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Local and Global Existence of Mild Solution for Impulsive Fractional Stochastic Differential Equations

P. Balasubramaniam · N. Kumaresan · K. Ratnavelu · P. Tamilalagan

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Abstract In this paper, the local and global existence of mild solutions are studied for impulsive fractional semilinear stochastic differential equation with nonlocal condition in a Hilbert space. The results are obtained by employing fixed-point technique and solution operator. In many existence results for stochastic fractional differential systems, the value of α is restricted to $\frac{1}{2} < \alpha \leq 1$; the aim of this manuscript is to extend the results which are valid for all values of $\alpha \in (0, 1)$. An example is provided to illustrate the obtained theoretical results.

Keywords Fractional stochastic differential equation · Mild solution · Fixed-point theorem · Impulsive condition

Mathematics Subject Classification 26A33 · 35R12 · 39A50 · 47H10

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P. Balasubramaniam (✉) · P. Tamilalagan
Department of Mathematics, Gandhigram Rural Institute-Deemed University,
Gandhigram 624 302, Tamil Nadu, India
e-mail: balugru@gmail.com

P. Tamilalagan
e-mail: mathematicstamil@gmail.com

N. Kumaresan · K. Ratnavelu
Institute of Mathematical Sciences, Faculty of Science, University of Malaya,
50603 Kuala Lumpur, Malaysia
e-mail: drnk2008@gmail.com

K. Ratnavelu
e-mail: kuru052001@gmail.com

1 Introduction

Fractional differential equations (FDEs) are viewed an excellent tool for describing real-life phenomena, which have memory and hereditary properties. The efficiency of describing the real-life phenomena by FDE is more accurate than the classical differential equations. Nowadays, it is the most attracted area of research. The existence of mild solution and other qualitative and quantitative properties of such FDE models of real-life phenomena have been studied by many researchers; as a part of it, several authors have established the existence of mild solutions for differential equations with fractional order see [3,6,8,10,15,16,20,25,31–34]. FDE are considered as an alternative model to nonlinear differential equations and can be found many applications in the areas of turbulence and fluid dynamics, stochastic dynamical systems, plasma physics and controlled thermonuclear fusion, nonlinear control theory, image processing, nonlinear biological systems, astrophysics, etc. (for more details, see [12,19,24]).

Stochastic differential equations play a vital role in mathematical modeling of real-life phenomena when noises are non-negligible. Besides, noise or stochastic perturbation is unavoidable and omnipresent in nature as well as in man-made systems [18]. Therefore, it is of great significance to import the stochastic effects into the investigation of fractional differential systems [2,23].

An ordinary differential equation coupled with impulsive effects is considered as impulsive differential equations, and it was introduced by Milman and Myshkis in the year 1960. The perturbations such as earthquake, harvesting, shock, etc. can be well-approximated as instantaneous change of state or impulses, and they can be modeled by impulsive differential equations. The dynamics of process in which sudden discontinuous jumps occurs in the real-world problems can be described by the impulsive differential equations. Such processes are naturally seen in biology, physics, engineering, etc. Moreover, a simple impulsive differential equation may exhibit several new phenomena such as rhythmical beating, merging of solution, and noncontinuability of solutions; hence, it has developed tremendously for more details see [4,5,11,13,14,17,21,22].

In [21], Ouahab studied the local and global existence and uniqueness results for first-order impulsive functional differential equations with multiple delay by means of the fixed-point theorems due to Schaefer and a nonlinear alternative of Leray–Schauder. The local and global existence of mild solution for a class of impulsive fractional semilinear integro-differential equations has been studied by Rashid and Al-Omari [25]. Very recently Chauhan and Dabas [5] discussed the local and global existence of mild solution for an impulsive fractional functional integro-differential equations with nonlocal condition. To the best of authors knowledge, there is no work still reported on the local and global existence of mild solution for impulsive fractional semilinear stochastic differential equation with nonlocal condition. Hence, the main objective of this manuscript is to fill this gap. Further, the existence results obtained in many works are valid only for $\frac{1}{2} < \alpha \leq 1$; the aim of this manuscript is to provide the result, which is valid for all values of $\alpha \in (0,\ 1)$.

In this paper, we study the local and global existence of mild solution for the following impulsive fractional semilinear stochastic differential equation (1) with nonlocal condition in a Hilbert space using fixed-point technique and solution operator:

$$\begin{cases} {}^cD_t^{\alpha}x(t) = -Ax(t) + f(t, x_t) + \int_0^t \sigma(t, s, x_s)\,\mathrm{d}w(s), \quad t \in J := [0, b], \quad t \neq t_k, \\ \triangle x(t_k) = I_k\left(x\left(t_k^-\right)\right), \quad k = 1, 2, \ldots, m, \\ h(x) = \phi_0 \quad \text{on } [-r, 0], \end{cases} , \tag{1}$$

where ${}^cD_t^{\alpha}x(t)$, $0 < \alpha < 1$ is the Caputo fractional derivative, $-A$ is sectorial operator, here $x(\cdot)$ takes the values in a Hilbert space H with inner product $\langle\cdot, \cdot\rangle$ and norm $\|\cdot\|$. Let K be another separable Hilbert space with inner product $\langle\cdot, \cdot\rangle_K$ and norm $\|\cdot\|_K$. Suppose $w(t)$ is K-valued Brownian motion or Wiener process with a finite trace nuclear covariance operator $Q \geq 0$. We employ the same notation $\|\cdot\|$ for the norm of $L(K, H)$, where $L(K, H)$ denotes the space of all bounded linear operators from K into H. Let the nonlinear maps $f : J \times \mathrm{PC}_0 \to H$ and $\sigma : J \times J \times \mathrm{PC}_0 \to L(K, H)$ be continuous, where $\mathrm{PC}_0 = \mathrm{PC}([-r, 0], H)$ and for any $x \in \mathrm{PC}_b = \mathrm{PC}([-r, b], H)$, $t \in J$, we define the element x_t of PC_0 by $x_t(\theta) = x(t+\theta)$, $\theta \in [-r, 0]$. The function $\phi_0 \in \mathrm{PC}_b$ and the map h is defined from PC_b into PC_b.

This paper is organized as follows: In Sect. 2, we give some preliminaries, basic definitions and results, which will be used throughout this paper. In Sect. 3, the proof for the local existence of mild solution is provided . In Sect. 4, we prove the global existence of mild solution. Section 5 illustrates our theoretical results by an example.

2 Preliminaries

In this section, some basic definitions, notations, and lemmas are provided that will be used in the sequel. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space furnished with a complete family of right continuous increasing sub-σ-algebras $\{\mathcal{F}_t : t \in J\}$ satisfying $\mathcal{F}_t \subset \mathcal{F}$. Let $x(t) : \Omega \to H$ be a continuous $\mathcal{F}_t$-adapted, H-valued stochastic process. Let $\{\zeta_n\}_{n=1}^{\infty}$ be a complete orthonormal basis of K. Suppose that $w(t)$, $t \geq 0$, is a cylindrical K-valued Wiener process with finite trace nuclear covariance operator $Q \geq 0$, denote $\mathrm{Tr}(Q) = \sum_{n=1}^{\infty} \lambda_n < \infty$, which satisfies that $Q\zeta_n = \lambda_n\zeta_n$. Indeed $w(t) = \sum_{n=1}^{\infty} \sqrt{\lambda_n} w_n(t)\zeta_n$, where $\{w_n(t)\}_{n=1}^{\infty}$ are mutually independent one-dimensional standard Wiener processes. Let $\varphi \in L(K, H)$, and define

$$\|\varphi\|_Q^2 = \mathrm{Tr}(\varphi Q \varphi^*) = \sum_{n=1}^{\infty} \left\|\sqrt{\lambda_n}\varphi\zeta_n\right\|^2.$$

If $\|\varphi\|_Q < \infty$, then φ is called Q-Hilbert–Schmidt operator. Let $L_Q(K, H)$ denote the space of all Q-Hilbert–Schmidt operators from $\varphi : K \to H$. The completion $L_Q(K, H)$ of $L(K, H)$ with respect to the topology induced by the norm $\|\cdot\|_Q$, where $\|\varphi\|_Q^2 = \langle\varphi, \varphi\rangle$ is a Hilbert space with the above norm topology. For more details of this section, the reader can refer [1,9,18,24].

Definition 2.1 [24] The Riemann–Liouville fractional integral of order $\alpha > 0$ for a function $f : \mathbb{R}_+ \to \mathbb{R}$ and $f \in L^1(\mathbb{R}_+, X)$ is defined as

$$J_t^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau) \mathrm{d}\tau, \quad \alpha > 0, \quad t > 0,$$

where $\Gamma(\cdot)$ is the Euler gamma function.

Definition 2.2 [24] The Caputo derivative of fractional order α of a function f : $[0, \infty) \to \mathbb{R}$ is defined as

$${}^c D_t^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{f^n(\tau)}{(t-\tau)^{\alpha-n+1}} \mathrm{d}\tau = I^{n-\alpha} f^{(n)}(t),$$

for $n-1 \leq \alpha < n$, $n \in N$.

Definition 2.3 [24] The two parameter function of the Mittag–Leffler type is defined as

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)} = \frac{1}{2\pi i} \int_C \frac{\mu^{\alpha-\beta} \mathrm{e}^\mu}{\mu^\alpha - z} \mathrm{d}\mu, \quad \alpha, \beta > 0, \quad z \in \mathbb{C},$$

where C is a contour which starts and ends at $-\infty$ and encircles the disk $|\mu| \leq |z|^{\frac{1}{2}}$ counter clockwise.

The Laplace transform of the Mittag–Leffler function is given by

$$\int_0^\infty \mathrm{e}^{-\lambda t} t^{\alpha k+\beta-1} E_{\alpha,\beta}^{(k)} \left(\pm a t^\alpha\right) \mathrm{d}t = \frac{k! \lambda^{\alpha-\beta}}{(\lambda^\alpha \mp a)^{k+1}}, \quad \mathrm{Re}(\lambda) > |a|^{\frac{1}{\alpha}}.$$

Definition 2.4 [9] A closed and linear operator A is said to be sectorial if there are constants $\omega \in \mathbb{R}$, $\theta \in [\frac{\pi}{2}, \pi]$, $M > 0$ such that the following conditions are satisfied

(i) $\rho(A) \subset \sum_{(\theta,\omega)} = \{\lambda \in \mathcal{C} : \lambda \neq \omega, |\arg(\lambda - \omega)| < \theta\}$,
(ii) $\|R(\lambda, A)\| \leq \frac{M}{|\lambda-\omega|}$, $\lambda \in \sum_{(\theta,\omega)}$.

Definition 2.5 [31] Let A be a closed and linear operator with the domain $D(A)$ defined in a Banach space X. Let $\rho(A)$ be the resolvent set of A. We say that A is the generator of an α-resolvent family if there exists $\omega \geq 0$ and a strongly continuous function $S_\alpha : \mathbb{R}_+ \to \mathcal{L}(X)$, where $\mathcal{L}(X)$ is a Banach space of all bounded linear operator from X into X and the corresponding norm is denoted by $\|\cdot\|$ such that $\{\lambda^\alpha : \mathrm{Re}(\lambda) > \omega\} \subset \rho(A)$ and

$$\left(\lambda^\alpha I - A\right)^{-1} x = \int_0^\infty \mathrm{e}^{-\lambda t} S_\alpha(t) x \mathrm{d}t, \quad \mathrm{Re}(\lambda) > \omega, \quad x \in X,$$

where $S_\alpha(t)$ is called the α-resolvent family generated by A.

Definition 2.6 [7] Let A be a closed linear operator with the domain $D(A)$ defined in a Banach space X and $\alpha > 0$. We say that A is the generator of a solution operator if there exists $\omega \geq 0$ and a strongly continuous function $T_\alpha : \mathbb{R}_+ \to \mathcal{L}(X)$ such that $\{\lambda^\alpha : \mathrm{Re}(\lambda) > \omega\} \subset \rho(A)$ and

$$\lambda^{\alpha-1}\big(\lambda^\alpha I - A\big)^{-1}x = \int_0^\infty \mathrm{e}^{-\lambda t} T_\alpha(t)x\mathrm{d}t, \quad \mathrm{Re}(\lambda) > \omega, \quad x \in X,$$

where $T_\alpha(t)$ is called the solution operator generated by A.

Definition 2.7 An $\mathcal{F}_t$ adapted stochastic process $x : [-r, b] \to H$ is called a mild solution of (1) if $x(t) = \psi(t)$ on $[-r, 0]$, where $\psi \in \mathrm{PC}_b$ such that $h(\psi) = \phi_0$ on $[-r, 0]$ and satisfies the following conditions:

(i) $x(t)$ is PC_b valued and the restrictions of $x(\cdot)$ to $(t_k, t_{k+1}]$, $k = 1, 2, \ldots, m$ is continuous.
(ii) For each $t \in J$, $x(t)$ satisfies the integral equation

$$x(t) = \begin{cases} T_\alpha(t)\psi(0) + \int_0^t S_\alpha(t-s)\left[f(s, x_s) + \int_0^s \sigma(s, \tau, x_\tau)\,\mathrm{d}w(\tau)\right]\mathrm{d}s, & t \in [0, t_1], \\ T_\alpha(t)\psi(0) + \sum_{i=1}^m T_\alpha(t-t_i)\, I_i\left(x\left(t_i^-\right)\right), & \\ + \int_0^t S_\alpha(t-s)\left[f(s, x_s) + \int_0^s \sigma(s, \tau, x_\tau)\,\mathrm{d}w(\tau)\right]\mathrm{d}s, & t \in (t_k, t_{k+1}], \quad k = 1, 2, \ldots, m, \end{cases}$$

where

$$T_\alpha(t) = E_{\alpha,1}\left(-At^\alpha\right) = \frac{1}{2\pi i}\int_{\hat{B}_r} \mathrm{e}^{\lambda t}\frac{\lambda^{\alpha-1}}{\lambda^\alpha I + A}\mathrm{d}\lambda,$$
$$S_\alpha(t) = t^{\alpha-1}E_{\alpha,\alpha}\left(-At^\alpha\right) = \frac{1}{2\pi i}\int_{\hat{B}_r} \mathrm{e}^{\lambda t}\frac{1}{\lambda^\alpha I + A}\mathrm{d}\lambda,$$

where $\hat{B}_r$ denotes the Bromwich path, $S_\alpha(t)$ is called α-resolvent family, and $T_\alpha(t)$ is the solution operator, both are generated by A.

The following assumptions are assumed to establish the main results:

(A_1) The function $f : J \times H \to H$ is continuous, and there exists a constant N_1 such that $E\|f(t, x) - f(t, y)\|^2 \leq N_1 E\|x - y\|^2$ for all $x, y \in H$.
(A_2) The function $g : D \times H \to L(K, H)$ is continuous, and there exists a constant N_2 such that $\int_0^t E\|\sigma(t, s, x) - \sigma(t, s, y)\|^2\mathrm{d}s \leq N_2 E\|x - y\|^2$ for all $x, y \in H$, where $D = J \times J = \{(t, s), t, s \in J\}$.
(A_3) The nonlinear map $h : \mathrm{PC}_b \to \mathrm{PC}_b$ is such that for any x_1 and x_2 in PC_b with $x_1 = x_2$ on $[-r, 0]$, $h(x_1) = h(x_2)$ on $[-r, 0]$.
(A_4) The functions $I_k : H \to H$ are continuous, and there exists a constant $\mu > 0$ such that $E\|I_k(x) - I_k(y)\|^2 \leq \mu E\|x - y\|^2$ for all $x, y \in H$, $k = 1, 2, \ldots, m$.
(A_5) The functions $I_k : H \to H$ are continuous, and there exists a constant $\rho > 0$ such that $E\|I_k(x)\|^2 \leq \rho E\|x\|^2$ for all $x \in H$, $k = 1, 2, \ldots, m$.

(A_6) The functions $f : J \times \mathrm{PC}_0 \to H$, $\sigma : D \times \mathrm{PC}_0 \to L(K, H)$, and $I_k : H \to H$, $k = 1, 2, \ldots, m$ are completely continuous.

(A_7) The operator family $\{T_\alpha(t)\}_{t\geq 0}$ and $\{\overline{S}_\alpha(t)\}_{t\geq 0}$ are compact, where $\overline{S}_\alpha(t) = t^{1-\alpha} S_\alpha(t)$. If $A \in \mathcal{A}^\alpha(\theta_0, \omega_0)$ then $\|T_\alpha(t)\|_{L(X)} \leq M e^{\omega t}$ and $\|S_\alpha(t)\|_{L(X)} \leq C e^{\omega t}(1 + t^{\alpha-1})$. Let $M_T = \sup_{0\leq t\leq b} \|T_\alpha(t)\|_{L(X)}$, $M_S = \sup_{0\leq t\leq b} C e^{\omega t}(1 + t^{1-\alpha})$, where $L(X)$ is the Banach space of bounded linear operators from X into X. So we have $\|T_\alpha(t)\|_{L(X)} \leq M_T$ and $\|S_\alpha(t)\|_{L(X)} \leq t^{\alpha-1} M_S$ (for more details see [32]).

3 Local Existence of Mild Solution

Theorem 3.1 *If the conditions (A_1)–(A_5) are satisfied and there exists $x_0 \in \mathrm{PC}_b$ such that $h(x_0) = \phi_0$ on $[-r, 0]$, then for every $\phi_0 \in \mathrm{PC}_b$ there exists a $\tau_0 = \tau_0(\phi_0)$, $0 < \tau_0 < b$ such that the initial value problem (1) has a unique mild solution $x \in \mathrm{PC}([-r, \tau_0], H)$.*

Proof Since we only consider the local solutions, and hence we may assume that $b < \infty$. Let $t' > 0$, $R > 0$ be such that $B_R(x_0) = \{x : E\|x - x_0\|^2_{t'} \leq R\}$, $E\|f(t, x)\|^2_H \leq N_1$ and $\int_0^t E\|\sigma(t, s, x_s)\|^2_{L(K,H)} \mathrm{d}s \leq N_2$ for $0 \leq t \leq t'$ and $x \in B_R(x_0)$. Choose $t'' > 0$ such that $E\|T_\alpha(t)x_0(0) - x_0(0)\|^2_H \leq \frac{R}{15}$ for $0 \leq t \leq t''$ and $E\|x_0(t) - x_0(0)\|^2_H \leq \frac{R}{15}$ for $0 \leq t \leq t''$ and we choose

$$\tau_0 = \min\left\{b,\, t',\, t'',\, \left[\frac{\frac{R}{15} - M_T^2 \rho m}{\frac{M_S^2}{\alpha^2}(N_1 + N_2 \mathrm{Tr}(Q))}\right]^{\frac{1}{2\alpha}}\right\},$$

set $Y = \mathrm{PC}_{\tau_0} = \mathrm{PC}([-r, \tau_0], H)$ and $Y_0 = \{x : x \in Y,\ x = x_0$ on $[-r, 0]$ and $x(t) \in B_R(x_0)$ for $0 \leq t \leq \tau_0\}$. It is clear that Y_0 is a bounded closed convex subset of Y.

We define a mapping $\Phi : Y_0 \to Y$ by

$$(\Phi x)(t) = \begin{cases} x_0(t); \quad t \in [-r, 0], \\ T_\alpha(t)x_0(0) + \sum_{0<t_i<t} T_\alpha(t - t_i) I_i\left(x\left(t_i^-\right)\right), \\ + \int_0^t S_\alpha(t - s)\left[f(s, x_s) + \int_0^s \sigma(s, \tau, x_\tau)\,\mathrm{d}w(\tau)\right]\mathrm{d}s; \quad t \in [0, \tau_0]. \end{cases}$$

For $x \in Y_0$, $t \in [0, \tau_0]$, we have

$$E\|(\Phi x)(t) - x_0(t)\|^2_H \leq 5\Bigg\{E\|T_\alpha(t)x_0(0) - x_0(0)\|^2_H + E\|x_0(t) - x_0(0)\|^2_H$$
$$+E\left\|\sum_{0<t_i<t} T_\alpha(t - t_i) I_i\left(x\left(t_i^-\right)\right)\right\|^2_H$$
$$+M_S^2 \int_0^t (t - s)^{\alpha-1}\mathrm{d}s \int_0^t (t - s)^{\alpha-1} E\|f(s, x_s)\|^2_H\,\mathrm{d}s$$

$$+M_S^2 \int_0^t (t-s)^{\alpha-1} \mathrm{d}s \int_0^t (t-s)^{\alpha-1} \mathrm{Tr}(Q) \int_0^s E \left\| \sigma(s, \tau, x_\tau) \right\|_{L(K,H)}^2 \mathrm{d}\tau \mathrm{d}s \Bigg\}$$

$$\leq 5 \left\{ \frac{R}{15} + \frac{R}{15} + \frac{M_S^2 \tau_0^{2\alpha}}{\alpha^2} (N_1 + N_2 \mathrm{Tr}(Q)) + M_T^2 \rho m \right\}$$

$$E \left\| (\Phi x)(t) - x_0(t) \right\|_H^2 \leq R.$$

Thus $\Phi : Y_0 \to Y_0$, if we choose $\tau_0 > 0$ such that

$$3 \left[M_T^2 \mu m + \frac{M_S^2 \tau_0^{2\alpha}}{\alpha^2} (N_1 + N_2 \mathrm{Tr}(Q)) \right] < 1. \tag{2}$$

Now, let $x, \ y \in Y_0$, then

$$E\|(\Phi x)(t) - (\Phi y)(t)\|_H^2 \leq 3 \Bigg\{ \sum_{0<t_i<t} E \left\| T_\alpha(t-t_i) I_i\left(x\left(t_i^-\right)\right) - T_\alpha(t-t_i) I_i\left(y\left(t_i^-\right)\right) \right\|^2$$

$$+ M_S^2 \int_0^t (t-s)^{\alpha-1} \mathrm{d}s \int_0^t (t-s)^{\alpha-1} E \left\| f(s, x_s) - f(s, y_s) \right\|^2 \mathrm{d}s$$

$$+ M_S^2 \int_0^t (t-s)^{\alpha-1} \mathrm{d}s \int_0^t (t-s)^{\alpha-1}$$

$$\times \mathrm{Tr}(Q) \int_0^s E \left\| \sigma(s, \tau, x_\tau) - \sigma(s, \tau, y_\tau) \right\|^2 \mathrm{d}\tau \mathrm{d}s \Bigg\}$$

$$E\|(\Phi x)(t) - (\Phi y)(t)\|_H^2 \leq 3 \left[M_T^2 \mu m + \frac{M_S^2 \tau_0^{2\alpha}}{\alpha^2} (N_1 + N_2 \mathrm{Tr}(Q)) \right] E\|x - y\|^2.$$

It follows from (2) and by Banach contraction mapping principle that there exists a unique $x \in Y_0$ such that x is a mild solution of (1) on $[-r, \ \tau_0]$. This completes the proof. □

Theorem 3.2 *If the conditions* (A_5)–(A_7) *are satisfied and there exists* $x_0 \in \mathrm{PC}_b$ *such that* $h(x_0) = \phi_0$ *on* $[-r, \ 0]$, *then for every* $\phi_0 \in \mathrm{PC}_b$, *there exists a* $\tau_0 = \tau_0(\phi_0)$, $0 < \tau_0 < b$ *such that the initial value problem* (*1*) *has a mild solution* $x \in \mathrm{PC}([-r, \ \tau_0], \ H)$.

Proof We use Schauder's fixed-point theorem for the proof of this theorem. Let $\Phi : Y_0 \to Y_0$ be defined as in Theorem 3.1.

Step 1: To show that, Φ is continuous from Y_0 into Y_0. Let $\{x^n\}$ be a sequence in Y_0, such that $x^n \to x$ in Y_0. Then $f(t, \ x_t^n) \to f(t, \ x_t)$ and $\sigma(t, \ s, \ x_s^n) \to \sigma(t, \ s, \ x_s)$ as $n \to \infty$, because the functions f and σ are continuous on $J \times \mathrm{PC}_0$ and $D \times \mathrm{PC}_0$, respectively. Now for every $t \in [0, \ \tau_0]$, we can estimate

$$E \left\| \left(\Phi x^n\right)(t) - (\Phi x)(t) \right\|_H^2 \leq 3 \Big\{ M_T^2 E \left\| I_i\left(x^n\left(t_i^-\right)\right) - I_i\left(x\left(t_i^-\right)\right) \right\|_H^2$$

$$+M_S^2 b \int_0^t (t-s)^{2(\alpha-1)} E \left\| f\left(s, \ x_s^n\right) - f(s, \ x_s) \right\|_H^2 \mathrm{d}s$$

$$+M_S^2 b \int_0^t (t-s)^{2(\alpha-1)}$$
$$\times \mathrm{Tr}(Q)\left[\int_0^s E\left\|\sigma\left(s,\ \tau,\ x_\tau^n\right)-\sigma\left(s,\ \tau,\ x_\tau\right)\right\|_H^2 \mathrm{d}\tau\right]\mathrm{d}s\Bigg\},$$

now, we use the fact that,

$$(t-s)^{2(\alpha-1)} E\left\|f\left(s,\ x_s^n\right)-f\left(s,\ x_s\right)\right\|^2$$
$$\leq 2N_1(t-s)^{2(\alpha-1)} \in L^1\left(J,\ \mathbb{R}^+\right),$$
$$(t-s)^{2(\alpha-1)} \int_0^s E\left\|\sigma\left(s,\ \tau,\ x_\tau^n\right)-\sigma\left(s,\ \tau,\ x_\tau\right)\right\|^2 \mathrm{d}\tau\mathrm{d}s$$
$$\leq 2N_2(t-s)^{2(\alpha-1)} \in L^1\left(J,\ \mathbb{R}^+\right),$$

and by means of Lebesgue dominated convergence theorem, we obtain

$$\int_0^t (t-s)^{2(\alpha-1)} E\left\|f\left(s,\ x_s^n\right)-f\left(s,\ x_s\right)\right\|_H^2 \mathrm{d}s \rightarrow 0,$$
$$\int_0^t (t-s)^{2(\alpha-1)} \int_0^s E\left\|\sigma\left(s,\ \tau,\ x_\tau^n\right)-\sigma\left(s,\ \tau,\ x_\tau\right)\right\|_H^2 \mathrm{d}\tau\mathrm{d}s \rightarrow 0.$$

Hence, $\lim_{n\to\infty} E\|\Phi x^n - \Phi x\|_{\tau_0}^2 = 0$. Since the functions $I_k,\ k = 1,\ 2, \ldots, m$ are continuous. This means that Φ is continuous.
Step 2: We show that $\Phi(Y_0) = \{\Phi x : x \in Y_0\}$ be an equicontinuous family of functions. For $\tau_0 > \tau_2 > \tau_1 > 0$, we have

$$E\left\|(\Phi x)(\tau_2)-(\Phi x)(\tau_1)\right\|_H^2$$
$$\leq 4\Bigg\{E\left\|T_\alpha(\tau_2)x_0(0)-T_\alpha(\tau_1)x_0(0)\right\|_H^2$$
$$+E\left\|\sum_{0<t_k<\tau_2} T_\alpha(\tau_2-t_k) I_k\left(x\left(t_k^-\right)\right)-\sum_{0<t_k<\tau_1} T_\alpha(\tau_1-t_k) I_k\left(x\left(t_k^-\right)\right)\right\|_H^2$$
$$+E\left\|\int_0^{\tau_2} S_\alpha(\tau_2-s) f(s,\ x_s)\mathrm{d}s-\int_0^{\tau_1} S_\alpha(\tau_1-s) f(s,\ x_s)\mathrm{d}s\right\|_H^2$$
$$+E\Bigg\|\int_0^{\tau_2} S_\alpha(\tau_2-s)\left(\int_0^s \sigma(s,\ \tau,\ x_\tau)\mathrm{d}w(\tau)\right)\mathrm{d}s$$
$$-\int_0^{\tau_1} S_\alpha(\tau_1-s)\left(\int_0^s \sigma(s,\ \tau,\ x_\tau)\mathrm{d}w(\tau)\right)\mathrm{d}s\Bigg\|_H^2\Bigg\}$$
$$\leq 6\Bigg\{E\left\|T_\alpha(\tau_2)x_0(0)-T_\alpha(\tau_1)x_0(0)\right\|_H^2$$
$$+\sum_{i=1}^k E\left\|\left(T_\alpha(\tau_2-t_i)-T_\alpha(\tau_1-t_i)\right) I_i\left(x\left(t_i^-\right)\right)\right\|_H^2$$

$$
\begin{aligned}
&+\int_0^{\tau_1} \|S_\alpha(\tau_2-s)-S_\alpha(\tau_1-s)\|_{L(H)}^2\,\mathrm{d}s\int_0^{\tau_1} E\,\|f(s,\,x_s)\|^2\,\mathrm{d}s\\
&+\int_{\tau_1}^{\tau_2} \|S_\alpha(\tau_2-s)\|_{L(H)}\,\mathrm{d}s\int_{\tau_1}^{\tau_2} \|S_\alpha(\tau_2-s)\|_{L(H)}\,E\,\|f(s,\,x_s)\|_H^2\,\mathrm{d}s\\
&+\int_0^{\tau_1} \|S_\alpha(\tau_2-s)-S_\alpha(\tau_1-s)\|_{L(H)}^2\,\mathrm{d}s\int_0^{\tau_1} \mathrm{Tr}(Q)\\
&\times\left(\int_0^s E\,\|\sigma(s,\,\tau,\,x_\tau)\|_{L(K,H)}^2\,\mathrm{d}\tau\right)\mathrm{d}s\\
&+\int_{\tau_1}^{\tau_2} \|S_\alpha(\tau_2-s)\|_{L(H)}\,\mathrm{d}s\int_{\tau_1}^{\tau_2} \|S_\alpha(\tau_2-s)\|_{L(H)}\,\mathrm{Tr}(Q)\\
&\times\left(\int_0^s E\,\|\sigma(s,\,\tau,\,x_\tau)\|_{L(K,H)}^2\,\mathrm{d}\tau\right)\mathrm{d}s\Bigg\}\\
&\le 6\sum_{j=1}^{6} J_j,
\end{aligned}
\tag{3}
$$

where

$$
\begin{aligned}
J_3 &= \int_0^{\tau_1} \|S_\alpha(\tau_2-s)-S_\alpha(\tau_1-s)\|_{L(H)}^2\,\mathrm{d}s\int_0^{\tau_1} E\,\|f(s,\,x_s)\|^2\,\mathrm{d}s,\\
J_3 &\le \tau_1 N_1\int_0^{\tau_1} \|S_\alpha(\tau_2-s)-S_\alpha(\tau_1-s)\|_{L(H)}^2\,\mathrm{d}s,
\end{aligned}
$$

and

$$
\begin{aligned}
J_5 = &\int_0^{\tau_1} \|S_\alpha(\tau_2-s)-S_\alpha(\tau_1-s)\|_{L(H)}^2\,\mathrm{d}s\\
&\times\int_0^{\tau_1} \mathrm{Tr}(Q)\left(\int_0^s E\,\|\sigma(s,\,\tau,\,x_\tau)\|_{L(K,H)}^2\,\mathrm{d}\tau\right)\mathrm{d}s\\
\le\ &\mathrm{Tr}(Q)\tau_1 N_2\int_0^{\tau_1} \|S_\alpha(\tau_2-s)-S_\alpha(\tau_1-s)\|_{L(H)}^2\,\mathrm{d}s.
\end{aligned}
$$

Since $\|S_\alpha(\tau_2-s)-S_\alpha(\tau_1-s)\|_{L(H)}^2 \le 2M_S^2(\tau_0-s)^{2(\alpha-1)} \in L^1(J,\,\mathbb{R}^+)$ for $s\in[0,\,\tau_0]$ and $S_\alpha(\tau_2-s)-S_\alpha(\tau_1-s)\to 0$ as $\tau_1\to\tau_2$ because $S_\alpha(\cdot)$ is strongly continuous. This implies that $\lim_{\tau_1\to\tau_2} J_3=\lim_{\tau_1\to\tau_2} J_5=0$. Also

$$J_4 = \int_{\tau_1}^{\tau_2} \|S_\alpha(\tau_2 - s)\|_{L(H)} \, ds \int_{\tau_1}^{\tau_2} \|S_\alpha(\tau_2 - s)\|_{L(H)} E \|f(s, x_s)\|_H^2 \, ds,$$
$$J_4 \le \frac{M_S^2 N_1 (\tau_2 - \tau_1)^{2\alpha}}{\alpha^2},$$

and

$$J_6 = \int_{\tau_1}^{\tau_2} \|S_\alpha(\tau_2 - s)\|_{L(H)} \, ds \int_{\tau_1}^{\tau_2} \|S_\alpha(\tau_2 - s)\|_{L(H)} \mathrm{Tr}(Q) \times \left(\int_0^s E \|\sigma(s, \tau, x_\tau)\|_{L(K,H)}^2 \, d\tau \right) ds,$$
$$J_6 \le \frac{M_S^2 N_2 \mathrm{Tr}(Q) (\tau_2 - \tau_1)^{2\alpha}}{\alpha^2}.$$

Hence, $\lim_{\tau_1 \to \tau_2} J_4 = 0$ and $\lim_{\tau_1 \to \tau_2} J_6 = 0$. Since T_α is strongly continuous, the continuity of $t \mapsto \|T_\alpha(t)\|_{L(H)}$ allows us to conclude that the right-hand side of (3) is zero as $\tau_1 \to \tau_2$, which implies that $\Phi(Y_0)$ is equicontinuous.
Step 3: Now, we prove Φ_1 is completely continuous operator on H by adopting the method used in [32]. Decompose Φ by $\Phi = \Phi_1 + \Phi_2$, where

$$(\Phi_1 x)(t) = \begin{cases} 0; \quad t \in [-r, 0], \\ \int_0^t S_\alpha(t-s) \left[f(s, x_s) + \int_0^s \sigma(s, \tau, x_\tau) \, dw(\tau) \right] ds; \quad t \in [0, \tau_0], \end{cases}$$
$$(\Phi_2 x)(t) = \begin{cases} x_0(t); \quad t \in [-r, 0], \\ T_\alpha(t) x_0(0) + \sum_{0 < t_i < t} T_\alpha(t - t_i) I_i\left(x\left(t_i^-\right)\right); \quad t \in [0, \tau_0]. \end{cases}$$

From the compactness of $\overline{S_\alpha(\cdot)}$ and (A_6), we can conclude that the set

$$\left\{ \overline{S_\alpha(t-s)} \left[f(s, x_s) + \int_0^s \sigma(s, \tau, x_\tau) \, dw(\tau) \right], \ t, \ s \in [0, \tau_0], \ x \in Y_0 \right\},$$

is relatively compact in H. Furthermore, using the mean value theorem for Bochner integral, we can conclude that $(\Phi_1 x)(t)$ belongs to the set

$$\frac{t^{\alpha+1}}{\alpha} \overline{\mathrm{conv} \left\{ \overline{S_\alpha(t-s)} \left[f(s, x_s) + \int_0^s \sigma(s, \tau, x_\tau) \, dw(\tau) \right], \ t, \ s \in [0, \tau_0], \ x \in Y_0 \right\}},$$

for all $t \in [0, \tau_0]$, where $\mathrm{conv}(\cdot)$ denotes the convex hull. Accordingly, the set $\{\Phi_1 x(t) : x \in Y_0\}$ is relatively compact. Now, for all $t \in [-r, 0]$, $(\Phi_2 x)(t) = x_0(t)$. Since $x_0(t)$ is a fixed function, it follows that $\{\Phi_2 x(t), \ t \in [-r, 0], \ x \in Y_0\}$ is a compact subset of H. But then, for $t \in [0, \tau_0]$ and $x \in Y_0$,

$$\Phi_2 x(t) = T_\alpha(t) x_0(0) + \sum_{0 < t_i < t} T_\alpha(t - t_i) I_i\left(x\left(t_i^-\right)\right).$$

Since $T_\alpha(t)$ is compact for all $t \in [0, \tau_0]$, it follows that the set $G(t) = \{(\Phi_2 x)(t) : t \in [-r, 0], x \in Y_0\}$ is precompact in H, Φ_2 is also compact. Therefore, $\Phi = \Phi_1 + \Phi_2$ is compact. As well, the set $E = \{x \in Y_0 : x = \lambda \Phi x$ for some $0 < \lambda < 1\}$ is bounded, since $E \subset Y_0$ and Y_0 is closed bounded convex set. By Schauder fixed-point theorem, we can conclude that Φ has a fixed point in Y_0 and any fixed point of Φ is a mild solution of (1) on $[-r, \tau_0]$. □

4 Global Existence of Mild Solution

This section consider the global existence of mild solution for the system (1).

Theorem 4.1 *Assume the hypothesis of Theorem 3.2, let $f : [-r, b) \times \mathrm{PC}_0 \to H$ and $\sigma : [-r, b) \times [-r, b) \times \mathrm{PC}_0 \to H$, $0 < b \le \infty$ are continuous and maps bounded sets in $[-r, b) \times \mathrm{PC}_0$ and $[-r, b) \times [-r, b) \times \mathrm{PC}_0$, respectively, into bounded sets in H, then for every $\Phi_0 \in \mathrm{PC}_b$ the initial value problem (1) has a mild solution x on a maximal interval of existence $[-r, t_{\max})$. If $t_{\max} < \infty$ then* $\lim t \uparrow t_{\max} E\|x(t)\|_H = \infty$.

Proof By defining $x(t + \tau_0) = V(t)$, the initial value problem (1) can be translated into the following form:

$$\begin{cases} {}^cD_t^\alpha V(t) + AV(t) = F(t, V_t) + \int_0^t G(t, s, V_s)\,\mathrm{d}w(s), & t \in [0, b - \tau_0], \quad t \ne t_k, \\ \Delta V\left(\widetilde{t_k}\right) = I_k\left(V\left(\widetilde{t_k}\right)\right), & k = 1, 2, \ldots, m, \\ \widetilde{h}(V(t)) = \widetilde{\phi}_0(t), & t \in [-r - \tau_0, 0], \end{cases} , \tag{4}$$

where

$$\begin{aligned} F(t, V_t) &= f(t + \tau_0, V_t), \quad t \in [0, b - \tau_0], \\ G(t, s, V_s) &= \sigma(t + \tau_0, s, V_s), \quad t \in [0, b - \tau_0], \\ \Delta V\left(\widetilde{t_k}\right) &= I_k\left(V\left(\widetilde{t_k}\right)\right), \quad k = 1, 2, \ldots, m, \\ \widetilde{\phi}(t) &= x(t + \tau_0), \end{aligned}$$

and $\widetilde{t_k} = t_k - \tau_0$. Since the functions F, G are bounded functions, by Theorem 3.2, there exists a function $V \in \mathrm{PC}([-r - \tau_0, b - \tau_0], H)$ such that V is a mild solution of (4) on $[-r - \tau_0, \tau_1]$ for some $0 < \tau_1 < b - \tau_0$ and given by

$$V(t) = \begin{cases} T_\alpha(t)V(0) + \sum_{0<\widetilde{t_k}<t} T_\alpha\left(t - \widetilde{t_k}\right) I_k\left(V\left(\widetilde{t_k}\right)\right), \\ + \int_0^t S_\alpha(t-s)\left[F(s, V_s) + \int_0^s G(s, \tau, V_\tau)\,\mathrm{d}w(\tau)\right]\mathrm{d}s, \quad t \in [0, \tau_1], \end{cases}$$

$\widetilde{h}(V(t)) = \widetilde{\phi}(t), \quad t \in [-r - \tau_0, 0]$.

Then

$$\widetilde{x}(t) = \begin{cases} x(t), & t \in [-r, \tau_0], \\ V(t - \tau_0), & t \in [\tau_0, \tau_0 + \tau_1], \end{cases}$$

is a mild solution of (1) on $[-r,\ \tau_0+\tau_1]$. Since $x(t+\tau_0) = V(t)$ thus for $t \in [\tau_0,\ \tau_0+\tau_1]$, we have

$$V\,(t-\tau_0) := x(t) = \begin{cases} T_\alpha\,(t-\tau_0)\,x\,(\tau_0) + \sum_{\tau_0<t_k<t} T_\alpha\,(t-t_k)\,I_k\left(x\left(t_k^-\right)\right), \\ + \int_{\tau_0}^t S_\alpha(t-s)\left[f\,(s,\ x_s) + \int_{\tau_0}^s \sigma\,(s,\ \tau,\ x_\tau)\,\mathrm{d}w(\tau)\right]\mathrm{d}s. \end{cases}$$

We can extend the solution of (1) to the maximal interval $[-r,\ t_{\max})$ by continuing in this way. Now, we can prove, if $t_{\max} < \infty$, then $E\|x(t)\|_H^2 \to \infty$ as $t \to t_{\max}$ by proving $t \to t_{\max}$ implies $\overline{\lim}_{t\to t_{\max}} E\|x(t)\|_H^2 = \infty$. Indeed, if $t \uparrow t_{\max}$ and $\overline{\lim}_{t\uparrow t_{\max}} E\|x(t)\|_H^2 < \infty$, we may assume that $\|T_\alpha(t)\|_{L(H)} \le M_T$ and $E\|x(t)\|_H^2 \le k_1$ for $0 \le t < t_{\max}$ where M_T and k_1 are constants. Now, if $0 < R < t < t' < t_{\max}$, then

$$\begin{aligned}
&E\|x(t') - x(t)\|_H^2 \\
&\quad\le 7\Bigg\{E\left\|T_\alpha(t')x_0(0) - T_\alpha(t)x_0(0)\right\|_H^2 \\
&\qquad + \sum_{t<t_i<t'} E\left\|T_\alpha\left(t'-t_i\right)I_i\left(x\left(t_i^-\right)\right)\right\|_H^2 \\
&\qquad + \sum_{0<t_i<t} \left\|T_\alpha\left(t'-t_i\right) - T_\alpha\,(t-t_i)\right\|_{L(H)}^2 E\left\|I_i\left(x\left(t_i^-\right)\right)\right\|_H^2 \\
&\qquad + \int_0^t \left\|S_\alpha(t'-s) - S_\alpha(t-s)\right\|_{L(H)}^2 \mathrm{d}s \int_0^t E\|f\,(s,\ x_s)\|_H^2\,\mathrm{d}s \\
&\qquad + \int_t^{t'} \left\|S_\alpha(t'-s)\right\|_{L(H)} \mathrm{d}s \int_t^{t'} \left\|S_\alpha(t'-s)\right\|_{L(H)} E\|f\,(s,\ x_s)\|_H^2\,\mathrm{d}s \\
&\qquad + \int_0^t \left\|S_\alpha(t'-s) - S_\alpha(t-s)\right\|_{L(H)}^2 \mathrm{d}s \int_0^t \mathrm{Tr}(Q)\left(\int_0^s E\|\sigma\,(s,\ \tau,\ x_\tau)\|_H^2\,\mathrm{d}\tau\right)\mathrm{d}s \\
&\qquad + \int_t^{t'} \left\|S_\alpha(t'-s)\right\|_{L(H)} \mathrm{d}s \int_t^{t'} \left\|S_\alpha(t'-s)\right\|_{L(H)} \mathrm{Tr}(Q)\left(\int_0^s E\|\sigma\,(s,\ \tau,\ x_\tau)\|_H^2\,\mathrm{d}\tau\right)\mathrm{d}s\Bigg\},
\end{aligned}$$

$$\begin{aligned}
E\left\|x(t')-x(t)\right\|_H^2 &\le 7\Bigg\{E\left\|T_\alpha(t')x_0(0)-T_\alpha(t)x_0(0)\right\|_H^2 + \rho\sum_{0<t_i<t}\left\|T_\alpha\left(t'-t_i\right)-T_\alpha\,(t-t_i)\right\|_{L(H)}^2 \\
&\quad + \rho\sum_{t<t_i<t'}\left\|T_\alpha\left(t'-t_i\right)\right\|_{L(H)}^2 + N_1 t_{\max}\int_0^t \left\|S_\alpha(t'-s) - S_\alpha(t-s)\right\|_{L(H)}^2 \mathrm{d}s \\
&\quad + \frac{N_1(t'-t)^{2\alpha}M_S^2}{\alpha^2} + N_2\mathrm{Tr}(Q)t_{\max}\int_0^t \left\|S_\alpha(t'-s) - S_\alpha(t-s)\right\|_{L(H)}^2 \mathrm{d}s \\
&\quad + \frac{\mathrm{Tr}(Q)N_2M_S^2(t'-t)^{2\alpha}}{\alpha^2}\Bigg\}. \qquad (5)
\end{aligned}$$

Since for arbitrary $t > R > 0$, in the uniform operator topology for $t \ge R > 0$, $T_\alpha(t)$, $S_\alpha(t)$ are continuous, which implies that the right-hand side of (5) tends to zero as t, t' tends to $t_{\max}$. Therefore, it proves that $\lim_{t\uparrow t_{\max}} x(t) = x(t_{\max})$ exists and the solution x can be extended beyond $t_{\max}$. Therefore, by assumption, $t_{\max} < \infty$

implies that $\overline{\lim}_{t\uparrow t_{\max}} E\|x(t)\|_H^2 = \infty$. Now the proof of the theorem can be concluded by showing $\lim_{t\uparrow t_{\max}} E\|x(t)\|_H^2 = \infty$. If it is not true, then there is a sequence $\tau_n \uparrow t_{\max}$ and a constant k_1 such that $E\|x(\tau_n)\|_H^2 \le k_1$ for all n. Let

$$\beta_1 = \sup\left\{E\,\|f\,(t,\,x_t)\|_H^2 : 0 \le t \le t_{\max},\; E\|x(t)\|_H^2 \le M_T^2\,(k_1+1)\right\},$$
$$\beta_2 = \sup\left\{\int_0^s E\,\|\sigma\,(s,\,\tau,\,x_\tau)\|_H^2\,\mathrm{d}\tau : 0 \le t \le t_{\max},\; E\|x(t)\|_H^2 \le M_T^2\,(k_1+1)\right\},$$

and choose ρ_1 such that $\rho_1 < \frac{1-6k_1}{8k_1 m}$.

Since $t \to E\|x(t)\|_H^2$ is continuous and $\overline{\lim}_{t\uparrow t_{\max}} E\|x(t)\|_H^2 = \infty$, we can find the sequence $\{\lambda_n\}$ with the following properties: $\lambda_n \to 0$ as $n \to \infty$, $E\|x(t)\|_H^2 \le M_T^2(k_1+1)$ for $\tau_n \le t \le \tau_n + \lambda_n$ and $E\|x(\tau_n+\lambda_n)\|_H^2 = M_T^2(k_1+1)$. On the other hand, we have

$$\begin{aligned}
M_T^2\,(k_1+1) &= E\,\|x\,(\tau_n+\lambda_n)\|_H^2\\
&\le 4\Bigg\{E\,\|T_\alpha\,(\lambda_n)\,x\,(\tau_n)\|_H^2 + \sum_{\tau_n<t_k<\tau_n+\lambda_n} \|T_\alpha\,(\tau_n+\lambda_n-t_k)\|_{L(H)}^2\, E\,\left\|I_k\left(x\left(t_k^-\right)\right)\right\|_H^2\\
&\quad + M_S^2 \int_{\tau_n}^{\tau_n+\lambda_n} (\tau_n+\lambda_n-s)^{\alpha-1}\,\mathrm{d}s \int_{\tau_n}^{\tau_n+\lambda_n} (\tau_n+\lambda_n-s)^{\alpha-1}\,E\,\|f\,(s,\,x_s)\|_H^2\,\mathrm{d}s\\
&\quad + M_S^2 \int_{\tau_n}^{\tau_n+\lambda_n} (\tau_n+\lambda_n-s)^{\alpha-1}\,\mathrm{d}s \int_{\tau_n}^{\tau_n+\lambda_n} (\tau_n+\lambda_n-s)^{\alpha-1}\\
&\quad \times \left(\mathrm{Tr}(Q)\int_0^s E\,\|\sigma\,(s,\,\tau,\,x_\tau)\|_{L(K,H)}^2\,\mathrm{d}\tau\right)\mathrm{d}s\Bigg\},\\
&\le 4\left\{M_T^2 E\|x\|_H^2 + M_T^2 m\rho E\|x\|_H^2 + M_S^2\frac{\lambda_n^{2\alpha}}{\alpha^2}\beta_1 + M_S^2\frac{\lambda_n^{2\alpha}}{\alpha^2}\mathrm{Tr}(Q)\beta_2\right\}\\
&\le 4M_T^2 k_1\,(1+m\rho_1) + \frac{4M_S^2\lambda_n^{2\alpha}}{\alpha^2}\{\beta_1+\mathrm{Tr}(Q)\beta_2\}\\
&\le 4M_T^2 k_1\left[1+m\left(\frac{1-6k_1}{8k_1 m}\right)\right] \quad \text{as } \lambda_n \to 0\\
&\le M_T^2\left(k_1+\frac{1}{2}\right),
\end{aligned}$$

which is absurd as $\lambda_n \to 0$. Therefore, we have $\lim_{t\to t_{\max}} E\|x(t)\|_H = \infty$. This completes the proof. □

5 Example

To illustrate our theoretical results, consider the following impulsive fractional semi-linear stochastic differential equation with nonlocal conditions

$$\frac{\partial^{\frac{1}{2}} u(t,\, x)}{\partial t^{\frac{1}{2}}} + \frac{\partial^2 u(t,\, x)}{\partial x^2} = \int_0^t \mathrm{e}^{\frac{(t-s)}{2}} \frac{\|u(s,\, x)\|}{25 + \|u(s,\, x)\|} \mathrm{d}s + \int_0^t \frac{\mathrm{e}^{t-s}}{4 + \|u(s,\, x)\|} \mathrm{d}w(s), \tag{6}$$

$$t \in J = [0,\, 1], \quad x \in (0,\, \pi), \quad t \neq \frac{1}{2}, \quad 0 < \alpha \leq 1,$$

$$u(t,\, 0) = u(t,\, \pi) = 0, \quad t \geq 0, \tag{7}$$

$$\Delta u|_{t=\frac{1^-}{2}} = \sin\left(\frac{1}{7}\left\| u\left(\frac{1^-}{2},\, x\right)\right\|\right), \tag{8}$$

$$\frac{1}{\tau}\int_{-\tau}^{0} \mathrm{e}^{2s} u(s,\, x)\mathrm{d}s = u_0(x), \quad 0 \leq x \leq \pi. \tag{9}$$

Let $H = L^2[0,\, \pi]$, $w(t)$ is standard cylindrical Wiener process defined on a stochastic basis $(\Omega,\, \mathcal{F},\, \{\mathcal{F}_t\}_{t\geq 0};\, \mathbb{P})$ and $A : D(A) \subset H \to H$ be defined by $Az = z^{''}$ with the domain $D(A) = \{z \in H\, z,\, z^{'}$ are absolutely continuous $z^{''} \in H,\, z(0) = z(\pi) = 0\}$ then

$$Az = \sum_{n=1}^{\infty} n^2 \langle z,\, z_n\rangle z_n, \quad z \in D(A),$$

where $z_n(x) = \sqrt{\frac{2}{\pi}} \sin(nx),\ n \in N$ is the orthonormal set of eigenvectors of A. It is well known that A generates an analytic semigroup $\{T(t)\}_{t\geq 0}$ and

$$T(t)z = \sum_{n=1}^{\infty} \mathrm{e}^{-n^2 t} \langle z,\, z_n\rangle z_n, \quad z \in H.$$

It follows from the above expression that $\{T(t)\}_{t\geq 0}$ is a uniformly bounded compact semigroup, so that $R(\lambda^\alpha,\, A) = (\lambda^\alpha I - A)^{-1}$ is a compact operator for all $\lambda \in \rho(A)$. In order to define the operator $Q : H \to H$, we choose a sequence $\{\xi_n\}$, set $Qz_n = \xi_n z_n$ and assume that

$$\mathrm{Tr}(Q) = \sum_{n=1}^{\infty} \sqrt{\xi_n} < 0.0718.$$

If we put $t - s = -\theta$ in the first and second term on the RHS of (6), and take $u(t,\, x) = u(t)x$, we get

$$\int_0^t \mathrm{e}^{\frac{(t-s)}{2}} \frac{\|u(s,\, x)\|}{25 + \|u(s,\, x)\|} \mathrm{d}s = \int_{-t}^{0} \mathrm{e}^{\frac{-\theta}{2}} \frac{\|u_t(\theta)(x)\|}{25 + \|u_t(\theta)(x)\|} \mathrm{d}\theta,$$

and

$$\int_0^t \frac{\mathrm{e}^{t-s}}{4 + \|u(s,\, x)\|} \mathrm{d}w(s) = \int_{-t}^{0} \frac{\mathrm{e}^{-\theta}}{4 + \|u_t(\theta)x\|} \mathrm{d}w(\theta),$$

then (6) takes the following abstract form

$$D_t^{\frac{1}{2}} u(t)x + Au(t)x = f\left(t,\, u_t\right)(x) + \int_0^t \sigma\left(t,\, s,\, u_s\right)(x)\mathrm{d}w(s),$$

where $f : [0,\, 1] \times \mathrm{PC}_0 \to H$ and $\sigma : [0,\, 1] \times [0,\, 1] \times \mathrm{PC}_0 \to L_2(K,\, H)$ given by

$$f(t,\, \phi)(x) = \int_{-t}^{0} \mathrm{e}^{\frac{-\theta}{2}} \frac{\|\phi(\theta)(x)\|}{25 + \|\phi(\theta)(x)\|}\mathrm{d}\theta,$$
$$\sigma(t,\, s,\, \phi)(x) = \int_{-t}^{0} \frac{\mathrm{e}^{-\theta}}{4 + \|\phi(\theta)x\|}\mathrm{d}w(\theta).$$

$I_k(u) = \sin(\frac{1}{7}\|u\|)$, $k = 1$, $h(u)(\theta) = \sigma(u)$ for $u \in \mathrm{PC}_1$, $\theta \in [-\tau,\, 0]$, $\phi(\theta) \equiv u_0$ for $\theta \in [-\tau,\, 0]$, where $\sigma : \mathrm{PC}_1 \to L^2([0,\, \pi])$ is such that

$$\sigma(u) = \frac{1}{\tau}\int_{-\tau}^{0} \mathrm{e}^{2s} u(s,\, x)\mathrm{d}s.$$

Then (6)–(9) can be written in the abstract form of (1). For $(t,\, \phi),\ (s,\, \psi) \in [0,\, 1]\times \mathrm{PC}_0$, we have

$$\begin{aligned} E\|f(t,\, \phi) - f(s,\, \psi)\|_H^2 &= \left\| \int_{-t}^{0} \mathrm{e}^{-\frac{\theta}{2}} \frac{\phi(\theta)(x)}{25+\phi(\theta)(x)}\mathrm{d}\theta - \int_{-s}^{0} \mathrm{e}^{-\frac{\theta}{2}} \frac{\psi(\theta)(x)}{25+\psi(\theta)(x)}\mathrm{d}\theta \right\|_H^2 \\ &\le 2\int_{-t}^{-s} \pi \mathrm{e}^{-\theta} \left\| \frac{\phi(\theta)(\cdot)}{25+\phi(\theta)(\cdot)} \right\|_H^2 \mathrm{d}\theta \\ &\quad + 2\int_{-s}^{0} \pi \mathrm{e}^{-\theta} \left\| \frac{\phi(\theta)(\cdot)}{25+\phi(\theta)(\cdot)} - \frac{\psi(\theta)(\cdot)}{25+\psi(\theta)(\cdot)} \right\|_H^2 \mathrm{d}\theta \\ &\le 2\pi \frac{(1+\mathrm{e}^1)|t-s|^2}{625} + \frac{2\pi(\mathrm{e}^1 - 1)}{625} E\|\phi - \psi\|_{\mathrm{PC}_0}^2 \\ &< 2\pi \frac{(1+e)|t-s|^2}{625} + \frac{2\pi(e-1)}{625} E\|\phi - \psi\|_{\mathrm{PC}_0}^2. \end{aligned}$$

Similarly,

$$\begin{aligned} E\|\sigma(t,\, \tau,\, \phi) - \sigma(s,\, \tau,\, \psi)\|_{L(K,H)}^2 \le& \frac{\pi \mathrm{e}^2 \mathrm{Tr}(Q)|t-s|^2}{8} \\ &+ \frac{\pi \mathrm{e}^2 \mathrm{Tr}(Q) E\|\phi - \psi\|_{\mathrm{PC}_0}^2}{8}, \end{aligned}$$

and

$$E\left\|I_k(u(t)) - I_k(v(t))\right\|_H^2 = E\left\|\sin\left(\frac{1}{7}u(t)\right) - \sin\left(\frac{1}{7}v(t)\right)\right\|_H^2,$$

$$\leq \frac{1}{49} E\|u(t) - v(t)\|^2_{\mathrm{PC}_1}, \quad u,\ v \in \mathrm{PC}_1,$$
$$\leq \frac{1}{49} E\|u\|^2_{\mathrm{PC}_1}, \quad u \in \mathrm{PC}_1 \quad \text{for} \quad t \in [0,\ 1].$$

Furthermore, for a defined h, we find $\eta(t) = \frac{u_0}{k^*} \in \mathrm{PC}_1$ on $[\tau,\ 0]$ with $k^* = \frac{1}{\tau}\int_0^\tau \mathrm{e}^{-2s}\mathrm{d}s \neq 0$ such that

$$h(\eta)(\theta) \equiv \sigma(\eta) = \frac{1}{\tau}\int_{-\tau}^{0} \mathrm{e}^{2s}\left(\frac{1}{k^*}u_0\right)\mathrm{d}s = u_0 \equiv \phi(\theta).$$

We have $h(\eta) = \phi$. Thus, (A_1)–(A_7) are satisfied with $N_1 = \frac{2\pi(1+e)}{625}$ and $N_2 = \frac{2\pi \mathrm{e}^2}{16}$, $\mu = \frac{1}{49}$, $m = 1$, $M_T = 1$, $M_S = \frac{1}{\Gamma(\frac{1}{2})}$, $\alpha = \frac{1}{2}$. Further

$$3\left[M_T^2 \mu m + \left(\frac{N_1}{\alpha^2} + \frac{N_2 \mathrm{Tr}(Q)}{\alpha^2}\right) M_S^2 \tau_0^{2\alpha}\right]$$
$$= 3\left[\frac{1}{49} + \frac{2\pi(1+e)4}{625\Gamma\left(\frac{1}{2}\right)^2} + \frac{2\pi \mathrm{e}^2(4)\mathrm{Tr}(Q)}{16\Gamma(\frac{1}{2})^2}\right] < 1.$$

Therefore by Theorem (3.1) the problem (6)–(9) has a unique mild solution on $[0,\ 1]$.

6 Conclusion

In this manuscript, we have studied the local and global existence of mild solutions for impulsive fractional semilinear stochastic differential equations with nonlocal condition in a Hilbert space. The local and global existence of mild solutions is proved, respectively, using the Banach contraction principle and Schauder fixed-point theorem. The fixed-point technique and solution operator are employed to obtain the results, and the obtained result is valid for all $\alpha \in (0,\ 1)$. To validate the obtained theoretical results, one numerical example is analyzed. The FDE are very efficient to describe the real-life phenomena; thus, it is essential to extend the present study to establish the other qualitative and quantitative properties such as stability and controllability. In future, we can extend this work with Poisson jump and study the existence, uniqueness, and stability properties as discussed in [26,28], and we could establish the asymptotic stability as discussed in [27,29,30]. The fractional Brownian motion is a generalization of the Brownian motion. Hence in our future work, we are interested to implement fractional Brownian motion to get more interesting results.

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