = Sz$, and therefore z is a common fixed point of the pair (A, S). Putting $x = u$ and $y = z$ in condition (2.1), we have

$$\int_0^{d(z,Bz)} \phi(t)\mathrm{d}t = \int_0^{d(Au,Bz)} \phi(t)\mathrm{d}t \leq M(u, z) - \varphi(M(u, z)), \qquad (2.10)$$

where

$$
\begin{aligned}
M(u, z) &= \int_0^{\max\{d(Au,Su),d(Bz,Tz),d(Su,Tz),[d(Au,Tz)+d(Bz,Su)]/2\}} \phi(t)\mathrm{d}t \\
&= \int_0^{\max\{d(z,z),d(Bz,Bz),d(z,Bz),[d(z,Bz)+d(Bz,z)]/2\}} \phi(t)\mathrm{d}t \\
&= \int_0^{d(z,Bz)} \phi(t)\mathrm{d}t.
\end{aligned}
$$

From (2.10), we obtain

$$\int_0^{d(z,Bz)} \phi(t)\mathrm{d}t \leq \int_0^{d(z,Bz)} \phi(t)\mathrm{d}t - \varphi\left(\int_0^{d(z,Bz)} \phi(t)\mathrm{d}t\right)$$

and then $z = Bz$. Therefore, $Bz = Tz = z$ and we can conclude that z is a common fixed point of A, B, S and T. The uniqueness of the common fixed point is an easy consequence of condition (2.1) and so, to avoid repetition, we omit the details. □

Theorem 3 *Let A, B, S, and T be self mappings of a metric space (X, d) satisfying all the hypotheses of Lemma 1. Then A, B, S, and T have a unique common fixed point provided both the pairs (A, S) and (B, T) are weakly compatible.*

Proof By Lemma 1, it is assured that the pairs (A, S) and (B, T) share the (CLR_{ST}) property, then there exist two sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$$\lim_{n \to +\infty} Ax_n = \lim_{n \to +\infty} Sx_n = \lim_{n \to +\infty} Ty_n = \lim_{n \to +\infty} By_n = z,$$

where $z \in S(X) \cap T(X)$. The rest of the proof runs on the lines of the proof of Theorem 2, therefore the details are avoided. □

By choosing A, B, S, and T suitably, we can deduce some corollaries for a pair as well as for a triode of self mappings. Here, as a sample, we give the following natural result for a pair of self mappings.

Corollary 1 *Let A and S be self mappings of a metric space (X, d). Suppose that*

(1) *the pair (A, S) satisfies the (CLR_S) property,*
(2) *there exists $\varphi \in \Phi$ such that*

$$\int_0^{d(Ax, Ay)} \phi(t)dt \leq M(x, y) - \varphi(M(x, y)), \tag{2.11}$$

for all $x, y \in X$, where

$$M(x, y) = \int_0^{\max\{d(Ax, Sx), d(Ay, Sy), d(Sx, Sy), [d(Ax, Sy) + d(Ay, Sx)]/2\}} \phi(t)dt,$$

and $\phi : [0, +\infty) \to [0, +\infty)$ is a Lebesgue-integrable mapping which is summable and such that (2.2) holds.

Then the pair (A, S) has a coincidence point. Moreover, if (A, S) is weakly compatible then it has a unique common fixed point in X.

Remark 1 Corollary 1 generalizes the results of Zhang and Song [79, Theorem 2.1] without any requirement on completeness of the space.

Now, we utilize Definition 6 to prove a common fixed point theorem for six mappings in a metric space.

Theorem 4 *Let A, B, H, R, S, and T be self mappings of a metric space (X, d). Suppose that*

(1) *the pairs (A, SR) and (B, TH) share the $(CLR_{(SR)(TH)})$ property*

(2) *there exists $\varphi \in \Phi$ such that*

$$\int_0^{d(Ax,By)} \phi(t)\mathrm{d}t \leq M(x,y) - \varphi(M(x,y)), \qquad (2.12)$$

for all $x, y \in X$, where

$$M(x,y) = \int_0^{\max\{d(Ax,SRx),d(By,THy),d(SRx,THy),[d(Ax,THy)+d(By,SRx)]/2\}} \phi(t)\mathrm{d}t,$$

and $\phi : [0,+\infty) \to [0,+\infty)$ is a Lebesgue-integrable mapping which is summable and such that (2.2) holds.

Then (A, SR) and (B, TH) have a coincidence point each. Moreover, A, B, H, R, S, and T have a unique common fixed point provided both the pairs (A, SR) and (B, TH) commute pairwise, that is, $AS = SA$, $AR = RA$, $SR = RS$, $BT = TB$, $BH = HB$, and $TH = HT$.

Proof Since the pairs (A, SR) and (B, TH) are commuting pairwise, obviously both the pairs are weakly compatible. By Theorem 2, A, B, SR, and TH have a unique common fixed point z in X. Now, we show that z is the unique common fixed point of the self mappings A, B, H, R, S, and T. Putting $x = Rz$ and $y = z$ in condition (2.12), we get

$$\int_0^{d(Rz,z)} \phi(t)\mathrm{d}t = \int_0^{d(A(Rz),Bz)} \phi(t)\mathrm{d}t \leq M(x,y) - \varphi(M(Rz,z)), \qquad (2.13)$$

where

$$M(Rz,z)$$
$$= \int_0^{\max\{d(A(Rz),SR(Rz)),d(Bz,THz),d(SR(Rz),THz),[d(A(Rz),THz)+d(Bz,SR(Rz))]/2\}} \phi(t)\mathrm{d}t$$
$$= \int_0^{\max\{d(Rz,Rz),d(z,z),d(Rz,z),[d(Rz,z)+d(z,Rz)]/2\}} \phi(t)\mathrm{d}t$$
$$= \int_0^{d(Rz,z)} \phi(t)\mathrm{d}t.$$

From (2.13), we obtain

$$\int_0^{d(Rz,z)} \phi(t)\mathrm{d}t \leq \int_0^{d(Rz,z)} \phi(t)\mathrm{d}t - \varphi\left(\int_0^{d(Rz,z)} \phi(t)\mathrm{d}t\right)$$

and then we have $Rz = z$, which implies $S(Rz) = Sz = z$. Similarly, one can prove that $z = Hz$, that is, $T(Hz) = Tz = z$. Hence $z = Az = Bz = Sz = Rz = Tz = Hz$, and z is the unique common fixed point of A, B, H, R, S and T. □

Remark 2 Theorem 4 generalizes the results of Ćirić et al. [23, Theorems 5–6] in the framework of integral settings as one never requires any condition on completeness (or closedness) of the underlying space (or subspaces), containment of ranges amongst involved mappings, and continuity of one or more mappings.

Corollary 2 *Let $\{A_i\}_{i=1}^{m}$, $\{B_r\}_{r=1}^{n}$, $\{S_k\}_{k=1}^{p}$, and $\{T_h\}_{h=1}^{q}$ be four finite families of self mappings of a metric space (X, d) with $A = A_1 A_2 \ldots A_m$, $B = B_1 B_2 \cdots B_n$, $S = S_1 S_2 \ldots S_p$, and $T = T_1 T_2 \cdots T_q$ satisfying hypothesis (5) of Lemma 1 such that the pairs (A, S) and (B, T) satisfy the (CLR_{ST}) property, then (A, S) and (B, T) have a point of coincidence each.*

Moreover, $\{A_i\}_{i=1}^{m}$, $\{B_r\}_{r=1}^{n}$, $\{S_k\}_{k=1}^{p}$, and $\{T_h\}_{h=1}^{q}$ have a unique common fixed point if the pairs of families $(\{A_i\}, \{S_k\})$ and $(\{B_r\}, \{T_h\})$ commute pairwise wherein $i \in \{1, 2, \ldots, m\}$, $k \in \{1, 2, \ldots, p\}$, $r \in \{1, 2, \ldots, n\}$, and $h \in \{1, 2, \ldots, q\}$.

Corollary 3 *Let A, B, S, and T be self mappings of a metric space (X, d). Suppose that*

(1) *the pairs (A^m, S^p) and (B^n, T^q) share the $(CLR_{S^p T^q})$ property,*
(2) *there exists $\varphi \in \Phi$ such that*

$$\int_0^{d(A^m x, B^n y)} \phi(t) \mathrm{d}t \leq M(x, y) - \varphi(M(x, y)), \qquad (2.14)$$

for all $x, y \in X$ and

$$M(x, y) = \int_0^{\max\{d(A^m x, S^p x), d(B^n y, T^q y), d(S^p x, T^q y), [d(A^m x, T^q y) + d(B^n y, S^p x)]/2\}} \phi(t) \mathrm{d}t,$$

where m, n, p, q are fixed positive integers and $\phi : [0, +\infty) \to [0, +\infty)$ is a Lebesgue-integrable mapping which is summable and such that (2.2) holds.

Then A, B, S, and T have a unique common fixed point provided $AS = SA$ and $BT = TB$.

Remark 3 Corollaries 2 and 3 improve the results of Razani and Yazdi [54, Theorems 2.5–2.7] for any finite number of mappings.

The conclusions of Lemma 1, Theorems 2–4, and Corollaries 1–3 remain true for $\phi(t) = 1$. In this case, the listing of possible corollaries are not presented here but, for a sample, we state the following theorem:

Theorem 5 *Let A, B, S and T be self mappings of a metric space (X, d). Suppose that*

(1) *the pairs (A, S) and (B, T) share the (CLR_{ST}) property,*
(2) *there exists $\varphi \in \Phi$ such that*

$$d(Ax, By) \leq M(x, y) - \varphi(M(x, y)), \qquad (2.15)$$

for all $x, y \in X$, where

$$M(x, y)= \max \{d(Ax, Sx), d(By, Ty), d(Sx, Ty), [d(Ax, Ty)+d(By, Sx)]/2\}.$$

Then (A, S) and (B, T) have a coincidence point each. Moreover, if the pairs (A, S) and (B, T) are weakly compatible then $A, B, S,$ and T have a unique common fixed point in X.

Remark 4 Results similar to Theorem 3, Corollaries 2 and 3 can be outlined in view of Theorem 5. Once again to avoid repetition, the details of possible corollaries are not included here.

Remark 5 Theorem 5 improves the results of Zhang and Song [79] and Razani and Yazdi [54].

We note that the main theorem of Altun et al. [6] is a consequence of Theorem 2 by taking $\psi(t) = t - \varphi(t)$.

Corollary 4 [6] *Let $A, B, S,$ and T be self mappings of a metric space (X, d). Suppose that the following hypotheses hold:*

(1) *$A(X) \subset T(X),\ B(X) \subset S(X)$,*
(2) *there exists a right continuous function $\psi : [0, \infty) \to [0, \infty)$ with $\psi(0) = 0$ and $\psi(s) < s$ for all $s > 0$ such that*

$$\int_0^{d(Ax, By)} \phi(t)dt \leq \psi \left(\int_0^{M(x, y)} \phi(t)dt \right), \qquad (2.16)$$

for all $x, y \in X$, where

$$M(x, y) = \int_0^{\max\{d(Ax, Sx), d(By, Ty), d(Sx, Ty), [d(Ax, Ty)+d(By, Sx)]/2\}} \phi(t)dt$$

and $\phi : [0, +\infty) \to [0, +\infty)$ is a Lebesgue-integrable mapping which is summable and such that (2.2) holds.

If one of $A(X),\ B(X),\ S(X)$ or $T(X)$ is a complete subspace of X, then

(a) *A and S have a coincidence point, or*
(b) *B and T have a coincidence point.*

Further, if S and A as well as T and B are weakly compatible, then

(c) *$A, B, S,$ and T have a unique common fixed point.*

Remark 6 Corollary 4 is a generalization of the main theorem of [17], Theorem 2 of [58], and Theorem 2 of [78].

3 Example and Application

3.1 Illustrative Example

Here we support our result by the following example.

Example 1 Let $X = \{0, 1, 2, \ldots\}$ and $d : X \times X \to X$ be given by

$$d(x, y) = \begin{cases} 0 & \text{if } x = y; \\ x + y & \text{if } x \neq y. \end{cases}$$

Also, define the mappings $A, B, S, T : X \to X$ by

$$Ax = \begin{cases} 0 & \text{if } x = 0; \\ x + 1 & \text{if } x \neq 0; \end{cases} \quad Bx = \begin{cases} 0 & \text{if } x = 0; \\ x + 2 & \text{if } x \neq 0; \end{cases}$$

$$Sx = \begin{cases} 0 & \text{if } x = 0; \\ 2x + 2 & \text{if } x \neq 0; \end{cases} \quad Tx = \begin{cases} 0 & \text{if } x = 0; \\ 2x + 1 & \text{if } x \neq 0. \end{cases}$$

Consider two functions $\phi, \varphi : [0, +\infty) \to [0, +\infty)$ given by $\phi(t) = 2t$ and $\varphi(t) = \sqrt{t}$. We will show that all the hypotheses of Theorem 2 are satisfied.

Proof The following facts are clear:

(1) (X, d) is a metric space;
(2) $\varphi \in \Phi$;
(3) ϕ is a Lebesgue-integrable mapping which is summable and non-negative such that (2.2) holds;
(4) the pairs (A, S) and (B, T) share the (CLR_{ST}) property.

Consequently, we have only to show that condition (2.1) holds. Then let $x, y \in X$ with $y \leq x$ and divide the proof into the following cases:

Case 1 Assume $y = x = 0$. In this case, condition (2.1) holds trivially since $Ax = By = Sx = Ty = 0$.

Case 2 Assume $y = 0$ and $x > 0$. Then we have $Ax = x + 1$, $By = 0$, $Sx = 2x + 2$, and $Ty = 0$. Consequently, we obtain

$$d(Ax, By) = d(x + 1, 0) = x + 1$$

and

$$\begin{aligned}
&\max\{d(Ax, Sx), d(By, Ty), d(Sx, Ty), [d(Ax, Ty) + d(By, Sx)]/2\} \\
&= \max\{d(x + 1, 2x + 2), d(B0, T0), d(2x + 2, 0), \\
&\quad [d(x + 1, 0) + d(0, 2x + 2)]/2\} \\
&= \max\{3x + 3, 0, 2x + 2, [3x + 3]/2\} \\
&= 3x + 3.
\end{aligned}$$

It follows that

$$\int_0^{d(Ax,By)} 2t\,dt = \int_0^{x+1} 2t\,dt = (x+1)^2, \ M(x,y) = \int_0^{3x+3} 2t\,dt = (3x+3)^2,$$

and $\varphi(M(x,y)) = 3x+3$.

Since $(x+1)^2 \leq (3x+3)^2 - (3x+3)$, then condition (2.1) holds.

Case 3 Assume $x > y > 0$. We need to consider two subcases:

Subcase 1. If $x = y+1$, or equivalently $y = x-1$, then we have

$$d(Ax, By) = d(Ax, B(x-1)) = d(x+1, x+1) = 0.$$

Therefore, condition (2.1) holds trivially again.

Subcase 2. If $x > y+1$, then we have $Ax = x+1$, $By = y+2$, $Sx = 2x+2$, and $Ty = 2y+1$.

Now, if $x = 2y$, then

$$d(Ax, By) = d(A(2y), By) = d(2y+1, y+2) = 3y+3$$

and

$$\begin{aligned}
&\max\{d(A(2y), S(2y)), d(By, Ty), d(S(2y), Ty),\\
&\quad [d(A(2y), Ty) + d(By, S(2y))]/2\}\\
&= \max\{d(2y+1, 4y+2), d(y+2, 2y+1), d(4y+2, 2y+1),\\
&\quad [d(2y+1, 2y+1) + d(y+2, 4y+2)]/2\}\\
&= \max\{6y+3, 3y+3, [9y+6]/2\}\\
&= 6y+3.
\end{aligned}$$

Therefore, we get

$$\int_0^{d(Ax,By)} 2t\,dt = \int_0^{3y+3} 2t\,dt = (3y+3)^2, \ M(x,y) = \int_0^{6y+3} 2t\,dt = (6y+3)^2$$

and $\varphi(M(x,y)) = 6y+3$.

Since $(3y+3)^2 \leq (6y+3)^2 - (6y+3)$, then condition (2.1) holds.

On the other hand, if $x < 2y$ then

$$d(Ax, By) = d(x+1, y+2) = x+y+3$$

and

$$\max\{d(Ax, Sx), d(By, Ty), d(Sx, Ty), [d(Ax, Ty) + d(By, Sx)]/2\}$$
$$= \max\{d(x + 1, 2x + 2), d(y + 2, 2y + 1), d(2x + 2, 2y + 1),$$
$$[d(x + 1, 2y + 1) + d(y + 2, 2x + 2)]/2\}$$
$$= \max\{3x + 3, 3y + 3, 2x + 2y + 3, [3x + 3y + 6]/2\}$$
$$= 2x + 2y + 3.$$

It follows that

$$\int_0^{d(Ax,By)} 2t \, dt = \int_0^{x+y+3} 2t \, dt = (x + y + 3)^2,$$

$$M(x, y) = \int_0^{2x+2y+3} 2t \, dt = (2x + 2y + 3)^2$$

and $\varphi(M(x, y)) = 2x + 2y + 3$.
Since $(x + y + 3)^2 \leq (2x + 2y + 3)^2 - (2x + 2y + 3)$, then condition
(2.1) holds.
Finally, if $x > 2y$, then

$$d(Ax, By) = d(x + 1, y + 2) = x + y + 3$$

and

$$\max\{d(Ax, Sx), d(By, Ty), d(Sx, Ty), [d(Ax, Ty) + d(By, Sx)]/2\}$$
$$= \max\{d(x + 1, 2x + 2), d(y + 2, 2y + 1), d(2x + 2, 2y + 1),$$
$$[d(x + 1, 2y + 1) + d(y + 2, 2x + 2)]/2\}$$
$$= \max\{3x + 3, 3y + 3, 2x + 2y + 3, [3x + 3y + 6]/2\}$$
$$= 3x + 3.$$

Therefore, we have

$$\int_0^{d(Ax,By)} 2t \, dt = \int_0^{x+y+3} 2t \, dt = (x + y + 3)^2,$$

$$M(x, y) = \int_0^{3x+3} 2t \, dt = (3x + 3)^2,$$

and $\varphi(M(x, y)) = 3x + 3$.
Since $y < x - 1$, then $(x + y + 3)^2 \leq (2x + 2)^2 \leq (3x + 3)^2 - (3x + 3)$
and therefore condition (2.1) holds.

Case 4 Assume $x = y > 0$. Consequently, we get

$$d(Ax, Bx) = d(x + 1, x + 2) = 2x + 3$$

and

$$\begin{aligned}
&\max\{d(Ax, Sx), d(Bx, Tx), d(Sx, Tx), [d(Ax, Tx) + d(Bx, Sx)]/2\}\\
&\quad = \max\{d(x + 1, 2x + 2), d(x + 2, 2x + 1), d(2x + 2, 2x + 1),\\
&\qquad [d(x + 1, 2x + 1) + d(x + 2, 2x + 2)]/2\}\\
&\quad = \max\{3x + 3, 4x + 3, [6x + 6]/2\}\\
&\quad = 4x + 3.
\end{aligned}$$

Therefore, we have

$$\int_0^{d(Ax, Bx)} 2t\,dt = \int_0^{2x+3} 2t\,dt = (2x + 3)^2, \ M(x, x) = \int_0^{4x+3} 2t\,dt = (4x + 3)^2$$

and $\varphi(M(x, x)) = 4x + 3$.
Since $(2x + 3)^2 \leq (4x + 3)^2 - (4x + 3)$, then condition (2.1) holds.

Thus, the mappings A, B, S, and T satisfy all the hypotheses of Theorem 2. Here 0 is the common fixed point of A, B, S, and T.   □

3.2 Application to Functional Equation

Let U and V be Banach spaces, $W \subseteq U$ be a state space and $D \subseteq V$ be a decision space. Now, using the fixed point theorems obtained in the previous Section, we study the solvability of the following functional equation arising in dynamic programming (see [11,13,14]):

$$Q(x) = \sup_{y \in D}\{f(x, y) + K(x, y, Q(\tau(x, y)))\}, \quad x \in W, \tag{3.1}$$

where $\tau : W \times D \to W$, $f : W \times D \to \mathbb{R}$, $K : W \times D \times \mathbb{R} \to \mathbb{R}$.
Let $B(W)$ denote the space of all bounded real-valued functions on W. Clearly, this space endowed with the metric given by

$$d(h, k) = \sup_{x \in W} |h(x) - k(x)|, \quad \text{for all } h, k \in B(W)$$

is a complete metric space.
We will prove the following theorem.

Theorem 6 *Let* $K : W \times D \times \mathbb{R} \to \mathbb{R}$ *and* $f : W \times D \to \mathbb{R}$ *be two bounded functions and let* $A : B(W) \to B(W)$ *be defined by*

$$Ah(x) = \sup_{y \in D} \{ f(x, y) + K(x, y, h(\tau(x, y))) \}, \tag{3.2}$$

for all $h \in B(W)$ *and* $x \in W$. *Assume that the following condition holds:*

$$\int_0^{|K(x,y,h(x)) - K(x,y,k(x))|} \phi(t) \mathrm{d}t \le \rho M(Ah, Ak), \tag{3.3}$$

where

$$M(Ah, Ak) = \int_0^{\max\{d(Ah(x),h(x)), d(Ak(x)k(x)), d(h(x),k(x)), [d(Ah(x),k(x)) + d(Ak(x),h(x))]/2\}} \phi(t) dt,$$

$h, k \in B(W)$, $x \in W$, $y \in D$, $\rho \in (0, 1)$, *and* $\phi : [0, +\infty) \to [0, +\infty)$ *is a Lebesgue-integrable mapping which is summable and such that* (2.2) *holds. Then the functional equation* (3.1) *has a unique bounded solution.*

Proof Since f and K are bounded, there exists a positive number Λ such that

$$\sup\{ |f(x, y)|, |K(x, y, z)| : (x, y, z) \in W \times D \times \mathbb{R} \} \le \Lambda.$$

Now, by using a property of the integration theory ([31], Theorem 12.34) and the properties of ϕ, we conclude that for each positive number ε, there exists a positive number $\delta(\varepsilon)$ such that

$$\int_\Omega \phi(t) \mathrm{d}t \le \varepsilon, \quad \text{for all } \Omega \subseteq [0, 2\Lambda] \text{ with } m(\Omega) \le \delta(\varepsilon), \tag{3.4}$$

where $m(\Omega)$ is the Lebesgue measure of Ω.

Let $x \in W$ and $h_1, h_2 \in B(W)$, then there exist $y_1, y_2 \in D$ such that

$$Ah_1(x) < f(x, y_1) + K(x, y_1, h_1(\tau(x, y_1))) + \delta(\varepsilon), \tag{3.5}$$
$$Ah_2(x) < f(x, y_2) + K(x, y_2, h_2(\tau(x, y_2))) + \delta(\varepsilon), \tag{3.6}$$
$$Ah_1(x) \ge f(x, y_2) + K(x, y_2, h_1(\tau(x, y_2))), \tag{3.7}$$
$$Ah_2(x) \ge f(x, y_1) + K(x, y_1, h_2(\tau(x, y_1))). \tag{3.8}$$

Then from (3.5) and (3.8), it follows easily that

$$Ah_1(x) - Ah_2(x) < K(x, y_1, h_1(\tau(x, y_1))) - K(x, y_1, h_2(\tau(x, y_1))) + \delta(\varepsilon)$$
$$\le |K(x, y_1, h_1(\tau(x, y_1))) - K(x, y_1, h_2(\tau(x, y_1)))| + \delta(\varepsilon).$$

Hence, we get

$$Ah_1(x) - Ah_2(x) < |K(x, y_1, h_1(\tau(x, y_1))) - K(x, y_1, h_2(\tau(x, y_1)))| + \delta(\varepsilon).$$
(3.9)

Similarly, from (3.6) and (3.7) we obtain

$$Ah_2(x) - Ah_1(x) < |K(x, y_2, h_1(\tau(x, y_2))) - K(x, y_2, h_2(\tau(x, y_2)))| + \delta(\varepsilon).$$
(3.10)

Therefore, from (3.9) and (3.10) we have

$$|Ah_1(x) - Ah_2(x)| < \sup_{y \in D} |K(x, y, h_1(\tau(x, y))) - K(x, y, h_2(\tau(x, y)))| + \delta(\varepsilon).$$
(3.11)

In view of (3.3), (3.4), and (3.11), it follows easily that

$$\int_0^{d(Ah_1(x), Ah_2(x))} \phi(t) \mathrm{d}t \le \rho M(Ah_1, Ah_2) + \varepsilon.$$

Since the above inequality is true for any $x \in W$ and $\varepsilon > 0$ is taken arbitrary, then we conclude immediately that

$$\int_0^{d(Ah_1(x), Ah_2(x))} \phi(t) \mathrm{d}t \le \rho M(Ah_1, Ah_2).$$

Thus, all the hypotheses of the Corollary 1 are satisfied with $S = I_{B(W)}$, the identity mapping on $B(W)$ and $\varphi : [0, +\infty) \to [0, +\infty)$ given by $\varphi(t) = (1 - \rho)t$ for all $t \ge 0$. Therefore, there is a unique bounded solution of the functional equation (3.1).
□

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