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Convergence Theorems for a Hybrid Pair of Generalized Nonexpansive Mappings in Banach Spaces

Izhar Uddin · M. Imdad · Javid Ali

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Abstract In this paper, we study convergence of modified Ishikawa iteration process involving a pair of hybrid mappings satisfying a generalized nonexpansive condition on Banach spaces. In process, several relevant results are generalized and improved.

Keywords Common fixed point · Generalized nonexpansive mapping · Demiclosedness · Convergence theorems

Mathematics Subject Classification 54H25 · 47H10 · 54E50

1 Introduction

Let X be a Banach space and K be a nonempty subset of X. Let $CB(K)$ be the family of nonempty closed bounded subsets of K, while $KC(K)$ be the family of nonempty compact convex subsets of K. A subset K of X is called proximinal if for each $x \in X$, there exists an element $k \in K$ such that

$$d(x,k) = dist(x,K) = \inf\{\|x-y\| : y \in K\}.$$

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I. Uddin · M. Imdad (✉) · J. Ali
Department of Mathematics, Aligarh Muslim University, Aligarh 202002, Uttar Pradesh, India
e-mail: mhimdad@yahoo.co.in

I. Uddin
e-mail: izharuddin_amu@yahoo.co.in

J. Ali
Department of Mathematics, Birla Institute of Technology & Science, Pilani 333031, Rajasthan, India
e-mail: javid@pilani.bits-pilani.ac.in; javid@amu.ac.in

It is well known that every closed convex subset of a uniformly convex Banach space is proximinal. We shall denote by $PB(K)$, the family of nonempty bounded proximinal subsets of K. The Hausdorff metric H on $CB(K)$ is defined as

$$H(A, B) = max\left\{ \sup_{x\in A} d(x, B), \sup_{y\in B} d(y, A)\right\} \text{ for } A,\ B \in CB(K).$$

A mapping $f : K \to K$ is said to be nonexpansive if

$$\|fx - fy\| \leq \|x - y\|, \text{ for all } x,\ y \in K,$$

while a multi-valued mapping $T : K \to CB(K)$ is said to be nonexpansive if

$$H(T(x), T(y)) \leq \|x - y\|, \text{ for all } x,\ y \in K.$$

We use the notation $Fix(T)$ for the set of fixed points of the mapping T, while $Fix(f) \bigcap Fix(T)$ denotes the set of common fixed points of f and T, i.e. a point x is said to be a common fixed point of f and T if $x = fx \in Tx$.

In 2008, Suzuki [1] introduced a new class of mappings, which is larger than the class of nonexpansive mappings and name the defining condition as condition (C) which runs follows: A mapping T defined on a subset K of a Banach space X is said to satisfy condition (C) if

$$\frac{1}{2}\|x - Tx\| \leq \|x - y\| \Rightarrow \|Tx - Ty\| \leq \|x - y\|,\ x, y \in K.$$

He then proved some fixed point as well as convergence theorems for such mappings. Inspired from this, recently García-Falset et al. [2] defined two new generalizations of condition (C) and term their conditions as condition (E) and Condition (C_λ) and studied the existence of fixed point besides their asymptotic behaviour for both the classes of mappings in Banach spaces.

In 2010, Sokhuma and Kaewkhao [3] introduced a modified Ishikawa iterative process involving a pair of single-valued and multi-valued nonexpansive mappings in Banach spaces and proved strong convergence theorems.

The purpose of this paper is to study modified Ishikawa iterative method for generalized nonexpansive mappings introduced by García-Falset et al. [2].

2 Preliminaries

With a view to make, our presentation self contained, we collect some basic definitions, needed results and some iterative methods which will be used frequently in the text later. We start with the following definitions:

Definition 2.1 [2] Let K be a nonempty subset of a Banach space X and $S : K \to X$ be a single-valued mapping. Then S is said to satisfy the condition (E_μ) for some $\mu \geq 1$ if

$$\|x - Sy\| \leq \mu\|x - Sx\| + \|x - y\|, \text{ for } x, \; y \in K.$$

We say that S satisfies condition (E) whenever S satisfies the condition (E_μ) for some $\mu \geq 1$.

Definition 2.2 [2] Let K be a nonempty subset of a Banach space X and $S : K \to X$ be a single-valued mapping. Then S is said to satisfy the condition (C_λ) for some $\lambda \in (0, 1)$ if

$$\lambda\|x - Sx\| \leq \|x - y\| \Rightarrow \|Sx - Sy\| \leq \|x - y\|, \text{ for } x, \; y \in K.$$

From the foregoing definitions, one can infer the following observations:

Lemma 2.1 [2] *If $S : K \to X$ is a nonexpansive mapping, then S satisfies the condition (E_1). But the converse statement is not true in general.*

Lemma 2.2 [2] *From Lemma 1 in* [1]*, one can see that if $S : K \to K$ satisfies the condition (C), then S satisfies condition (E_3). But the converse statement is not true in general.*

The following example supports two preceding facts.

Example 2.1 [2] Consider $X = C[0, 1]$ and the nonempty subset K as follows:

$$K = \{f \in C[0, 1] : 0 = f(0) \leq f(x) \leq f(1) = 1\}.$$

For any $g \in K$, we construct the function $F_g : K \to K$

$$F_g h(t) = (g \circ h)(t) = g(h(t)).$$

It is easy to verify that F_g satisfies the condition (E_1) and but does not satisfy the condition (C).

By choosing $\lambda = \frac{1}{2}$, we can retrieve the class of Suzuki's generalized nonexpansive mappings. From the definition of condition (C_λ), one can easily see that condition (C_{λ_1}) implies the condition (C_{λ_2}) for $0 < \lambda_1 < \lambda_2 < 1$. The following example shows that the converse fails.

Example 2.2 [2] For a given $\lambda \in (0, 1)$ define a self-mapping S on $[0, 1]$ by

$$S(x) = \begin{cases} \frac{x}{2} & \text{if } x \neq 1 \\ \\ \frac{1+\lambda}{2+\lambda} & \text{if } x = 1. \end{cases}$$

Then the mapping S satisfies condition (C_λ) but it fails condition (C_{λ_1}) whenever $0 < \lambda_1 < \lambda$. Moreover S satisfies condition (E_μ) for $\mu = \frac{2+\lambda}{2}$.

In the same paper, García-Falset et al. used the concept of strongly demiclosed to prove a weaker version of famous principle of demiclosedness for generalized nonexpansive mappings in which weak convergence has been replaced by strong convergence. Definition runs as follows:

Definition 2.3 [2] Given a mapping $S : K \to X$, we say that $(I - S)$ is strongly demiclosed at 0 if for every sequence $\{x_n\}$ in K strongly convergent to $z \in K$ and such that $x_n - Sx_n \to 0$ we have that $z = Sz$.

Lemma 2.3 [2] *Let K be a nonempty subset of a Banach space X. If $S : K \to X$ satisfies condition(E) on K, then $(I - S)$ is strongly demiclosed at 0.*

Kaewcharoen and Panyanak [4] gave the multi-valued version of Condition (C_λ).

Definition 2.4 [4] Let $T : K \to CB(X)$ be a multi-valued mapping. Then T is said to satisfy condition (C_λ) for some $\lambda \in (0, 1)$ if each $x,\ y \in K$,

$$\lambda\, dist(x, Tx) \leq \|x - y\| \text{ implies } H(Tx, Ty) \leq \|x - y\|.$$

As the single-valued case, one can easy verify that condition for $0 < \lambda_1 < \lambda_2 < 1$, (C_{λ_1}) implies condition (C_{λ_2}).

Lemma 2.4 [4] *Let K be a nonempty subset of a Banach space X and $T : K \to CB(X)$ be a mapping. If T is a nonexpansive, then T satisfies condition (C_λ).*

The following example exhibits that the converse statement need not be true.

Example 2.3 [4] Define a mapping $T : [0, 3] \to CB([0, 3])$ by

$$T(x) = \begin{cases} \{0\}, & \text{if } x \neq 3, \\ [0.5, 1], & \text{if } x = 3. \end{cases}$$

T satisfies condition (C_λ) for $\lambda = \frac{1}{2}$ but T is not nonexpansive.

Now we list the following important property of a uniformly convex Banach space due to Schu [5] and an important lemma due to Sokhuma and Kaewkhao [3].

Lemma 2.5 *Let X be a uniformly convex Banach space, let $\{u_n\}$ be a sequence of real numbers such that $0 < b \leq u_n \leq c < 1$ for all $n \geq 1$, and let $\{x_n\}$ and $\{y_n\}$ be sequences in X such that $\limsup_{n\to\infty} \|x_n\| \leq a$, $\limsup_{n\to\infty} \|y_n\| \leq a$, and $\lim_{n\to\infty} \|u_n x_n + (1 - u_n)y_n\| = a$ for some $a > 0$. Then, $\lim_{n\to\infty} \|x_n - y_n\| = 0$.*

Lemma 2.6 *Let X be a Banach space, and let K be a nonempty closed convex subset of X. Then,*

$$dist(y, Ty) \leq \|y - x\| + dist(x, Tx) + H(Tx, Ty),$$

where $x, y \in K$ and T is a multi-valued nonexpansive mapping from K into $CB(K)$.

In 1974, Ishikawa [6] introduced an iterative procedure for approximating fixed point as follows: For $x_0 \in K$, the sequence $\{x_n\}$ of Ishikawa iteration defined by

$$\begin{aligned} y_n &= (1-\beta_n)x_n + \beta_n S x_n, \\ x_{n+1} &= (1-\alpha_n)x_n + \alpha_n S y_n, \; n \geq 0, \end{aligned}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are real sequences in [0, 1].

In 2005, Sastry and Babu [7] defined Ishikawa iteration scheme for multi-valued mappings. Let $T : K \to PB(K)$ a multi-valued mapping and fix $p \in F(T)$. Then the sequence of Ishikawa iteration is defined as follows:
Choose $x_0 \in K$,

$$y_n = \beta_n z_n + (1-\beta_n)x_n, \; \beta_n \in [0,1], \quad n \geq 0,$$

where $z_n \in T(x_n)$ such that $\|z_n - p\| = \operatorname{dist}(p, Tx_n)$ and

$$x_{n+1} = \alpha_n z_n' + (1-\alpha_n)x_n, \quad \alpha_n \in [0,1], \; n \geq 0,$$

where $z_n' \in T(y_n)$ such that $\|z_n' - p\| = \operatorname{dist}(p, Ty_n)$.

Sastry and Babu [7] proved that Ishikawa iteration scheme for a multi-valued nonexpansive mapping T with a fixed point p converges to a fixed point q of T under certain conditions. In 2007, Panyanak [8] extended the results of Sastry and Babu to uniformly convex Banach space for multi-valued nonexpansive mappings. Panyanak also modified the iteration scheme of Sastry and Babu and imposed the question of convergence of this scheme. He introduced the following modified Ishikawa iteration method,
Choose $x_0 \in K$, then

$$y_n = \beta_n z_n + (1-\beta_n)x_n, \; \beta_n \in [a,b], \; 0 < a < b < 1, \; n \geq 0,$$

where $z_n \in Tx_n$ is such that $\|z_n - u_n\| = \operatorname{dist}(u_n, Tx_n)$, and $u_n \in F(T)$ such that $\|x_n - u_n\| = \operatorname{dist}(x_n, F(T))$, and

$$x_{n+1} = \alpha_n z_n' + (1-\alpha_n)x_n \, \alpha_n \in [a,b],$$

where $z_n' \in T(y_n)$ such that $\|z_n' - v_n\| = \operatorname{dist}(v_n, Ty_n)$, and $v_n \in F(T)$ such that $\|y_n - v_n\| = \operatorname{dist}(y_n, F(T))$.

In 2009, Song and Wang [9] pointed out the gap in the result of Panyanak [8]. They solved/revised the gap and gave the partial answer to the question raised by Panyanak using the following iteration scheme.
Let $\alpha_n, \; \beta_n \in [0,1]$ and $\gamma_n \in (0,\infty)$ such that $\lim_{n\to\infty} \gamma_n = 0$. Choose $x_0 \in K$, then

$$\begin{aligned} y_n &= \beta_n z_n + (1-\beta_n)x_n, \\ x_{n+1} &= \alpha_n z_n' + (1-\alpha_n)x_n, \end{aligned}$$

where $\|z_n - z'_n\| \le H(Tx_n, Ty_n) + \gamma_n$ and $\|z_{n+1} - z'_n\| \le H(Tx_{n+1}, Ty_n) + \gamma_n$ for $z_n \in Tx_n$ and $z'_n \in Ty_n$.
Simultaneously, Shahzad and Zegeye [11] extended the results of Sastry and Babu, Panyanak, and Song and Wang to quasi nonexpansive multi-valued mappings and also relaxed the end point condition and compactness of the domain using the following modified iteration scheme and gave the affirmative answer to the Panyanak question in a more general setting.

$$\begin{aligned} y_n &= \beta_n z_n + (1-\beta_n)x_n, \quad \beta_n \in [0,1], \quad n \ge 0, \\ x_{n+1} &= \alpha_n z'_n + (1-\alpha_n)x_n, \quad \alpha_n \in [0,1], \quad n \ge 0, \end{aligned}$$

where $z_n \in Tx_n$ and $z'_n \in Ty_n$. Recently, Sokhuma and Kaewkhao [3] introduced the following modified Ishikawa iteration scheme for a pair of single-valued and multi-valued mapping.
Let K be a nonempty closed and bounded convex subset of Banach space X, let $S : K \to K$ be a single-valued nonexpansive mapping and let $T : K \to CB(K)$ be a multi-valued nonexpansive mapping. The sequence $\{x_n\}$ of the modified Ishikawa iteration is defined by

$$\begin{cases} y_n = \beta_n z_n + (1-\beta_n)x_n, \\ x_{n+1} = \alpha_n S y_n + (1-\alpha_n)x_n, \end{cases} \tag{1}$$

where $x_0 \in K$, $z_n \in Tx_n$ and $0 < a \le \alpha_n,\ \beta_n \le b < 1$.
Furthermore, they proved the following strong convergence theorem:

Theorem 2.7 *Let K be a nonempty compact convex subset of a uniformly convex Banach space X, and let $S : K \to K$ and $T : K \to CB(K)$ be a single-valued and a multi-valued nonexpansive mapping, respectively, and $Fix(S) \bigcap Fix(T) \neq \emptyset$ satisfying $Tw = \{w\}$ for all $w \in Fix(S) \bigcap Fix(T)$. Let $\{x_n\}$ be the sequence of the modified Ishikawa iteration defined by (1) with* $0 < a \le \alpha_n,\ \beta_n \le b < 1$. *Then $\{x_n\}$ converges strongly to a common fixed point of S and T.*

3 Main Results

Before establishing our main results we prove following lemmas. These results generalize the results of [3,10] and will be used to accomplish our main Theorem 3.6.

Lemma 3.1 *Let K be a nonempty compact convex subset of a uniformly convex Banach space X, and let $S : K \to K$ and $T : K \to CB(K)$ be a single-valued and a multi-valued mapping such that both satisfy the condition (C_λ), respectively, and $Fix(S) \cap Fix(T) \neq \emptyset$ satisfying $Tw = \{w\}$ for all $w \in Fix(S) \cap Fix(T)$. Let $\{x_n\}$ be the sequence of the modified Ishikawa iteration defined by (1). Then, $\lim_{n\to\infty} \|x_n - w\|$ exists for all $w \in Fix(S) \cap Fix(T)$.*

Proof Letting $w \in Fix(S) \cap Fix(T)$. Since for $\lambda \in (0,\ 1)$,

$$\lambda\|w - Sw\| = 0 \leq \|((1-\beta_n)x_n + \beta_n z_n) - w\|,$$

using definition of Condition (C_λ), we get

$$\|S((1-\beta_n)x_n + \beta_n z_n) - Sw\| \leq \|(1-\beta_n)x_n + \beta_n z_n - w\|.$$

Similarly, since λ dist$(w, Tw) = 0 \leq \|x_n - w\|$, then we get, $H(Tx_n, Tw) \leq \|x_n - w\|$. Consider

$$\begin{aligned}
\|x_{n+1} - w\| &= \|(1-\alpha_n)x_n + \alpha_n S((1-\beta_n)x_n + \beta_n z_n) - w\| \\
&= \|(1-\alpha_n)x_n + \alpha_n S((1-\beta_n)x_n + \beta_n z_n) - (1-\alpha_n)w - \alpha_n w\| \\
&\leq (1-\alpha_n)\|x_n - w\| + \alpha_n\|S((1-\beta_n)x_n + \beta_n z_n) - w\| \\
&= (1-\alpha_n)\|x_n - w\| + \alpha_n\|S((1-\beta_n)x_n + \beta_n z_n) - Sw\| \\
&\leq (1-\alpha_n)\|x_n - w\| + \alpha_n\|(1-\beta_n)x_n + \beta_n z_n - w\| \\
&= (1-\alpha_n)\|x_n - w\| + \alpha_n\|(1-\beta_n)x_n + \beta_n z_n - (1-\beta_n)w - \beta_n w\| \\
&\leq (1-\alpha_n)\|x_n - w\| + \alpha_n\{(1-\beta_n)\|x_n - w\| + \beta_n\|z_n - w\|\} \\
&= (1-\alpha_n)\|x_n - w\| + \alpha_n\{(1-\beta_n)\|x_n - w\| + \beta_n dist(z_n, Tw)\} \\
&\leq (1-\alpha_n)\|x_n - w\| + \alpha_n\{(1-\beta_n)\|x_n - w\| + \beta_n H(Tx_n, Tw)\} \\
&\leq (1-\alpha_n)\|x_n - w\| + \alpha_n\{(1-\beta_n)\|x_n - w\| + \beta_n\|x_n - w\|\} \\
&= \|x_n - w\|.
\end{aligned}$$

Since $\{\|x_n - w\|\}$ is a bounded and decreasing sequence, hence the limit of $\{\|x_n - w\|\}$ exists.

Using Lemma 2.5 we get the following results: □

Lemma 3.2 *Let K be a nonempty compact convex subset of a uniformly convex Banach space X, and let $S : K \to K$ and $T : K \to CB(K)$ be a single-valued and a multi-valued mapping such that both satisfy the condition (C_λ), respectively, and $Fix(S) \cap Fix(T) \neq \emptyset$ satisfying $Tw = \{w\}$ for all $w \in Fix(S) \cap Fix(T)$. Let $\{x_n\}$ be the sequence of the modified Ishikawa iteration defined by (1). If $0 < a \leq \alpha_n, \beta_n \leq b < 1$ for some $a, b \in \mathbb{R}$, then $\lim_{n\to\infty} \|Sy_n - x_n\| = 0$.*

Proof Let $w \in Fix(S) \cap Fix(T)$.
Since for any $\lambda \in (0, 1)$, $\lambda\|w - Sw\| = 0 \leq \|y_n - w\|$, therefore using the definition of condition (C_λ), we get $\|Sy_n - Sw\| \leq \|y_n - w\|$. Similarly, since λdist$(w, Tw) = 0 \leq \|x_n - w\|$ then we get $H(Tx_n, Tw) \leq \|x_n - w\|$.
By Lemma 3.1, $\lim_{n\to\infty} \|x_n - w\|$ exists, say c, and consider,

$$\begin{aligned}
\|Sy_n - w\| &= \|Sy_n - Sw\| \\
&\leq \|y_n - w\| \\
&= \|(1-\beta_n)x_n + \beta_n z_n - w\|
\end{aligned}$$

$$
\begin{aligned}
&\leq (1-\beta_n)\|x_n - w\| + \beta_n \mathrm{dist}(z_n, Tw)\\
&\leq (1-\beta_n)\|x_n - w\| + \beta_n H(Tx_n, Tw)\\
&\leq (1-\beta_n)\|x_n - w\| + \beta_n \|x_n - w\|\\
&= \|x_n - w\|.
\end{aligned}
$$

Therefore, we have

$$
\limsup_{n\to\infty} \|Sy_n - w\| \leq \limsup_{n\to\infty} \|y_n - w\| \leq \limsup_{n\to\infty} \|x_n - w\| = c. \tag{2}
$$

Further, we have

$$
\begin{aligned}
c &= \lim_{n\to\infty} \|x_{n+1} - w\|\\
&= \lim_{n\to\infty} \|(1-\alpha_n)x_n + \alpha_n Sy_n - w\|\\
&= \lim_{n\to\infty} \|\alpha_n Sy_n - \alpha_n w + x_n - \alpha_n x_n + \alpha_n w - w\|\\
&= \lim_{n\to\infty} \|\alpha_n (Sy_n - w) + (1-\alpha_n)(x_n - w)\|.
\end{aligned}
$$

By Lemma 2.5, we obtain $\lim_{n\to\infty} \|(Sy_n - w) - (x_n - w)\| = \lim_{n\to\infty} \|Sy_n - x_n\| = 0.$ □

Lemma 3.3 *Let K be a nonempty compact convex subset of a uniformly convex Banach space X, and let $S : K \to K$ and $T : K \to CB(K)$ be a single-valued and a multi-valued mapping such that both satisfy the condition (C_λ), respectively, and $Fix(S) \cap Fix(T) \neq \emptyset$ satisfying $Tw = \{w\}$ for all $w \in Fix(S) \cap Fix(T)$. Let $\{x_n\}$ be the sequence of the modified Ishikawa iteration defined by (1). If $0 < a \leq \alpha_n, \ \beta_n \leq b < 1$ for some $a, \ b \in \mathbb{R}$, then $\lim_{n\to\infty} \|x_n - z_n\| = 0$.*

Proof Let $w \in Fix(S) \cap Fix(T)$.
Since $\lambda\|w - Sw\| = 0 \leq \|y_n - w\|$, we get $\|Sy_n - Sw\| \leq \|y_n - w\|$. Similarly, since $\lambda dist(w, Tw) = 0 \leq \|x_n - w\|$ then we get $H(Tx_n, Tw) \leq \|x_n - w\|$. We put $\lim_{n\to\infty} \|x_n - w\| = c$. For $n \geq 0$, we have

$$
\begin{aligned}
\|x_{n+1} - w\| &= \|(1-\alpha_n)x_n + \alpha_n Sy_n - w\|\\
&= \|(1-\alpha_n)x_n + \alpha_n Sy_n - (1-\alpha_n)w - \alpha_n w\|\\
&\leq (1-\alpha_n)\|x_n - w\| + \alpha_n \|Sy_n - w\|\\
&= (1-\alpha_n)\|x_n - w\| + \alpha_n \|Sy_n - Sw\|\\
&\leq (1-\alpha_n)\|x_n - w\| + \alpha_n \|y_n - w\|
\end{aligned}
$$

and therefore

$$
\begin{aligned}
\|x_{n+1} - w\| - \|x_n - w\| &\leq \alpha_n (\|y_n - w\| - \|x_n - w\|),\\
\frac{\|x_{n+1} - w\| - \|x_n - w\|}{\alpha_n} &\leq \|y_n - w\| - \|x_n - w\|.
\end{aligned}
$$

Since $0 < a \leq \alpha_n \leq b < 1$, we have

$$\liminf_{n\to\infty} \left\{ \left(\frac{\|x_{n+1} - w\| - \|x_n - w\|}{\alpha_n} \right) + \|x_n - w\| \right\} \leq \liminf_{n\to\infty} \|y_n - w\|.$$

It follows that

$$c \leq \liminf_{n\to\infty} \|y_n - w\|.$$

Since, from (2) $\limsup_{n\to\infty} \|y_n - w\| \leq c$, hence we have

$$\begin{aligned} c &= \lim_{n\to\infty} \|y_n - w\| \\ &= \lim_{n\to\infty} \|(1-\beta_n)x_n + \beta_n z_n - w\| \\ &= \lim_{n\to\infty} \|(1-\beta_n)(x_n - w) + \beta_n(z_n - w)\|. \end{aligned} \tag{3}$$

Recall that $\|z_n - w\| = \text{dist}(z_n, Tw) \leq H(Tx_n, Tw) \leq \|x_n - w\|$. Thus, we have

$$\limsup_{n\to\infty} \|z_n - w\| \leq \limsup_{n\to\infty} \|x_n - w\| = c.$$

Since $0 < a \leq \beta_n \leq b < 1$ and by Lemma 2.5 and (3), we obtain $\lim_{n\to\infty} \|x_n - z_n\| = 0$. □

Lemma 3.4 *Let K be a nonempty compact convex subset of a uniformly convex Banach space X. Let $S : K \to K$ be a single-valued mapping which satisfies conditions (C_λ) and (E) while $T : K \to CB(K)$ be a multi-valued mapping which satisfies condition (C_λ). Also, $Fix(S) \bigcap Fix(T) \neq \emptyset$ and $Tw = \{w\}$ for $w \in Fix(S) \bigcap Fix(T)$. Let $\{x_n\}$ be the sequence of the modified Ishikawa iteration defined by (1). If $0 < a \leq \alpha_n, \beta_n \leq b < 1$, then $\lim_{n\to\infty} \|Sx_n - x_n\| = 0$.*

Proof As S satisfies the condition (E), there exists $\mu \geq 1$, such that

$$\begin{aligned} \|Sx_n - x_n\| &= \|Sx_n - y_n + y_n - x_n\| \\ &\leq \|Sx_n - y_n\| + \|y_n - x_n\| \\ &\leq \mu\|Sy_n - y_n\| + 2\|y_n - x_n\| \\ &\leq \mu\|Sy_n - x_n\| + (\mu + 2)\|y_n - x_n\| \\ &= \mu\|Sy_n - x_n\| + (\mu + 2)\|(1-\beta_n)x_n + \beta_n z_n - x_n\| \\ &= \mu\|Sy_n - x_n\| + (\mu + 2)\beta_n\|x_n - z_n\|, \end{aligned}$$

therefore,

$$\lim_{n\to\infty} \|Sx_n - x_n\| \leq \lim_{n\to\infty} \mu\|Sy_n - x_n\| + \lim_{n\to\infty} (\mu + 2)\beta_n\|x_n - z_n\|.$$

Thus, by Lemma 3.2 and Lemma 3.3,

$$\lim_{n\to\infty} \|Sx_n - x_n\| = 0.$$

Now, we are equipped to prove the our main results of this section. □

Theorem 3.5. Let K be a nonempty compact convex subset of a uniformly convex Banach space X. Let $S : K \to K$ be a single-valued mapping which satisfies conditions (C_λ) and (E) while $T : K \to CB(K)$ be a multi-valued mapping which satisfies condition (C_λ). Also, $Fix(S) \bigcap Fix(T) \neq \emptyset$ and $Tw = \{w\}$ for $w \in Fix(S) \bigcap Fix(T)$. Let $\{x_n\}$ be the sequence of the modified Ishikawa iteration defined by (1). If $0 < a \leq \alpha_n, \ \beta_n \leq b < 1$, then $\{x_{n_i}\} \to y$ for some subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $\{x_{n_i}\} \in Tx_{n_i}$ for all n_i implies $y \in Fix(S) \bigcap Fix(T)$.

Proof Assume that $\lim_{i\to\infty} \|x_{n_i} - y\| = 0$. By Lemma 3.4, we obtain $0 = \lim_{i\to\infty} \|Sx_{n_i} - x_{n_i}\| = \lim_{i\to\infty} \|(I - S)(x_{n_i})\|$. Since $I - S$ is strongly demiclosed at 0, we have $(I - S)(y) = 0$. Thus $y = Sy$, i.e. $y \in Fix(S)$. Since $\lambda dist(x_{n_i}, Tx_{n_i}) = 0 \leq \|x_{n_i} - y\|$, then we have $H(Tx_{n_i}, Ty) \leq \|x_{n_i} - y\|$. By Lemma 2.6, we have

$$\begin{aligned} dist(y, Ty) &\leq \|y - x_{n_i}\| + dist(x_{n_i}, Tx_{n_i}) + H(Tx_{n_i}, Ty) \\ &\leq \|y - x_{n_i}\| + dist(x_{n_i}, Tx_{n_i}) + \|x_{n_i} - y\| \ \to 0 \end{aligned}$$

as $i \to \infty$. It follows that $y \in Fix(T)$. Thus $y \in Fix(S) \bigcap Fix(T)$. □

Theorem 3.5 *Let K be a nonempty compact convex subset of a uniformly convex Banach space X. Let $S : K \to K$ be a single-valued mapping which satisfies conditions (C_λ) and (E) while $T : K \to CB(K)$ be a multi-valued mapping which satisfies condition (C_λ). Also, $Fix(S) \bigcap Fix(T) \neq \emptyset$ and $Tw = \{w\}$ for $w \in Fix(S) \bigcap Fix(T)$. Let $\{x_n\}$ be the sequence of the modified Ishikawa iteration defined by (1) with $0 < a \leq \alpha_n, \ \beta_n \leq b < 1$. Then $\{x_n\}$ converges strongly to a common fixed point of S and T.*

Proof Since $\{x_n\}$ is contained in K which is compact, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $\{x_{n_i}\}$ converges strongly to some point $y \in K$, that is, $\lim_{i\to\infty} \|x_{n_i} - y\| = 0$. By Theorem 2.7, we get $y \in Fix(S) \bigcap Fix(T)$, and by Lemma 3.1, we have that $\lim_{n\to\infty} \|x_n - y\|$ exists. It must be the case in which $\lim_{n\to\infty} \|x_n - y\| = \lim_{i\to\infty} \|x_{n_i} - y\| = 0$. Thus, $\{x_n\}$ converges strongly to $y \in Fix(S) \bigcap Fix(T)$. □

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