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Hopf Hypersurfaces in Complex Two-Plane Grassmannians with Reeb Parallel Shape Operator

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Abstract In this paper, we consider a new notion of *Reeb parallel shape operator* for real hypersurfaces M in complex two-plane Grassmannians $G_2(\mathbb{C}^{m+2})$. When M has Reeb parallel shape operator and non-vanishing geodesic Reeb flow, it becomes a real hypersurface of Type (A) with exactly four distinct constant principal curvatures. Moreover, if M has vanishing geodesic Reeb flow and Reeb parallel shape operator, then M is model space of Type (A) with the radius $r = \frac{\pi}{4\sqrt{2}}$.

Keywords Complex two-plane Grassmannians · Hopf hypersurface · Shape operator · Parallel shape operator · $\mathfrak{F}$-parallel shape operator · Reeb parallel shape operator

Mathematics Subject Classification Primary 53C40 · Secondary 53C15

1 Introduction

We denote by $G_2(\mathbb{C}^{m+2})$ the set of all complex two-dimensional linear subspaces in $\mathbb{C}^{m+2}$. This Riemannian symmetric space $G_2(\mathbb{C}^{m+2})$ has a remarkable geometric

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structure. It is the unique compact irreducible Riemannian manifold with both a Kähler structure J and a quaternionic Kähler structure $\mathfrak{J}$ not containing J. Namely, $G_2(\mathbb{C}^{m+2})$ is a unique compact, irreducible, Kähler, quaternionic Kähler manifold which is not a hyper-Kähler manifold. Accordingly, in $G_2(\mathbb{C}^{m+2})$ we have two natural geometric conditions for real hypersurfaces M: that the 1-dimensional distribution $[\xi] = \text{Span}\{\xi\}$ and the 3-dimensional distribution $\mathfrak{D}^\perp = \text{Span}\{\xi_1, \xi_2, \xi_3\}$ are invariant under the shape operator A of M (see [2,3] and [4]).

The almost contact structure vector field ξ is defined by $\xi = -JN$ and is said to be a *Reeb* vector field, where N denotes a local unit normal vector field of M in $G_2(\mathbb{C}^{m+2})$. The *almost contact 3-structure* vector fields $\{\xi_1, \xi_2, \xi_3\}$ for the 3-dimensional distribution $\mathfrak{D}^\perp$ of M in $G_2(\mathbb{C}^{m+2})$ are defined by $\xi_\nu = -J_\nu N$ $(\nu = 1, 2, 3)$, where J_ν denotes a canonical local basis of a quaternionic Kähler structure $\mathfrak{J}$ and $T_x M = \mathfrak{D} \oplus \mathfrak{D}^\perp$, $x \in M$.

Using two invariant conditions mentioned above and the result in Alekseevskii [1], Berndt and Suh [3] proved the following:

Theorem A *Let M be a connected orientable real hypersurface in $G_2(\mathbb{C}^{m+2})$, $m \geq 3$. Then both $[\xi]$ and $\mathfrak{D}^\perp$ are invariant under the shape operator of M if and only if*

(A) *M is an open part of a tube around a totally geodesic $G_2(\mathbb{C}^{m+1})$ in $G_2(\mathbb{C}^{m+2})$, or*

(B) *m is even, say $m = 2n$, and M is an open part of a tube around a totally geodesic $\mathbb{H}P^n$ in $G_2(\mathbb{C}^{m+2})$.*

Furthermore, the Reeb vector field ξ is said to be *Hopf* if it is invariant under the shape operator A. The one dimensional foliation of M by the integral manifolds of the Reeb vector field ξ is said to be a *Hopf foliation* of M. We say that M is a *Hopf hypersurface* in $G_2(\mathbb{C}^{m+2})$ if and only if the Hopf foliation of M is totally geodesic. By the formulas in Sect. 3, it can be easily checked that M is Hopf if and only if the Reeb vector field ξ is Hopf. In particular, M is said to be a real hypersurface with *non-vanishing geodesic Reeb flow* in $G_2(\mathbb{C}^{m+2})$ if it has a nonzero principal curvature for the Reeb vector field ξ, that is, $A\xi = \alpha\xi$ where $\alpha = g(A\xi, \xi) \neq 0$.

Using Theorem A, many geometers have given characterizations for Hopf hypersurfaces in $G_2(\mathbb{C}^{m+2})$ under certain assumption for various geometry quantities, for instance, shape operator, normal (or structure) Jacobi operator, structure tensor, and so on.

In [4], Berndt and Suh considered some equivalent conditions of isometric Reeb flow. Here, the Reeb flow on M in $G_2(\mathbb{C}^{m+2})$ is *isometric* means the Reeb vector field ξ on M is Killing. Using this notion, they gave a characterization of real hypersurfaces of Type (A) in Theorem A as follows:

Theorem B *Let M be a connected orientable real hypersurface in $G_2(\mathbb{C}^{m+2})$, $m \geq 3$. Then the Reeb flow on M is isometric if and only if M is an open part of a tube around a totally geodesic $G_2(\mathbb{C}^{m+1})$ in $G_2(\mathbb{C}^{m+2})$.*

Among the equivalent conditions of isometric Reeb flow in [4], it is very useful to our proof in Sect. 5 that the Reeb flow on M is isometric if and only if the shape operator A and the structure tensor field ϕ commute with each other, that is, $A\phi = \phi A$.

Moreover, Lee and Suh [9] gave a characterization of real hypersurfaces of Type (B) in $G_2(\mathbb{C}^{m+2})$ in terms of the Reeb vector field ξ as follows:

Theorem C *Let M be a connected orientable Hopf hypersurface in $G_2(\mathbb{C}^{m+2})$, $m \geq 3$. Then the Reeb vector field ξ belongs to the distribution $\mathfrak{D}$ if and only if M is locally congruent to an open part of a tube around a totally geodesic $\mathbb{H}P^n$ in $G_2(\mathbb{C}^{m+2})$, $m = 2n$, where the distribution $\mathfrak{D}$ denotes the orthogonal complement of $\mathfrak{D}^\perp = \mathrm{Span}\{\xi_1, \xi_2, \xi_3\}$.*

In [11], Suh proved the non-existence of real hypersurfaces in $G_2(\mathbb{C}^{m+2})$ with *parallel shape operator*, that is, $(\nabla_X A)Y = 0$, where X and Y are any tangent vector field on M. Moreover, he [12] also considered a new condition which is to restrict X to a distribution $\mathfrak{F} = [\xi] \cup \mathfrak{D}^\perp$, namely $\mathfrak{F}$-*parallel shape operator*, and gave two non-existence theorems to the following two cases of real hypersurfaces M in $G_2(\mathbb{C}^{m+2})$ with $\mathfrak{F}$-parallel shape operator: One is when M is a Hopf hypersurface. Another is when M is a real hypersurface in $G_2(\mathbb{C}^{m+2})$ satisfying $\mathfrak{D}^\perp$-invariance under the shape operator, that is, $A\mathfrak{D}^\perp \subset \mathfrak{D}^\perp$. As regards a weaker condition for a real hypersurface in $G_2(\mathbb{C}^{m+2})$ with parallel shape operator, in [6] and [8] Kim, Yang and the first author considered *recurrent* and *η-parallel shape operator* and gave non-existence theorems of Hopf hypersurfaces in $G_2(\mathbb{C}^{m+2})$ satisfying such weaker parallelism conditions, respectively.

Motivated by these notions, it is natural to consider a condition weaker than parallel shape operator for real hypersurfaces M in $G_2(\mathbb{C}^{m+2})$. From such a point of view, the authors in [5] studied a generalized parallelness for the shape operator of M in $G_2(\mathbb{C}^{m+2})$, namely η-parallel shape operator of M. They defined the *η-parallel* shape operator of M in $G_2(\mathbb{C}^{m+2})$ if the shape operator A of M satisfies $g((\nabla_X A)Y, Z) = 0$ for any tangent vectors $X, Y, Z \in \mathfrak{h}$, where $\mathfrak{h}$ denotes the set of all tangent vectors being orthogonal to the Reeb vector ξ in T_xM, $x \in M$. From this definition, we see that it becomes a weaker condition than parallel shape operator.

Accordingly, we consider a new notion weaker than parallel shape operator, that is, *Reeb parallel shape operator* which is defined by

$$(\nabla_\xi A)Y = 0 \tag{*}$$

for any tangent vector field Y on M.

In this paper, we give a classification of real hypersurfaces in $G_2(\mathbb{C}^{m+2})$ with *Reeb parallel shape operator* as follows:

Theorem 1 *Let M be a connected orientable real hypersurface in $G_2(\mathbb{C}^{m+2})$, $m \geq 3$, with non-vanishing geodesic Reeb flow. Then the shape operator of M is Reeb parallel if and only if M is an open part of a tube of some radius $r \in (0, \frac{\pi}{4\sqrt{2}}) \cup (\frac{\pi}{4\sqrt{2}}, \frac{\pi}{2\sqrt{2}})$ around a totally geodesic $G_2(\mathbb{C}^{m+1})$ in $G_2(\mathbb{C}^{m+2})$.*

Actually, when the Reeb vector field ξ belongs to the distribution $\mathfrak{D}^\perp$, the shape operator A for a Hopf hypersurface M in $G_2(\mathbb{C}^{m+2})$ with vanishing geodesic Reeb flow satisfies automatically the Reeb parallelness (see Sect. 6). Using this fact, we give:

Theorem 2 *Let M be a connected orientable real hypersurface in* $G_2(\mathbb{C}^{m+2})$, $m \geq 3$, *with Reeb parallel shape operator and vanishing geodesic Reeb flow. If the squared norm of the shape operator satisfies* $TrA^2 = ||A||^2 \leq 4m$, *then M is locally congruent to an open part of a tube of radius* $r = \frac{\pi}{4\sqrt{2}}$ *around a totally geodesic* $G_2(\mathbb{C}^{m+1})$ *in* $G_2(\mathbb{C}^{m+2})$.

In order to give the proof of our theorems, in Sect. 2 we recall Riemannian geometry of complex two-plane Grassmannians $G_2(\mathbb{C}^{m+2})$. In Sect. 3, some fundamental formulas including the Codazzi and Gauss equations for real hypersurfaces in $G_2(\mathbb{C}^{m+2})$ will be also recalled. In Sect. 4, we will prove that the Reeb vector field ξ of a Hopf hypersurface in $G_2(\mathbb{C}^{m+2})$ with Reeb parallel shape operator belongs to either the distribution $\mathfrak{D}$ or the distribution $\mathfrak{D}^{\perp}$. And in the same section, we will check whether real hypersurfaces of Type (A) or Type (B) in Theorem A satisfy the condition (*) or not. In Sect. 5, we will give a complete proof of our Theorem 1 according to the non-vanishing geodesic Reeb flow. Finally, we will give the proof of Theorem 2 in Sect. 6.

2 Riemannian Geometry of $G_2(\mathbb{C}^{m+2})$

In this section, we summarize basic material about $G_2(\mathbb{C}^{m+2})$, for details we refer to [2,3], and [4]. By $G_2(\mathbb{C}^{m+2})$ we denote the set of all complex two-dimensional linear subspaces in $\mathbb{C}^{m+2}$. The special unitary group $G = SU(m+2)$ acts transitively on $G_2(\mathbb{C}^{m+2})$ with stabilizer isomorphic to $K = S(U(2) \times U(m)) \subset G$. Then $G_2(\mathbb{C}^{m+2})$ can be identified with the homogeneous space G/K, which we equip with the unique analytic structure for which the natural action of G on $G_2(\mathbb{C}^{m+2})$ becomes analytic. Denote by $\mathfrak{g}$ and $\mathfrak{k}$ the Lie algebra of G and K, respectively, and by $\mathfrak{m}$ the orthogonal complement of $\mathfrak{k}$ in $\mathfrak{g}$ with respect to the Cartan-Killing form B of $\mathfrak{g}$. Then $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$ is an $Ad(K)$-invariant reductive decomposition of $\mathfrak{g}$. We put $o = eK$ and identify $T_oG_2(\mathbb{C}^{m+2})$ with $\mathfrak{m}$ in the usual manner. Since B is negative definite on $\mathfrak{g}$, its negative restricted to $\mathfrak{m} \times \mathfrak{m}$ yields a positive definite inner product on $\mathfrak{m}$. By $Ad(K)$-invariance of B this inner product can be extended to a G-invariant Riemannian metric g on $G_2(\mathbb{C}^{m+2})$. In this way, $G_2(\mathbb{C}^{m+2})$ becomes a Riemannian homogeneous space, even a Riemannian symmetric space. For computational reasons, we normalize g such that the maximal sectional curvature of $(G_2(\mathbb{C}^{m+2}), g)$ is eight.

When $m = 1$, $G_2(\mathbb{C}^3)$ is isometric to the two-dimensional complex projective space $\mathbb{C}P^2$ with constant holomorphic sectional curvature eight.

When $m = 2$, we note that the isomorphism $Spin(6) \simeq SU(4)$ yields an isometry between $G_2(\mathbb{C}^4)$ and the real Grassmann manifold $G_2^+(\mathbb{R}^6)$ of oriented two-dimensional linear subspaces in $\mathbb{R}^6$. In this paper, we will assume $m \geq 3$.

The Lie algebra $\mathfrak{k}$ of K has the direct sum decomposition $\mathfrak{k} = \mathfrak{su}(m) \oplus \mathfrak{su}(2) \oplus \mathfrak{R}$, where $\mathfrak{R}$ denotes the center of $\mathfrak{k}$. Viewing $\mathfrak{k}$ as the holonomy algebra of $G_2(\mathbb{C}^{m+2})$, the center $\mathfrak{R}$ induces a Kähler structure J and the $\mathfrak{su}(2)$-part a quaternionic Kähler structure $\mathfrak{J}$ on $G_2(\mathbb{C}^{m+2})$. If J_ν is any almost Hermitian structure in $\mathfrak{J}$, then $JJ_\nu = J_\nu J$, and JJ_ν is a symmetric endomorphism with $(JJ_\nu)^2 = I$ and $\text{tr}(JJ_\nu) = 0$ for $\nu = 1, 2, 3$.

A canonical local basis $\{J_1, J_2, J_3\}$ of $\mathfrak{J}$ consists of three local almost Hermitian structures J_ν in $\mathfrak{J}$ such that $J_\nu J_{\nu+1} = J_{\nu+2} = -J_{\nu+1}J_\nu$, where the index ν is taken modulo three. Since $\mathfrak{J}$ is parallel with respect to the Riemannian connection $\widetilde{\nabla}$ of $(G_2(\mathbb{C}^{m+2}), g)$, there exist for any canonical local basis $\{J_1, J_2, J_3\}$ of $\mathfrak{J}$ three local one-forms q_1, q_2, q_3 such that

$$\widetilde{\nabla}_X J_\nu = q_{\nu+2}(X)J_{\nu+1} - q_{\nu+1}(X)J_{\nu+2} \tag{2.1}$$

for all vector fields X on $G_2(\mathbb{C}^{m+2})$.

The Riemannian curvature tensor $\widetilde{R}$ of $G_2(\mathbb{C}^{m+2})$ is locally given by

$$\begin{aligned}\widetilde{R}(X,Y)Z = {} & g(Y,Z)X - g(X,Z)Y + g(JY,Z)JX \\ & - g(JX,Z)JY - 2g(JX,Y)JZ \\ & + \sum_{\nu=1}^{3}\Big\{g(J_\nu Y,Z)J_\nu X - g(J_\nu X,Z)J_\nu Y - 2g(J_\nu X,Y)J_\nu Z\Big\} \\ & + \sum_{\nu=1}^{3}\Big\{g(J_\nu JY,Z)J_\nu JX - g(J_\nu JX,Z)J_\nu JY\Big\},\end{aligned} \tag{2.2}$$

where $\{J_1, J_2, J_3\}$ denotes a canonical local basis of $\mathfrak{J}$.

3 Some Fundamental Formulas

In this section, we derive some basic formulas and the Codazzi equation for a real hypersurface in $G_2(\mathbb{C}^{m+2})$ (see [9–12] and [7]).

Let M be a real hypersurface of $G_2(\mathbb{C}^{m+2})$, that is, a submanifold of $G_2(\mathbb{C}^{m+2})$ with real codimension one. The induced Riemannian metric on M will also be denoted by g, and ∇ denotes the Riemannian connection of (M, g). Let N be a local unit normal vector field of M and A the shape operator of M with respect to N.

Now let us put

$$JX = \phi X + \eta(X)N, \quad J_\nu X = \phi_\nu X + \eta_\nu(X)N \tag{3.1}$$

for any tangent vector field X of a real hypersurface M in $G_2(\mathbb{C}^{m+2})$. From the Kähler structure J of $G_2(\mathbb{C}^{m+2})$ there exists an almost contact metric structure (ϕ, ξ, η, g) induced on M in such a way that

$$\phi^2 X = -X + \eta(X)\xi, \quad \eta(\xi) = 1, \quad \phi\xi = 0, \quad \eta(X) = g(X, \xi) \tag{3.2}$$

for any vector field X on M. Furthermore, let $\{J_1, J_2, J_3\}$ be a canonical local basis of $\mathfrak{J}$. Then the quaternionic Kähler structure J_ν of $G_2(\mathbb{C}^{m+2})$, together with the condition $J_\nu J_{\nu+1} = J_{\nu+2} = -J_{\nu+1}J_\nu$ in Sect. 1, induces an almost contact metric 3-structure $(\phi_\nu, \xi_\nu, \eta_\nu, g)$ on M as follows:

$$\begin{aligned}
&\phi_\nu^2 X = -X + \eta_\nu(X)\xi_\nu, \quad \eta_\nu(\xi_\nu) = 1, \quad \phi_\nu \xi_\nu = 0, \\
&\phi_{\nu+1}\xi_\nu = -\xi_{\nu+2}, \quad \phi_\nu \xi_{\nu+1} = \xi_{\nu+2}, \\
&\phi_\nu \phi_{\nu+1} X = \phi_{\nu+2} X + \eta_{\nu+1}(X)\xi_\nu, \\
&\phi_{\nu+1}\phi_\nu X = -\phi_{\nu+2} X + \eta_\nu(X)\xi_{\nu+1}
\end{aligned} \tag{3.3}$$

for any vector field X tangent to M. Moreover, from the commuting property of $J_\nu J = J J_\nu$, $\nu = 1, 2, 3$ in Sect. 2 and (3.1), the relation between these two contact metric structures (ϕ, ξ, η, g) and $(\phi_\nu, \xi_\nu, \eta_\nu, g)$, $\nu = 1, 2, 3$, can be given by

$$\begin{aligned}
\phi\phi_\nu X &= \phi_\nu \phi X + \eta_\nu(X)\xi - \eta(X)\xi_\nu, \\
\eta_\nu(\phi X) &= \eta(\phi_\nu X), \quad \phi\xi_\nu = \phi_\nu \xi.
\end{aligned} \tag{3.4}$$

On the other hand, from the parallelism of Kähler structure J, that is, $\widetilde{\nabla} J = 0$ and the quaternionic Kähler structure $\mathfrak{J}$ (see (2.1)), together with Gauss and Weingarten formulas it follows that

$$(\nabla_X \phi)Y = \eta(Y)AX - g(AX, Y)\xi, \quad \nabla_X \xi = \phi AX, \tag{3.5}$$

$$\nabla_X \xi_\nu = q_{\nu+2}(X)\xi_{\nu+1} - q_{\nu+1}(X)\xi_{\nu+2} + \phi_\nu AX, \tag{3.6}$$

$$\begin{aligned}
(\nabla_X \phi_\nu)Y = &-q_{\nu+1}(X)\phi_{\nu+2}Y + q_{\nu+2}(X)\phi_{\nu+1}Y + \eta_\nu(Y)AX \\
&- g(AX, Y)\xi_\nu.
\end{aligned} \tag{3.7}$$

Combining these formulas, we find the following:

$$\begin{aligned}
\nabla_X(\phi_\nu \xi) &= \nabla_X(\phi \xi_\nu) \\
&= (\nabla_X \phi)\xi_\nu + \phi(\nabla_X \xi_\nu) \\
&= q_{\nu+2}(X)\phi_{\nu+1}\xi - q_{\nu+1}(X)\phi_{\nu+2}\xi + \phi_\nu \phi AX \\
&\quad - g(AX, \xi)\xi_\nu + \eta(\xi_\nu)AX.
\end{aligned} \tag{3.8}$$

Using the above expression (2.2) for the curvature tensor $\widetilde{R}$ of $G_2(\mathbb{C}^{m+2})$, the equations of Codazzi and Gauss are respectively given by

$$\begin{aligned}
(\nabla_X A)Y - (\nabla_Y A)X = &\ \eta(X)\phi Y - \eta(Y)\phi X - 2g(\phi X, Y)\xi \\
&+ \sum_{\nu=1}^{3} \Big\{ \eta_\nu(X)\phi_\nu Y - \eta_\nu(Y)\phi_\nu X - 2g(\phi_\nu X, Y)\xi_\nu \Big\} \\
&+ \sum_{\nu=1}^{3} \Big\{ \eta_\nu(\phi X)\phi_\nu \phi Y - \eta_\nu(\phi Y)\phi_\nu \phi X \Big\} \\
&+ \sum_{\nu=1}^{3} \Big\{ \eta(X)\eta_\nu(\phi Y) - \eta(Y)\eta_\nu(\phi X) \Big\}\xi_\nu
\end{aligned} \tag{3.9}$$

and

$$\begin{aligned} R(X,Y)Z &= g(Y,Z)X - g(X,Z)Y \\ &\quad + g(\phi Y,Z)\phi X - g(\phi X,Z)\phi Y - 2g(\phi X,Y)\phi Z \\ &\quad + \sum_{\nu=1}^{3}\Big\{g(\phi_\nu Y,Z)\phi_\nu X - g(\phi_\nu X,Z)\phi_\nu Y - 2g(\phi_\nu X,Y)\phi_\nu Z\Big\} \\ &\quad + \sum_{\nu=1}^{3}\Big\{g(\phi_\nu\phi Y,Z)\phi_\nu\phi X - g(\phi_\nu\phi X,Z)\phi_\nu\phi Y\Big\} \\ &\quad - \sum_{\nu=1}^{3}\Big\{\eta(Y)\eta_\nu(Z)\phi_\nu\phi X - \eta(X)\eta_\nu(Z)\phi_\nu\phi Y\Big\} \\ &\quad - \sum_{\nu=1}^{3}\Big\{\eta(X)g(\phi_\nu\phi Y,Z) - \eta(Y)g(\phi_\nu\phi X,Z)\Big\}\xi_\nu \\ &\quad + g(AY,Z)AX - g(AX,Z)AY, \end{aligned} \tag{3.10}$$

where R denotes the curvature tensor of a real hypersurface M in $G_2(\mathbb{C}^{m+2})$.

4 Hopf Hypersurfaces in $G_2(\mathbb{C}^{m+2})$ with Reeb Parallel Shape Operator

From now on, we assume that M is a Hopf hypersurface in $G_2(\mathbb{C}^{m+2})$ with Reeb parallel shape operator, that is, the shape operator A of M satisfies:

$$(\nabla_\xi A)Y = 0 \tag{*}$$

for any tangent vector field Y on M.

Then from the equation of Codazzi (3.9), we have

$$(\nabla_\xi A)Y = (\nabla_Y A)\xi + \phi Y + \sum_{\nu=1}^{3}\big\{\eta_\nu(\xi)\phi_\nu Y - \eta_\nu(Y)\phi_\nu\xi + 3\eta_\nu(\phi Y)\xi_\nu\big\}$$

for any tangent vector field Y on M.

Since $(\nabla_Y A)\xi = (Y\alpha)\xi + \alpha\phi AY - A\phi AY$, the condition (*) can be written as

$$\begin{aligned} &(Y\alpha)\xi + \alpha\phi AY - A\phi AY + \phi Y \\ &\quad + \sum_{\nu=1}^{3}\big\{\eta_\nu(\xi)\phi_\nu Y - \eta_\nu(Y)\phi_\nu\xi + 3\eta_\nu(\phi Y)\xi_\nu\big\} = 0 \end{aligned} \tag{4.1}$$

for any tangent vector field Y on M.

Substituting $Y = \xi$ in above equation, we have $(\xi\alpha)\xi = 0$. From this, we obtain the following result:

Lemma 4.1 *Let M be a Hopf hypersurface in $G_2(\mathbb{C}^{m+2})$, $m \geq 3$, with Reeb parallel shape operator. Then the principal curvature α is constant along the direction of ξ, that is, $\xi\alpha = 0$.*

In this section, our main purpose is to show that the Reeb vector field ξ belongs to either the distribution $\mathfrak{D}$ or the orthogonal complement $\mathfrak{D}^\perp$ such that $T_x M = \mathfrak{D} \oplus \mathfrak{D}^\perp$ for any point $x \in M$.

To show this fact, unless otherwise stated in this section, we consider that the Reeb vector field ξ satisfies

$$\xi = \eta(X_0)X_0 + \eta(\xi_1)\xi_1 \tag{**}$$

for some unit vectors $X_0 \in \mathfrak{D}$ and $\xi_1 \in \mathfrak{D}^\perp$ and $\eta(X_0)\eta(\xi_1) \neq 0$.

Remark 4.2 Under this situation, in [7] the authors proved that $\mathfrak{D}$ and $\mathfrak{D}^\perp$-components of the Reeb vector field ξ are invariant under the shape operator when a Hopf hypersurface in $G_2(\mathbb{C}^{m+2})$ satisfies the condition $\xi\alpha = 0$.

On the other hand, using the notion of the geodesic Reeb flow, Berndt and Suh ([3,4]) proved the following:

Lemma A *If M is a connected orientable real hypersurface in $G_2(\mathbb{C}^{m+2})$ with geodesic Reeb flow, then we have the following two equations:*

$$Y\alpha = (\xi\alpha)\eta(Y) - 4\sum_{\nu=1}^{3} \eta_\nu(\xi)\eta_\nu(\phi Y), \tag{4.2}$$

and

$$\begin{aligned}\alpha A\phi Y + \alpha\phi AY - 2A\phi AY + 2\phi Y = 2\sum_{\nu=1}^{3}\Big\{&-\eta_\nu(Y)\phi\xi_\nu - \eta_\nu(\phi Y)\xi_\nu \\ &-\eta_\nu(\xi)\phi_\nu Y + 2\eta(Y)\eta_\nu(\xi)\phi\xi_\nu + 2\eta_\nu(\phi Y)\eta_\nu(\xi)\xi\Big\}\end{aligned} \tag{4.3}$$

for any tangent vector field Y on M.

Remark 4.3 Assume that the $\mathfrak{D}$-component of ξ is invariant under the shape operator A, that is, $AX_0 = \alpha X_0$. By putting $Y = X_0$ in (4.3) and using the fact $\phi X_0 = -\eta(\xi_1)\phi_1 X_0$ which is induced by $\phi\xi = 0$, we see that

$$\alpha A\phi X_0 = \left(\alpha^2 + 4\eta^2(X_0)\right)\phi X_0. \tag{4.4}$$

Now, using these facts, we prove the following proposition:

Proposition 4.4 *Let M be a real hypersurface in complex two-plane Grassmannians $G_2(\mathbb{C}^{m+2})$, $m \geq 3$ with Reeb parallel shape operator. Then the Reeb vector field ξ belongs to either the distribution $\mathfrak{D}$ or the distribution $\mathfrak{D}^\perp$.*

Proof Under our assumption, M is a Hopf hypersurface with Reeb parallel shape operator, we see that $\xi\alpha = 0$ (Lemma 4.1). Moreover, we know that $\mathfrak{D}$ and $\mathfrak{D}^{\perp}$-components of the Reeb vector field ξ are invariant under the shape operator A of M, that is, $A\xi_1 = \alpha\xi_1$ and $AX_0 = \alpha X_0$ (see Remark 4.2).

Actually, when the smooth function $\alpha = g(A\xi, \xi)$ identically vanishes, this proposition can be verified directly from (4.2).

Thus, in this proof, we consider only the case that the function α is non-vanishing. In order to prove our proposition, we put $Y = X_0$ in (4.1). It follows

$$(X_0\alpha)\xi + \alpha\phi AX_0 - A\phi AX_0 + \phi X_0 + \sum_{\nu=1}^{3}\left\{\eta_\nu(\xi)\phi_\nu X_0 + 3\eta_\nu(\phi X_0)\xi_\nu\right\} = 0.$$

Since $\phi X_0 \in \mathfrak{D}$ and $AX_0 = \alpha X_0$, it becomes

$$(X_0\alpha)\xi + \alpha^2\phi X_0 - \alpha A\phi X_0 + \phi X_0 + \sum_{\nu=1}^{3}\eta_\nu(\xi)\phi_\nu X_0 = 0.$$

Moreover, using (**), we have

$$\eta(X_0)(X_0\alpha)X_0 + \eta(\xi_1)(X_0\alpha)\xi_1 + \alpha^2\phi X_0 - \alpha A\phi X_0 + \phi X_0 + \eta(\xi_1)\phi_1 X_0 = 0.$$

From (4.4), we obtain

$$\eta(X_0)(X_0\alpha)X_0 + \eta(\xi_1)(X_0\alpha)\xi_1 - 4\eta^2(X_0)\phi X_0 + \phi X_0 + \eta(\xi_1)\phi_1 X_0 = 0.$$

Using $\phi\xi = 0$ and (**), we see that $\phi X_0 = -\eta(\xi_1)\phi_1 X_0$. It implies that

$$\eta(X_0)(X_0\alpha)X_0 + \eta(\xi_1)(X_0\alpha)\xi_1 + 4\eta^2(X_0)\eta(\xi_1)\phi_1 X_0 = 0. \tag{4.5}$$

Taking the inner product with $\phi_1 X_0$ in (4.5), we get $4\eta^2(X_0)\eta(\xi_1) = 0$. It contradicts our assumption $\eta(X_0)\eta(\xi_1) \neq 0$. Accordingly, we get a complete proof of our Proposition 4.4. □

Before giving the proofs of our Theorems in the introduction, let us check whether the shape operator A of real hypersurfaces of Type (A) and of Type (B) in Theorem A satisfies the condition (*) or not for any tangent vector field $Y \in TM$.

First, let us check our problem for the case that M is locally congruent to a real hypersurface of Type (A), that is, an open part of a tube around a totally geodesic $G_2(\mathbb{C}^{m+1})$ in $G_2(\mathbb{C}^{m+2})$ with some radius $r \in (0, \frac{\pi}{2\sqrt{2}})$. In order to do this, we recall a proposition due to Berndt and Suh [3] as follows:

Proposition A *Let M be a connected real hypersurface of $G_2(\mathbb{C}^{m+2})$. Suppose that $A\mathfrak{D} \subset \mathfrak{D}$, $A\xi = \alpha\xi$, and ξ is tangent to $\mathfrak{D}^{\perp}$. Let $J_1 \in \mathfrak{J}$ be the almost Hermitian* *structure such that* $JN = J_1N$. *Then* M *has three (if* $r = \pi/2\sqrt{8}$*) or four (otherwise) distinct constant principal curvatures*

$$\alpha = \sqrt{8}\cot(\sqrt{8}r), \quad \beta = \sqrt{2}\cot(\sqrt{2}r), \quad \lambda = -\sqrt{2}\tan(\sqrt{2}r), \quad \mu = 0$$

with some $r \in (0, \pi/\sqrt{8})$. *The corresponding multiplicities are*

$$m(\alpha) = 1, \quad m(\beta) = 2, \quad m(\lambda) = 2m - 2 = m(\mu),$$

and the corresponding eigenspaces are

$$\begin{aligned} T_\alpha &= \mathbb{R}\xi = \mathbb{R}JN = \mathbb{R}\xi_1 = \mathrm{Span}\{\xi\} = \mathrm{Span}\{\xi_1\}, \\ T_\beta &= \mathbb{C}^\perp\xi = \mathbb{C}^\perp N = \mathbb{R}\xi_2 \oplus \mathbb{R}\xi_3 = \mathrm{Span}\{\xi_2,\ \xi_3\}, \\ T_\lambda &= \{X \mid X \perp \mathbb{H}\xi,\ JX = J_1X\}, \\ T_\mu &= \{X \mid X \perp \mathbb{H}\xi,\ JX = -J_1X\}, \end{aligned}$$

where $\mathbb{R}\xi$, $\mathbb{C}\xi$, *and* $\mathbb{H}\xi$, *respectively, denote real, complex, and quaternionic span of the structure vector field* ξ, *and* $\mathbb{C}^\perp\xi$ *denotes the orthogonal complement of* $\mathbb{C}\xi$ *in* $\mathbb{H}\xi$.

For our convenience, let M_A be a real hypersurface of Type (A) in $G_2(\mathbb{C}^{m+2})$. Using the equation of Codazzi (3.9) and the fact that the principal curvature α of ξ is a constant, we have the following equation.

$$(\nabla_\xi A)Y = \alpha\phi AY - A\phi AY + \phi Y + \phi_1 Y + 2\eta_3(Y)\xi_2 - 2\eta_2(Y)\xi_3 \tag{4.6}$$

for any tangent vector field Y on M.

From now on, using (4.6), let us check whether each eigenspace, T_α, T_β, T_λ, and T_μ of M_A in $G_2(\mathbb{C}^{m+2})$, has Reeb parallel shape operator or not.

Case A-1 : $Y = \xi (= \xi_1) \in T_\alpha$.

By putting $Y = \xi$ into (4.6), we know that the shape operator A becomes *Reeb parallel*, that is, $(\nabla_\xi A)\xi = 0$.

Case A-2 : $Y \in T_\beta$ where $T_\beta = \mathrm{Span}\{\xi_2, \xi_3\}$.

Since T_β is spanned by ξ_2 and ξ_3, we put $Y = \xi_2$ and $Y = \xi_3$ in (4.6). Then we have

$$(\nabla_\xi A)\xi_2 = (\beta^2 - \alpha\beta - 2)\xi_3$$

and

$$(\nabla_\xi A)\xi_3 = -(\beta^2 - \alpha\beta - 2)\xi_2,$$

respectively. On the other hand, we know that

$$\beta^2 - \alpha\beta - 2 = 0,$$

because $\alpha = \sqrt{8}\cot(\sqrt{8}r)$ and $\beta = \sqrt{2}\cot(\sqrt{2}r)$ in Proposition A. So we conclude that the shape operator of M_A also satisfies $(\nabla_\xi A)Y = 0$ for any eigenvector $Y \in T_\beta$.

Case A-3 : $Y \in T_\lambda = \{Y \mid Y \in \mathfrak{D},\ JY = J_1Y\}$.

We naturally see that if $Y \in T_\lambda$ then $\phi Y = \phi_1 Y$. Moreover, the vector ϕY also belong to the eigenspace T_λ for any $Y \in T_\lambda$, that is, $\phi T_\lambda \subset T_\lambda$. Putting $Y \in T_\lambda$ in (4.6) and together with these facts, we obtain

$$(\nabla_\xi A)Y = -(\lambda^2 - \alpha\lambda - 2)\phi Y.$$

But in Proposition A, since the principal curvatures α and λ are given by $\alpha = \sqrt{8}\cot(\sqrt{8}r)$ and $\lambda = -\sqrt{2}\tan(\sqrt{2}r)$ for $r \in (0, \pi/\sqrt{8})$, respectively, we get

$$\lambda^2 - \alpha\lambda - 2 = 0.$$

It implies that $(\nabla_\xi A)Y = 0$ for any tangent vector field $Y \in T_\lambda$.

Case A-4: $Y \in T_\mu = \{\, Y \mid Y \in \mathfrak{D},\ JY = -J_1 Y \,\}$.

In this case, if $Y \in T_\mu$ then $\phi Y = -\phi_1 Y$. Moreover, we see $\phi T_\mu \subset T_\mu$. So, (4.6) is reduced to $(\nabla_\xi A)X = 0$, because $\mu = 0$.

Summing up all cases mentioned above, we can assert that:

Remark 4.5 The shape operator A of real hypersurfaces of Type (A) in $G_2(\mathbb{C}^{m+2})$ is *Reeb parallel*.

Next, let us check whether the shape operator A of real hypersurfaces of Type (B) satisfies the condition (*) for any tangent vector field $Y \in TM$. From now on, we will denote such real hypersurfaces by M_B for the sake of convenience. As is well known, M_B has five distinct constant principal curvatures as follows [3]:

Proposition B *Let M be a connected real hypersurface of $G_2(\mathbb{C}^{m+2})$. Suppose that $A\mathfrak{D} \subset \mathfrak{D}$, $A\xi = \alpha\xi$, and ξ is tangent to $\mathfrak{D}$. Then the quaternionic dimension m of $G_2(\mathbb{C}^{m+2})$ is even, say $m = 2n$, and M has five distinct constant principal curvatures*

$$\alpha = -2\tan(2r), \quad \beta = 2\cot(2r), \quad \gamma = 0, \quad \lambda = \cot(r), \quad \mu = -\tan(r)$$

with some $r \in (0, \pi/4)$. The corresponding multiplicities are

$$m(\alpha) = 1, \quad m(\beta) = 3 = m(\gamma), \quad m(\lambda) = 4n - 4 = m(\mu)$$

and the corresponding eigenspaces are

$$\begin{aligned} T_\alpha &= \mathbb{R}\xi = \mathrm{Span}\{\xi\}, \\ T_\beta &= \mathfrak{J}J\xi = \mathrm{Span}\{\xi_\nu \mid \nu = 1, 2, 3\}, \\ T_\gamma &= \mathfrak{J}\xi = \mathrm{Span}\{\phi_\nu\xi \mid \nu = 1, 2, 3\}, \\ &\quad T_\lambda, \quad T_\mu, \end{aligned}$$

where

$$T_\lambda \oplus T_\mu = (\mathbb{H}\mathbb{C}\xi)^\perp, \quad \mathfrak{J}T_\lambda = T_\lambda, \quad \mathfrak{J}T_\mu = T_\mu, \quad JT_\lambda = T_\mu.$$

The distribution $(\mathbb{H}\mathbb{C}\xi)^\perp$ is the orthogonal complement of $\mathbb{H}\mathbb{C}\xi$ where

$$\mathbb{H}\mathbb{C}\xi = \mathbb{R}\xi \oplus \mathbb{R}J\xi \oplus \mathfrak{J}\xi \oplus \mathfrak{J}J\xi.$$

Now, to prove the claim, we suppose that M_B has Reeb parallel shape operator. Then, since $\xi \in \mathfrak{D}$, M_B satisfies the following equation:

$$\alpha\phi AY - A\phi AY + \phi Y - \sum_{\nu=1}^{3} \left\{ \eta_\nu(Y)\phi_\nu\xi - 3\eta_\nu(\phi Y)\xi_\nu \right\} = 0, \quad \forall Y \in TM,$$

from (4.1).

If we put $Y = \xi_2 \in T_\beta$ in above equation, it becomes

$$\alpha\beta\phi\xi_2 = 0$$

because $A\phi_2\xi = \gamma\phi_2\xi$ and $\gamma = 0$. From this, it follows that

$$\alpha\beta = 0.$$

But, from Proposition B, we see that $\alpha\beta = -4$ for some $r \in (0, \pi/4)$. This is a contradiction. So this case can not occur.

Therefore we also have the following:

Remark 4.6 The shape operator A of real hypersurfaces of Type (B) in $G_2(\mathbb{C}^{m+2})$ does not satisfy the *Reeb parallel* condition (*).

5 The Proof of Theorem 1

In this section, let M be a real hypersurface in $G_2(\mathbb{C}^{m+2})$ with Reeb parallel shape operator and non-vanishing geodesic Reeb flow.

In [4], Berndt and Suh proved that

Lemma B *Let M be a connected orientable real hypersurface in Kähler manifolds. Then the following statements are equivalent:*

(a) *The Reeb flow on M is geodesic;*
(b) *The Reeb vector field ξ is a principal curvature vector of M everywhere;*
(c) *The maximal complex subbundle $\mathfrak{B}$ of TM is invariant under the shape operator A of M.*

From this, we see that a real hypersurface M satisfying our condition becomes Hopf, since our real hypersurface M has non-vanishing geodesic Reeb flow. Thus by Proposition 4.4, we consider the following two cases:

- **Case I :** the Reeb vector field ξ belongs to the distribution $\mathfrak{D}^\perp$,
- **Case II :** the Reeb vector field ξ belongs to the distribution $\mathfrak{D}$.

First of all, let us consider the Case I, that is, $\xi \in \mathfrak{D}^\perp$. Accordingly, we may put $\xi = \xi_1$. Under these assumptions, we prove the following:

Proposition 5.1 *Let M be a real hypersurface in $G_2(\mathbb{C}^{m+2})$, $m \geq 3$, with Reeb parallel shape operator and non-vanishing geodesic Reeb flow. If the Reeb vector field ξ belongs to the distribution $\mathfrak{D}^\perp$, then the shape operator A commutes with the structure tensor field ϕ.*

Proof From our assumptions, (4.1) can be written as

$$(Y\alpha)\xi + \alpha\phi AY - A\phi AY + \phi Y + \phi_1 Y + 2\eta_3(Y)\xi_2 - 2\eta_2(Y)\xi_3 = 0$$

for any tangent vector field Y on M. It follows that

$$2A\phi AY = 2(Y\alpha)\xi + 2\alpha\phi AY + 2\phi Y + 2\phi_1 Y + 4\eta_3(Y)\xi_2 - 4\eta_2(Y)\xi_3.$$

On the other hand, from (4.3) we also obtain

$$2A\phi AY = \alpha A\phi Y + \alpha\phi AY + 2\phi Y + 2\phi_1 Y + 4\eta_3(Y)\xi_2 - 4\eta_2(Y)\xi_3.$$

Thus from the preceding two equations, we have finally

$$2(Y\alpha)\xi + \alpha\phi AY - \alpha A\phi Y = 0 \tag{5.1}$$

for any tangent vector field Y on M.

But, under our assumptions, we have already seen that $\xi\alpha = 0$ (see Lemma 4.1). From this fact, (4.2) can be written as

$$Y\alpha = -4\sum_{\nu=1}^{3}\eta_\nu(\xi)\eta_\nu(\phi Y)$$

for any $Y \in TM$. Therefore since $\xi = \xi_1$, it follows that $Y\alpha = 0$. Substituting this result into (5.1), it follows that

$$\alpha(\phi A - A\phi)Y = 0$$

for any tangent vector field Y on M. It means that the shape operator A commutes with the structure tensor field ϕ on M in $G_2(\mathbb{C}^{m+2})$, since M has non-vanishing geodesic Reeb flow. It completes the proof of our Proposition 5.1.

Remark 5.2 As mentioned in the introduction, the structure tensor field ϕ and the shape operator A of M commute with each other if and only if the Reeb flow on M is isometric (see [4]).

Therefore from Theorem B and Remark 4.5, we have the following:

Theorem 5.3 *Let M be a connected real hypersurface in $G_2(\mathbb{C}^{m+2})$, $m \geq 3$ with non-vanishing geodesic Reeb flow. The shape operator A of M is Reeb parallel and the Reeb vector field ξ belongs to the distribution $\mathfrak{D}^\perp$ if and only if M is locally congruent to an open part of a tube around radius r on a totally geodesic $G_2(\mathbb{C}^{m+1})$ in $G_2(\mathbb{C}^{m+2})$ where $r \in (0, \frac{\pi}{4\sqrt{2}}) \cup (\frac{\pi}{4\sqrt{2}}, \frac{\pi}{2\sqrt{2}})$.*

Next, we consider the case $\xi \in \mathfrak{D}$. By Theorem C, we see that M is locally congruent to a real hypersurface of Type (B) under our assumptions. But as mentioned in Sect. 4, a real hypersurface of Type (B) does not have Reeb parallel shape operator (see Remark 4.6). From these facts, we obtain the following theorem:

Theorem 5.4 *There does not exist any real hypersurface in* $G_2(\mathbb{C}^{m+2})$, $m \geq 3$, *with Reeb parallel shape operator and non-vanishing geodesic Reeb flow when the Reeb vector field* ξ *belongs to the distribution* $\mathfrak{D}$.

Combining Proposition 4.4, and Theorems 5.3 and 5.4, this completes the proof of our Theorem 1 in the introduction.

6 The Proof of Theorem 2

From now on, let M be a real hypersurface in $G_2(\mathbb{C}^{m+2})$, $m \geq 3$, with Reeb parallel shape operator and vanishing geodesic Reeb flow. By virtue of Lemma B given in the previous section, M must be Hopf, that is, $A\xi = \alpha\xi$ where $\alpha = g(A\xi, \xi) = 0$. Then by Proposition 4.4, we consider the following two cases:

- **Case I :** the Reeb vector field ξ belongs to the distribution $\mathfrak{D}^\perp$,
- **Case II :** the Reeb vector field ξ belongs to the distribution $\mathfrak{D}$.

By virtue of Theorem C and Proposition B, we assert that *when the Reeb vector field* ξ *belongs to the distribution* $\mathfrak{D}$, *there does not exist any real hypersurface in* $G_2(\mathbb{C}^{m+2})$, $m \geq 3$, *with Reeb parallel shape operator and vanishing geodesic Reeb flow*. In fact, a real hypersurface of Type (B) in Theorem A due to Berndt and Suh [3] does not have vanishing geodesic Reeb flow (see Proposition B in Sect. 4).

From such a point of view, from now on we only consider the Case I, that is, $\xi \in \mathfrak{D}^\perp$. Accordingly, we may put $\xi = \xi_1$. Under these assumptions, we prove the following:

Proposition 6.1 *Let* M *be a real hypersurface in* $G_2(\mathbb{C}^{m+2})$, $m \geq 3$, *with Reeb parallel shape operator and vanishing geodesic Reeb flow. If the Reeb vector field* ξ *belongs to the distribution* $\mathfrak{D}^\perp$ *and the squared norm of the shape operator satisfies* $||A||^2 \leq 4m$, *then the Reeb flow on* M *is isometric.*

Proof Since M has vanishing geodesic Reeb flow, that is, $A\xi = 0$, we obtain

$$(\nabla_X A)\xi = -A\phi AX$$

for any tangent vector field X on M. Using the equation of Codazzi (3.9), we get

$$(\nabla_X A)\xi = -\phi X - \phi_1 X - 2\eta_3(X)\xi_2 + 2\eta_2(X)\xi_3$$

together with our assumptions that M has Reeb parallel shape operator and $\xi = \xi_1$ (since we now consider the case $\xi \in \mathfrak{D}^\perp$, we may put $\xi = \xi_1$). Hence the above two equations give us

$$A\phi AX = \phi X + \phi_1 X + 2\eta_3(X)\xi_2 - 2\eta_2(X)\xi_3 \tag{6.1}$$

for any vector field $X \in TM$.

Moreover, applying the structure tensor ϕ to (6.1), it can be written as

$$\begin{aligned}\phi A\phi AX &= \phi^2 X + \phi\phi_1 X + 2\eta_3(X)\phi\xi_2 - 2\eta_2(X)\phi\xi_3\\ &= -X + \eta(X)\xi + \phi\phi_1 X - 2\eta_3(X)\xi_3 - 2\eta_2(X)\xi_2\end{aligned}$$

for any tangent vector field X on M.

Let $\{e_1, e_2, \cdots, e_{4m-1}\}$ be an orthonormal basis for T_xM where x is any point of M. Then we get

$$\phi A\phi Ae_i = -e_i + \eta(e_i)\xi + \phi\phi_1 e_i - 2\eta_3(e_i)\xi_3 - 2\eta_2(e_i)\xi_2 \tag{6.2}$$

for $i = 1, 2, \cdots, 4m-1$. From this, we calculate the trace of the matrix $\phi A\phi A$, that is,

$$\begin{aligned}
\mathrm{Tr}(\phi A\phi A) &= \sum_{i=1}^{4m-1} g(\phi A\phi Ae_i, e_i) \\
&= -\sum_{i=1}^{4m-1} g(e_i, e_i) + \sum_{i=1}^{4m-1} \eta(e_i)g(\xi, e_i) + \sum_{i=1}^{4m-1} g(\phi\phi_1 e_i, e_i) \\
&\quad -2\sum_{i=1}^{4m-1} \eta_3(e_i)g(\xi_3, e_i) - 2\sum_{i=1}^{4m-1} \eta_2(e_i)g(\xi_2, e_i) \\
&= -(4m-1) + g(\xi,\xi) + \mathrm{Tr}(\phi\phi_1) - 2g(\xi_3,\xi_3) - 2g(\xi_2,\xi_2) \\
&= -4m,
\end{aligned} \tag{6.3}$$

together with $\mathrm{Tr}(\phi\phi_\nu) = 2\eta_\nu(\xi)$, $\nu = 1, 2, 3$ (see [13]).

On the other hand, we are able to calculate the following:

$$\begin{aligned}
||\phi A - A\phi||^2 &= \sum_{i=1}^{4m-1} g\big((\phi A - A\phi)e_i, (\phi A - A\phi)e_i\big) \\
&= -\sum_{i=1}^{4m-1} g(A\phi^2 Ae_i, e_i) + \sum_{i=1}^{4m-1} g(\phi A\phi Ae_i, e_i) \\
&\quad + \sum_{i=1}^{4m-1} g(A\phi A\phi e_i, e_i) - \sum_{i=1}^{4m-1} g(\phi A^2\phi e_i, e_i) \\
&= \sum_{i=1}^{4m-1} g(A^2 e_i, e_i) - \sum_{i=1}^{4m-1} \eta(Ae_i)g(A\xi, e_i) \\
&\quad +2\sum_{i=1}^{4m-1} g(A\phi A\phi e_i, e_i) - \sum_{i=1}^{4m-1} g(\phi A^2\phi e_i, e_i) \\
&= \mathrm{Tr}A^2 + 2\mathrm{Tr}(A\phi A\phi) - \mathrm{Tr}(\phi A^2\phi) \\
&= \mathrm{Tr}A^2 + 2\mathrm{Tr}(\phi A\phi A) - \mathrm{Tr}(A^2\phi^2) \\
&= 2\mathrm{Tr}A^2 + 2\mathrm{Tr}(\phi A\phi A),
\end{aligned} \tag{6.4}$$

using the facts, $A\xi = 0$ and $\mathrm{Tr}(AB) = \mathrm{Tr}(BA)$ for any two matrices A, B with same size.

From this, together with (6.3) and our assumption for the squared norm of shape operator A of M, the left side of (6.4) should vanish for a real hypersurface M in $G_2(\mathbb{C}^{m+2})$ with $\alpha = 0$ and $\nabla_\xi A = 0$. This gives that the shape operator A commutes with the structure tensor ϕ, that is, $A\phi = \phi A$. According to the result due to Berndt and Suh [4], the Reeb flow on M becomes isometric. It completes a proof of our proposition. □

Hence from Theorem B, we can assert that *if a real hypersurface M in $G_2(\mathbb{C}^{m+2})$ satisfies the conditions in Proposition 6.1, then M becomes a model space of Type* (A) *in Theorem* A. To serve the convenience of notation, a model space of Type (A) with radius r is denoted by M_A or $M_A(r)$. From this let us now check if the model space M_A satisfies the assumptions in Proposition 6.1 or not.

First, we can state that M_A has Reeb parallel shape operator from the observation given in Sect. 4. Moreover, we see that a model space M_A becomes an open part of a tube around a totally geodesic $G_2(\mathbb{C}^{m+1})$ in $G_2(\mathbb{C}^{m+2})$ with radius $r = \frac{\pi}{4\sqrt{2}}$, because the principal curvature α of ξ on M_A must be zero. From this and Proposition A in Sect. 4, we have the following three distinct principal curvatures and the corresponding multiplicities with respect to the eigenspaces of $M_A(\frac{\pi}{4\sqrt{2}})$:

Principal curvature	Multiplicity	Eigenspace
$\alpha = 0$	1	$T_\alpha = \mathrm{Span}\{\xi\}$
$\beta = \sqrt{2}$	2	$T_\beta = \mathrm{Span}\{\xi_2, \xi_3\}$
$\lambda = -\sqrt{2}$	$2(m-1)$	$T_\lambda = \{X \mid X \perp \mathbb{H}\xi,\ JX = J_1 X\}$
$\mu = 0$	$2(m-1)$	$T_\mu = \{X \mid X \perp \mathbb{H}\xi,\ JX = -J_1 X\}$

By this table and a straightforward calculation we have the squared norm of the shape operator A on $M_A(\frac{\pi}{4\sqrt{2}})$ as follows.

$$
\begin{aligned}
||A^2|| &= \sum_{i=1}^{4m-1} g(Ae_i, Ae_i) \\
&= \sum_{i=1}^{2m-2} g(Ae_i, Ae_i) + \sum_{i=2m-1}^{4m-4} g(Ae_i, Ae_i) + g(Ae_{4m-3}, Ae_{4m-3}) \\
&\quad + g(Ae_{4m-2}, Ae_{4m-2}) + g(Ae_{4m-1}, Ae_{4m-1}) \\
&= \sum_{i=1}^{2m-2} \lambda^2 g(e_i, e_i) + \sum_{i=2m-1}^{4m-4} \mu^2 g(e_i, e_i) + g(A\xi, A\xi) \\
&\quad + g(A\xi_2, A\xi_2) + g(A\xi_3, A\xi_3) \\
&= 2(m-1)\lambda^2 + 2(m-1)\mu^2 + \alpha^2 + 2\beta^2 \\
&= 4(m-1) + 4 \\
&= 4m,
\end{aligned}
$$

where $e_1, e_2, \cdots, e_{2m-2} \in T_\lambda$, $e_{2m-1}, \cdots, e_{4m-4} \in T_\mu$, $e_{4m-3} = \xi = \xi_1$, $e_{4m-2} = \xi_2$, $e_{4m-1} = \xi_3$. From this calculation, we see that $M_A(\frac{\pi}{4\sqrt{2}})$ also satisfies our assumption in Proposition 6.1.

Summing up these discussions, we obtain our Theorem 2 mentioned in the introduction. □

Lastly, we will give a proof for our assertion given in the introduction as follows.

Lemma 6.2 *Let M be a real hypersurface in $G_2(\mathbb{C}^{m+2})$, $m \geq 3$, with vanishing geodesic Reeb flow. If the Reeb vector field ξ belongs to the distribution $\mathfrak{D}^\perp$, then the shape operator A of M is Reeb parallel.*

Proof Using the equation of Codazzi (3.9), we obtain that

$$(\nabla_\xi A)Y - (\nabla_Y A)\xi = \phi Y + \sum_{\nu=1}^{3} \left\{\eta_\nu(\xi)\phi_\nu Y - \eta_\nu(Y)\phi_\nu \xi + 3\eta_\nu(\phi Y)\xi_\nu\right\}$$

for any tangent vector field Y on M.

From our assumptions, $A\xi = 0$ and $\xi = \xi_1$, we have

$$(\nabla_\xi A)Y + A\phi AY = \phi Y + \phi_1 Y + 2\eta_2(Y)\xi_2 - 2\eta_3(Y)\xi_3. \tag{6.5}$$

Moreover, since M is Hopf, (4.3) implies that

$$A\phi AY = \phi Y + \phi_1 Y + 2\eta_2(Y)\xi_2 - 2\eta_3(Y)\xi_3, \tag{6.6}$$

together with $\alpha = 0$ and $\xi = \xi_1$.

From (6.5) and (6.6), we get $(\nabla_\xi A)Y = 0$ for any tangent vector field Y on M. That is, a real hypersurface M in $G_2(\mathbb{C}^{m+2})$ with vanishing geodesic Reeb flow, that is, $\alpha = g(A\xi, \xi) = 0$ and $\xi \in \mathfrak{D}^\perp$ has automatically Reeb parallel shape operator.

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