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Total Coloring of Planar Graphs Without Some Chordal 6-cycles

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Abstract A k-total-coloring of a graph G is a coloring of vertex set and edge set using k colors such that no two adjacent or incident elements receive the same color. In this paper, we prove that if G is a planar graph with maximum $\Delta \geq 8$ and every 6-cycle of G contains at most one chord or any chordal 6-cycles are not adjacent, then G has a $(\Delta + 1)$-total-coloring.

Keywords Planar graph · Total coloring · Cycle

Mathematics Subject Classification 05C15

1 Introduction

All graphs considered in this paper are simple, finite and undirected, and we follow [2] for terminologies and notations not defined here. Let G be a graph. We use $V(G)$, $E(G)$, $\Delta(G)$ and $\delta(G)$ (or simply V, E, Δ and δ) to denote the vertex set, the edge set, the maximum degree and the minimum degree of G, respectively. For a vertex $v \in V$, let $N(v)$ denote the set of vertices adjacent to v and let $d(v) = |N(v)|$ denote the degree of v. A *k-vertex*, a *k^--vertex* or a *k^+-vertex* is a vertex of degree k, at most k or at least k, respectively. A *k-cycle* is a cycle of length k. We use $(v_1, v_2, \ldots, v_d)$ to denote a cycle (or a face) whose boundary vertices are $v_1, v_2, \ldots, v_d$ in the clockwise order. Note that all the subscripts in the paper are taken modulo d. We say that two

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cycles are *adjacent* (or *intersecting*) if they share at least one edge (or one vertex, respectively). Let $C = (v_1, v_2, \dots, v_k)(k \geq 4)$ be a cycle. If there is an edge $v_i v_j$ with $j \not\equiv i \pm 1 \pmod{k}$, then the edge $v_i v_j$ is called a *chord* of C.

A *k-total-coloring* of a graph $G = (V, E)$ is a coloring of $V \cup E$ using k colors such that no two adjacent or incident elements receive the same color. A graph G is *total-k-colorable* if it admits a k-total-coloring. The *total chromatic number* $\chi'(G)$ of G is the smallest integer k such that G has a k-total-coloring. Clearly, $\chi''(G) \geq \Delta+1$. Behzad [1] and Vizing [11] conjectured independently that $\chi''(G) \leq \Delta + 2$ for each graph G. This conjecture was confirmed for graphs with $\Delta \leq 5$. For planar graphs the only open case is that of $\Delta = 6$ (see [6,9]). In recent years, the study of total colorings planar graphs has attracted considerable attention. For planar graphs with large maximum degree, it is possible to determine $\chi''(G) = \Delta + 1$. This first result was given in [3] for $\Delta \geq 14$, which was finally extended to $\Delta \geq 9$ in [7]. Zhu [8] proved that if G is a planar graph with maximum degree 8, and for each vertex x, there is an integer $k_x \in \{3, 4, 5, 6, 7, 8\}$ such that there is no k_x-cycle which contains x, then $\chi''(G) = 9$. Wang et al. [13] extended this result to that there is at most one k_x-cycle which contains x. Chang [4] proved that for planar graph G with $\Delta \geq 7$, if there is an integer $k_x \in \{3, 4, 5, 6\}$ such that there is no k_x-cycle which contains x for each $x \in V(G)$, then $\chi''(G) = \Delta + 1$. Wang et al. [12] proved $\chi''(G) = \Delta + 1$ for some planar graphs with small maximum degree. Hou et al. [5] proved that every planar graphs with $\Delta \geq 8$ and without 6-cycles are total-9-colorable. Shen and Wang [10] extended this result to planar graphs without chordal 6-cycles. In this paper, we extend this result and get the following theorem.

Theorem 1 *Let G be a planar graph with maximum degree $\Delta \geq 8$. If every 6-cycle of G contains at most one chord or chordal 6-cycles are not adjacent in G, then $\chi''(G) = \Delta + 1$.*

2 Proof of Theorem 1

First, we introduce additional notations and definitions here for convenience. Let G be a planar graph having a plane drawing and let F be the face set of G. For a face f of G, the *degree* $d(f)$ is the number of edges incident with it, where each cut-edge is counted twice. A *k-face*, a *k^--face* or a *k^+-face* is a face of degree k, at most k or at least k, respectively. Denote by $n_d(v)$ the number of d-vertices adjacent to the vertex v, $f_d(v)$ the number of d-faces incident with v.

Now, we begin to prove Theorem 1. According to [7], the theorem is true for $\Delta \geq 9$. So we assume in the following that $\Delta = 8$. Let $G = (V, E)$ be a minimal counterexample to the planar graph G with maximum degree $\Delta = 8$, such that $|V|+|E|$ is minimal and G has been embedded in the plane. Then every proper subgraph of G is total-9-colorable. First we give some lemmas for G.

Lemma 1 *[3]*

(a) G is 2-connected.
(b) If uv is an edge of G with $d(u) \leq 4$, then $d(u) + d(v) \geq \Delta + 2 = 10$.

By Lemma 1(b), any two neighbors of a 2-vertex are 8-vertices.

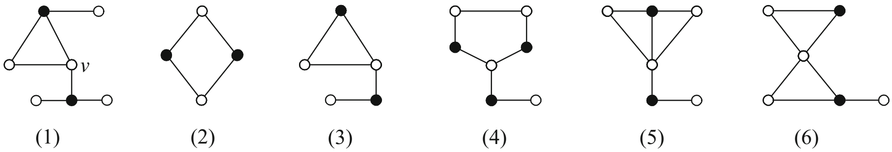

Fig. 1 Forbidden configurations in G

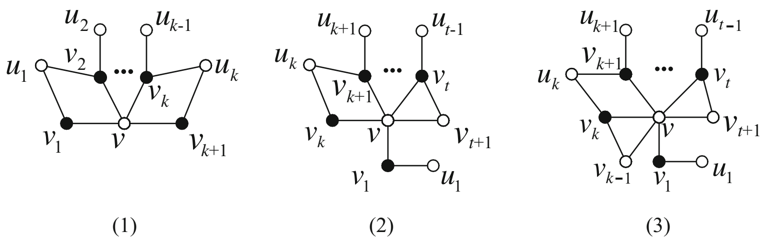

Fig. 2 Forbidden configurations in G

Note that in all figures of the paper, vertices marked • have no edges of G incident with them other than those shown and vertices marked ∘ are 3^+-vertices.

Lemma 2 *G has no configurations depicted in Fig. 1, where v denotes the vertex of degree of 7.*

Proof The proof of (1), (2), (4) and (6) can be found in [14], (3) can be found in [10], (5) can be found in [7]. □

Lemma 3 *Suppose v is a d-vertex of G with $d \geq 5$. Let $v_1, \dots, v_d$ be the neighbor of v and $f_1, f_2, \dots, f_d$ be faces incident with v, such that f_i is incident with v_i and v_{i+1}, for $i \in \{1, 2, \dots, d\}$. Let $d(v_1) = 2$ and $\{v, u_1\} = N(v_1)$. Then G does not satisfy one of the following conditions (see Fig. 2).*

(1) there exists an integer k $(2 \leq k \leq d-1)$ such that $d(v_{k+1}) = 2$, $d(v_i) = 3$ $(2 \leq i \leq k)$ and $d(f_j) = 4$ $(1 \leq j \leq k)$.
(2) there exist two integers k and t $(2 \leq k < t \leq d-1)$ such that $d(v_k) = 2$, $d(v_i) = 3$ $(k+1 \leq i \leq t)$, $d(f_t) = 3$ and $d(f_j) = 4$ $(k \leq j \leq t-1)$.

(3) there exist two integers k and t $(3 \leq k \leq t \leq d-1)$ such that $d(v_i) = 3$ $(k \leq i \leq t)$, $d(f_{k-1}) = d(f_t) = 3$ and $d(f_j) = 4$ $(k \leq j \leq t-1)$.

Proof Suppose G satisfies all of the conditions (1)-(3). If $d(f_i) = 4$, then let u_i be adjacent to v_i and v_{i+1}. By the minimality of G, $G' = G - vv_1$ has a $(\Delta+1)$-total-coloring ϕ. Let $C(x) = \{\phi(xy) : y \in N(x)\} \cup \{\phi(x)\}$. First we erase the colors on all 3^--vertices adjacent to v. We have $\phi(v_1u_1) \notin C(v)$, for otherwise, the number of the forbidden colors for vv_1 is at most Δ, so vv_1 can be properly colored and by properly recoloring the erased vertices, we get a $(\Delta+1)$-total-coloring of G, a contradiction. Without loss of generality, assume that $C(v) = \{1, 2, \dots, d\}$ with $\phi(vv_i) = i$ $(i \in \{2, 3, \dots, d\})$, $\phi(v_1u_1) = d+1$ and $\phi(v) = 1$. Thus we have

$d+1 \in C(v_i)$ for all $i \in \{2, 3, \ldots, d\}$, for otherwise, we can recolor vv_i with $d+1$ and color vv_1 with i, and by properly recoloring the erased vertices, we get a $(\Delta+1)$-total-coloring of G, a contradiction, too. In the following we consider (1)-(3) one by one.

(1) Since $d+1 \in C(v_i)$ for all $i \in \{2, 3, \ldots, d\}$, there is a vertex u_s $(1 \leq s \leq k)$ such that $d+1$ appears at least twice on u_s, a contradiction to ϕ.

(2) Since $d+1 \in C(v_i)$ for all $i \in \{2, 3, \ldots, d\}$, $\phi(v_k u_k) = \phi(v_{k+1} u_{k+1}) = \cdots = \phi(v_{t-1} u_{t-1}) = \phi(v_t v_{t+1}) = d+1$. We also have $\phi(v_t u_{t-1}) = t+1$. For otherwise, we can recolor $v_t v_{t+1}$ with $t+1$, vv_{t+1} with $d+1$ and color vv_1 with $t+1$. By properly recoloring the erased vertices, we get a $(\Delta+1)$-total-coloring of G, a contradiction. Similarly, $\phi(v_{t-1} u_{t-2}) = \phi(v_{t-2} u_{t-3}) = \cdots = \phi(v_{k+1} u_k) = t+1$. So we can recolor vv_{t+1} with $d+1$, $v_t v_{t+1}$ with $t+1$, $v_t u_{t-1}$ with $d+1$, $v_{t-1} u_{t-1}$ with $t+1$,. . ., $v_{k+1} u_{k+1}$ with $t+1$, $v_{k+1} u_k$ with $d+1$, $v_k u_k$ with $t+1$ and color vv_1 with $t+1$. By properly recoloring the erased vertices, we get a $(\Delta+1)$-total-coloring of G, also a contradiction.

(3) If $d+1 \notin \{\phi(v_{k-1} v_k) \cup \phi(v_t v_{t+1})\}$, then there is a vertex u_s $(k \leq s \leq t-1)$ such that $d+1$ appears at least twice on u_s, a contradiction to ϕ. So without loss of generality, assume $\phi(v_{k-1} v_k) = d+1$. If $\phi(v_{k+1} u_k) = d+1$, then $\phi(v_{k+2} u_{k+1}) = \phi(v_{k+3} u_{k+2}) = \cdots = \phi(v_t u_{t-1}) = d+1$. By the discussion of (2), we also have $\phi(v_k u_k) = \phi(v_{k+1} u_{k+1}) = \cdots = \phi(v_{t-1} u_{t-1}) = \phi(v_t v_{t+1}) = k-1$. Then we can recolor vv_{k-1} with $d+1$, $v_{k-1} v_k$ with $k-1$, $v_k u_k$ with $d+1$, $v_{k+1} u_k$ with $k-1$, …, $v_{t-1} u_{t-1}$ with $d+1$, $v_t u_{t-1}$ with $k-1$, $v_t v_{t+1}$ with $t+1$, vv_{t+1} with $k-1$ and color vv_1 with $t+1$. By properly recoloring the erased vertices, we get a $(\Delta+1)$-total-coloring of G, a contradiction. If $\phi(v_{k+1} u_{k+1}) = d+1$, then $\phi(v_{k+2} u_{k+2}) = \phi(v_{k+3} u_{k+3}) = \cdots = \phi(v_{t-1} u_{t-1}) = \phi(v_t v_{t+1}) = d+1$. Similarly, we have $\phi(v_t u_{t-1}) = \phi(v_{t-1} u_{t-2}) = \cdots = \phi(v_{k+1} u_k) = t+1$. Let $\phi(v_k u_k) = s$. Then we can recolor vv_{t+1} with $d+1$, $v_t v_{t+1}$ with $t+1$, $v_t u_{t-1}$ with $d+1$, $v_{t-1} u_{t-1}$ with $t+1, \ldots$, $v_{k+1} u_{k+1}$ with $t+1$, $v_{k+1} u_k$ with s, $v_k u_k$ with $t+1$, and color vv_1 with $t+1$. By properly recoloring the erased vertices, we get a $(\Delta+1)$-total-coloring of G, a contradiction, too. □

By the Euler's formula $|V| - |E| + |F| = 2$, we have

$$\sum_{v \in V} (2d(v) - 6) + \sum_{f \in F} (d(f) - 6) = -6(|V| - |E| + |F|) = -12 < 0$$

We define ch the initial charge that $ch(x) = 2d(x) - 6$ for each $x \in V$ and $ch(x) = d(x) - 6$ for each $x \in F$. So $\sum_{x \in V \cup F} ch(x) = -12 < 0$. In the following, we will reassign a new charge denoted by $ch'(x)$ to each $x \in V \cup F$ according to the discharging rules. If we can show that $ch'(x) \geq 0$ for each $x \in V \cup F$, then we get an obvious contradiction to $0 \leq \sum_{x \in V \cup F} ch'(x) = \sum_{x \in V \cup F} ch(x) = -12$, which completes our proof. Now we define the discharging rules as follows.

R1. Each 2-vertex receives 1 from each of its neighbors.

R2. Let f be a 3-face. If f is incident with a 3^--vertex, then it receives $\frac{3}{2}$ from each of its two incident 7^+-vertices. If f is incident with a 4-vertex, then it receives $\frac{5}{4}$ from each of its two incident 6^+-vertices. If f is not incident with any 4^--vertex, then it receives 1 from each of its two incident 5^+-vertices.

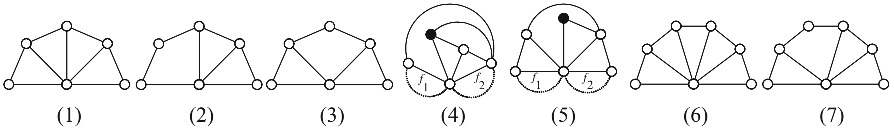

Fig. 3 Forbidden configurations in G

R3. Let f be a 4-face. If f is incident with two 3^--vertices, then it receives 1 from each of its two incident 7^+-vertices. If f is incident with only one 3^--vertex, then it receives $\frac{3}{4}$ from each of its two incident 7^+-vertices; and $\frac{1}{2}$ from the left incident 4^+-vertex. If f is not incident with any 3^--vertex, then it receives $\frac{1}{2}$ from each of its incident 4^+-vertices.

R4. Each 5-face receives $\frac{1}{3}$ from each of its incident 4^+-vertices.

Next, we show that $ch^{'}(x) \geq 0$ for all $x \in V \cup F$. It is easy to check that $ch^{'}(f) \geq 0$ for all $f \in F$ and $ch^{'}(v) \geq 0$ for all 2-vertices $v \in V$ by the above discharging rules. If $d(v) = 3$, then $ch^{'}(v) = ch(v) = 0$. If $d(v) = 4$, then $ch^{'}(v) \geq ch(v) - \frac{1}{2} \times 4 = 0$ by R2 and R3. For $d(v) \geq 5$, we need the following structural lemma.

Lemma 4 *(1) Suppose that every* 6*-cycle of G contains at most one chord. Then we have the following results.*

(a) G has no configurations depicted in Fig. 3(1), Fig. 3(2) and Fig. 3(3);

(b) Suppose G has a subgraph isomorphic to Fig. 3(4). Then $d(f_1) \geq 4$ *and* $d(f_2) \neq 4$*. More over if* $d(f_1) = 4$*, then* $d(f_2) \geq 5$*;*

(c) If G has a subgraph isomorphic to Fig. 3(5), then $d(f_1) \geq 5$ *and* $d(f_2) \geq 5$*.*

(2) Suppose that all chordal 6*-cycles are not adjacent. Then we have the following results.*

(d) If G has a subgraph isomorphic to Fig. 3(5), then $\max\{d(f_1), d(f_2)\} \geq 4$*;*

(e) G has no configurations depicted in Fig. 3(6) and Fig. 3(7).

Suppose $d(v) = 5$. Then $f_3(v) \leq 4$ by Lemma 4. If $f_3(v) = 4$, then $f_{6^+}(v) \geq 1$, so $ch^{'}(v) \geq ch(v) - 1 \times 4 = 0$. If $f_3(v) \leq 3$, then $ch'(v) \geq ch(v) - 1 \times f_3(v) - \frac{1}{2} \times (5 - f_3(v)) = \frac{3-f_3(v)}{2} \geq 0$. Suppose $d(v) = 6$. Then $f_3(v) \leq 4$ and $ch^{'}(v) \geq ch(v) - \frac{5}{4} \times f_3(v) - \frac{1}{2} \times (6 - f_3(v)) = \frac{3(4-f_3(v))}{4} \geq 0$. Suppose $d(v) = 7$. Then $f_3(v) \leq 5$. By Lemma 2(1), v is incident with at most two 3-faces are incident with a 3^--vertex, that is, v sends $\frac{3}{2}$ to each of the two 3-faces and at most $\frac{5}{4}$ to each other 3-face. If $f_3(v) = 5$, then $f_{5^+}(v) \geq 1$, and $ch'(v) \geq ch(v) - \frac{3}{2} \times 2 - \frac{5}{4} \times 3 - \frac{3}{4} \times 1 - \frac{1}{3} \times 1 = \frac{1}{6} > 0$. If $2 \leq f_3(v) \leq 4$, then $ch'(v) \geq ch(v) - \frac{3}{2} \times 2 - \frac{5}{4} \times (f_3(v) - 2) - 1 \times (5 - f_3(v)) - \frac{3}{4} \times 2 = \frac{4-f_3(v)}{4} \geq 0$. If $f_3(v) \leq 2$, then $ch^{'}(v) \geq ch(v) - \frac{3}{2} \times f_3(v) - 1 \times (7 - f_3(v)) = \frac{2-f_3(v)}{2} > 0$.

Suppose $d(v) = 8$. Then $ch(v) = 10$. Let $v_1, \ldots, v_8$ be neighbors of v in the clockwise order and $f_1, f_2, \ldots, f_8$ be faces incident with v, such that f_i is incident with v_i and v_{i+1}, for $i \in \{1, 2, \ldots, 8\}$, and $f_9 = f_1$.

Suppose $n_2(v) = 0$. Then $f_3(v) \leq 6$. If $f_3(v) = 6$, then $f_{5^+}(v) \geq 2$, so $ch'(v) \geq 10 - \frac{3}{2} \times 6 - \frac{1}{3} \times 2 = \frac{1}{3} > 0$. If $f_3(v) = 5$, then $f_{5^+}(v) \geq 1$, so $ch'(v) \geq 10 - \frac{3}{2} \times 5 - 1 \times 2 - \frac{1}{3} \times 1 = \frac{1}{6} > 0$. If $f_3(v) \leq 4$, then $ch'(v) \geq 10 - \frac{3}{2} \times f_3(v) - 1 \times (8 - f_3(v)) \geq 0$.

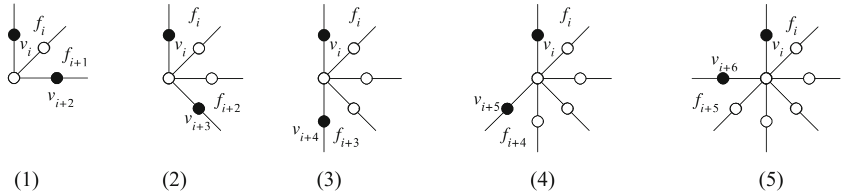

Fig. 4 Claim

Suppose $n_2(v) = 1$. Without loss of generality, assume $d(v_1) = 2$.

Suppose v_1 is incident with a 3-cycle f_1.

By Lemma 4, $f_3(v) \leq 6$ and all 3-faces incident with no 3^--vertex except f_1 by Lemma 2(6). If $f_3(v) = 6$, then $f_{5^+}(v) \geq 2$, so $ch'(v) \geq 10 - 1 - \frac{3}{2} \times 1 - \frac{5}{4} \times 5 - \frac{1}{3} \times 2 = \frac{7}{12} > 0$. If $4 \leq f_3(v) \leq 5$, then $ch'(v) \geq 10 - 1 - \frac{3}{2} \times 1 - \frac{5}{4} \times (f_3(v) - 1) - 1 \times (6 - f_3(v)) - \frac{3}{4} \times 2 = \frac{5 - f_3(v)}{4} \geq 0$. If $1 \leq f_3(v) \leq 3$, then $ch'(v) \geq 10 - 1 - \frac{3}{2} \times 1 - \frac{5}{4} \times (f_3(v) - 1) - 1 \times (8 - f_3(v)) = \frac{3 - f_3(v)}{4} \geq 0$.

Suppose v_1 is not incident with a 3-cycle.

Suppose every 6-cycle of G contains at most one chord. Then $f_3(v) \leq 5$ by Lemma 2(2)–(4). If $4 \leq f_3(v) \leq 5$, then $f_{5^+}(v) \geq 2$, so $ch'(v) \geq 10 - 1 - \frac{3}{2} \times (f_3(v) - 1) - 1 \times 1 - 1 \times (6 - f_3(v)) - \frac{1}{3} \times 2 = \frac{17 - 3f_3(v)}{6} > 0$. If $f_3(v) = 3$, then $f_{5^+}(v) \geq 1$, so $ch'(v) \geq 10 - 1 - \frac{3}{2} \times 3 - 1 \times 4 - \frac{1}{3} \times 1 = \frac{1}{6} > 0$. If $f_3(v) = 2$, then $ch'(v) \geq 10 - 1 - \frac{3}{2} \times 2 - 1 \times 6 = 0$. If $f_3(v) = 1$, then without loss of generality, $d(f_2) = 3$, i.e. $d(v_3) = 3$ and $d(v_2) \geq 7$, so $ch'(v) \geq 10 - 1 - \frac{3}{2} \times 1 - 1 \times 6 - \frac{3}{4} \times 1 = \frac{3}{4} > 0$. If $f_3(v) = 0$, then $ch'(v) \geq 10 - 1 - 1 \times 8 = 1 > 0$.

Suppose any two chordal 6-cycles are not adjacent. Then $f_3(v) \leq 5$ by Lemma 2(2)–(4). If $f_3(v) \geq 4$, then $ch'(v) \geq 10 - 1 - \frac{3}{2} \times 2 - \frac{5}{4} \times (f_3(v)) - \frac{3}{4} \times (8 - f_3(v)) = \frac{5 - f_3(v)}{2} \geq 0$. If $f_3(v) = 3$, then $ch'(v) \geq 10 - 1 - \frac{3}{2} \times 3 - \frac{3}{4} \times 5 = \frac{3}{4} > 0$. If $1 \leq f_3(v) \leq 2$, then $ch'(v) \geq 10 - 1 - \frac{3}{2} \times f_3(v) - 1 \times (6 - 2f_3(v)) - \frac{3}{4} \times (2 + f_3(v)) = \frac{6 - f_3(v)}{4} > 0$. If $f_3(v) = 0$, then $ch'(v) \geq 10 - 1 - 1 \times 8 = 1 > 0$.

Note that next Lemma 5 is also true for general planar graphs if we just use the above discharging rules.

Lemma 5 *Suppose $d(v) = 8$ and $2 \leq n_2(v) \leq 8$. Then $ch'(v) \geq 0$.*

Proof Since $d(v) = 8$, then $ch(v) = 10$. First we give a Claim for convenience.

Claim *Suppose that $d(v_i) = d(v_{i+k+1}) = 2$ and $d(v_j) \geq 3$ for $i + 1 \leq j \leq i + k$. Then v sends at most ϕ (in total) to f_i and $f_{i+1}, f_{i+2}, \ldots, f_{i+k}$, where $\phi = \frac{5k+1}{4}$ $(k = 1, 2, 3, 4, 5)$, see Fig. 4.*

By Lemma 2, $d(f_i) \geq 4$ and $d(f_{i+k}) \geq 4$.

Case 1. $k = 1$ By Lemma 3(1), we have $d(v_{i+1}) \geq 4$ or $\max\{d(f_i),\ d(f_{i+1})\} \geq 5$, so $\phi \leq \max\{\frac{3}{4} \times 2,\ 1 + \frac{1}{3}\} = \frac{3}{2}$ by R3.

Case 2. $k = 2$ If $d(f_{i+1}) = 3$, then $\min\{d(v_{i+1}),\ d(v_{i+2})\} \geq 4$ or $\max\{d(f_i),\ d(f_{i+2})\} \geq 5$ by Lemma 3(2), and it follows that $\phi \leq \max\{\frac{3}{4} + \frac{5}{4} +$

$\frac{3}{4},\ \frac{1}{3}+\frac{3}{2}+\frac{3}{4}\} = \frac{11}{4}$. Otherwise, $d(f_{i+1}) \geq 4$, then $\min\{d(v_{i+1}),\ d(v_{i+2})\} \geq 4$ or $\max\{d(f_i),\ d(f_{i+1}),\ d(f_{i+2})\} \geq 5$ by Lemma 3(1), and it follows that $\phi \leq \max\{1+\frac{3}{4}\times 2,\ 1\times 2+\frac{1}{3}\} = \frac{5}{2} < \frac{11}{4}$.

Case 3. $k = 3$ Suppose $d(f_{i+1}) = d(f_{i+2}) = 3$. Then $d(v_{i+2}) \geq 4$. If $d(v_{i+1}) = d(v_{i+3}) = 3$, then $d(f_i) \geq 5$ and $d(f_{i+3}) \geq 5$, so $\phi \leq \frac{3}{2}\times 2+\frac{1}{3}\times 2 = \frac{11}{3}$. If $\min\{d(v_{i+1}),\ d(v_{i+3})\} \geq 4$, then $\phi \leq \frac{5}{4}\times 2+\frac{3}{4}\times 2 = 4$. Suppose $d(f_{i+1}) = 3$ and $d(f_{i+2}) \geq 4$. If $d(v_{i+1}) = 3$, then $d(v_{i+2}) \geq 7$ and $d(f_i) \geq 5$, so $\phi \leq \frac{1}{3}+\frac{3}{2}+\frac{3}{4}+1 = \frac{43}{12}$. If $d(v_{i+2}) = 3$, then $d(v_{i+1}) \geq 7$ and $d(v_{i+3}) \geq 4$, so $\phi \leq \frac{3}{4}+\frac{3}{2}+\frac{3}{4}+\frac{3}{4} = \frac{15}{4}$. If $\min\{d(v_{i+1}),\ d(v_{i+2})\} \geq 4$, $\phi \leq \frac{3}{4}+\frac{5}{4}+\frac{3}{4}+1 = \frac{15}{4}$. It is similar with $d(f_{i+2}) = 3$ and $d(f_{i+1}) \geq 4$. Suppose $\min\{d(f_{i+1}),\ d(f_{i+2})\} \geq 4$. Then $\max\{d(v_{i+1}),\ d(v_{i+2}),\ d(v_{i+3})\} \geq 4$ or $\max\{d(f_i),\ d(f_{i+1}),\ d(f_{i+2}),\ d(f_{i+3})\} \geq 5$, so $\phi \leq \max\{1\times 2+\frac{3}{4}\times 2,\ 1\times 3+\frac{1}{3}\} = \frac{7}{2}$. So $\phi \leq \max\{\frac{11}{3},\ 4,\ \frac{43}{12},\ \frac{15}{4},\ \frac{7}{2}\} = 4$.

Case 4. $k = 4$ Suppose $d(f_{i+1}) = d(f_{i+2}) = d(f_{i+3}) = 3$. Then $\min\{d(v_{i+2}),\ d(v_{i+3})\} \geq 4$. If $d(v_{i+1}) = d(v_{i+4}) = 3$, then $d(f_i) \geq 5$ and $d(f_{i+4}) \geq 5$, so $\phi \leq \frac{3}{2}\times 2+1\times 1+\frac{1}{3}\times 2 = \frac{14}{3}$. If $\min\{d(v_{i+1}),\ d(v_{i+4})\} \geq 4$, then $\phi \leq \frac{5}{4}\times 3+\frac{3}{4}\times 2 = \frac{21}{4}$. Suppose $d(f_{i+1}) = d(f_{i+2}) = 3$, $d(f_{i+3}) \geq 4$. Then $d(v_{i+2}) \geq 4$. If $d(v_{i+1}) = d(v_{i+3}) = 3$, then $d(v_{i+4}) \geq 4$ and $d(f_i) \geq 5$, so $\phi \leq \frac{3}{2}\times 2+\frac{3}{4}\times 2+\frac{1}{3}\times 1 = \frac{29}{6}$. If $\min\{d(v_{i+1}),\ d(v_{i+3})\} \geq 4$, then $\phi \leq \frac{5}{4}\times 2+1\times 1+\frac{3}{4}\times 2 = 5$. Similar with $d(f_{i+2}) = d(f_{i+3}) = 3$, $d(f_{i+1}) \geq 4$. Suppose $d(f_{i+1}) = d(f_{i+3}) = 3$, $d(f_{i+2}) \geq 4$. Then $\max\{d(v_{i+2}),\ d(v_{i+3})\} \geq 4$ by Lemma 3(3), so $\phi \leq \frac{3}{2}\times 1+\frac{5}{4}\times 1+\frac{3}{4}\times 3 = 5$. Suppose $d(f_{i+1}) = 3$, $d(f_{i+2}) \geq 4$ and $d(f_{i+3}) \geq 4$. If $d(v_{i+1}) = 3$, then $d(v_{i+2}) \geq 7$ and $d(f_i) \geq 5$, so $\phi \leq \frac{3}{2}+1\times 2+\frac{3}{4}\times 1+\frac{1}{3}\times 1 = \frac{55}{12}$. If $d(v_{i+2}) = 3$, then $d(v_{i+1}) \geq 7$ and $\max\{d(v_{i+3}),\ d(v_{i+4})\} \geq 4$, so $\phi \leq \frac{3}{2}\times 1+1\times 1+\frac{3}{4}\times 3 = \frac{19}{4}$. Otherwise, $\phi \leq \frac{5}{4}\times 1+1\times 2+\frac{3}{4}\times 2 = \frac{19}{4}$. It is similar with $d(f_{i+3}) = 3$, $d(f_{i+1}) \geq 4$ and $d(f_{i+2}) \geq 4$. Suppose $d(f_{i+2}) = 3$, $d(f_{i+1}) \geq 4$ and $d(f_{i+3}) \geq 4$. If $d(v_{i+2}) = 3$ or $d(v_{i+3}) = 3$, then $\phi \leq \frac{3}{2}\times 1+1\times 1+\frac{3}{4}\times 3 = \frac{19}{4}$. Otherwise, $\phi \leq \frac{5}{4}\times 1+1\times 2+\frac{3}{4}\times 2 = \frac{19}{4}$. Suppose $\min\{d(f_{i+1}),\ d(f_{i+2}),\ d(f_{i+3})\} \geq 4$. Then $\max\{d(v_{i+1}),\ d(v_{i+2}),\ d(v_{i+3}),\ d(v_{i+4})\} \geq 4$ or $\max\{d(f_i),\ d(f_{i+1}),\ d(f_{i+2}),\ d(f_{i+3}),\ d(f_{i+4})\} \geq 5$, so $\phi \leq \max\{1\times 3+\frac{3}{4}\times 2,\ 1\times 4+\frac{1}{3}\} = \frac{9}{2}$. So $\phi \leq \max\{\frac{14}{3},\ \frac{21}{4},\ \frac{29}{6},\ 5,\ \frac{55}{12},\ \frac{19}{4},\ \frac{9}{2}\} = \frac{21}{4}$.

Case 5. $k = 5$ If $k = 5$, then $\phi \leq \frac{13}{2}$. It is similar to prove (e), we omit it here.

Next, we prove the Lemma.

If $n_2(v) = 8$, then all faces incident with v are 6^+-faces by Lemma 2(2)–(4), that is, $f_{6^+}(v) = 8$, so $ch'(v) = 10 - 1\times 8 = 2 > 0$. If $n_2(v) = 7$, then $f_{6^+}(v) \geq 6$ and $f_3(v) = 0$, so $ch'(v) \geq 10 - 1\times 7 - \frac{3}{2} = \frac{3}{2} > 0$ by Claim (a).

Suppose $n_2(v) \leq 6$. The possible distributions of 2-vertices adjacent to v are illustrated in Fig. 5. For Fig. 5(1), we have $f_{6^+}(v) \geq 5$ and $ch'(v) \geq 10-1\times 6-\frac{11}{4} = \frac{5}{4} > 0$ by Claim (b).

For Fig. 5(2)–(4), we have $f_{6^+}(v) \geq 4$ and $ch'(v) \geq 10-1\times 6-\frac{3}{2}\times 2 = 1 > 0$. For Fig. 5(5), we have $f_{6^+}(v) \geq 4$ and $ch'(v) \geq 10 - 1\times 5 - 4 = 1 > 0$ by Claim (c). For Fig. 5(6)–(7), we have $f_{6^+}(v) \geq 3$ and $ch'(v) \geq 10 - 1\times 5 - \frac{3}{2} - \frac{11}{4} = \frac{3}{4} > 0$. For Fig. 5(8)–(9), we have $f_{6^+}(v) \geq 2$ and $ch'(v) \geq 10-1\times 5-\frac{3}{2}\times 3 = \frac{1}{2} > 0$. For

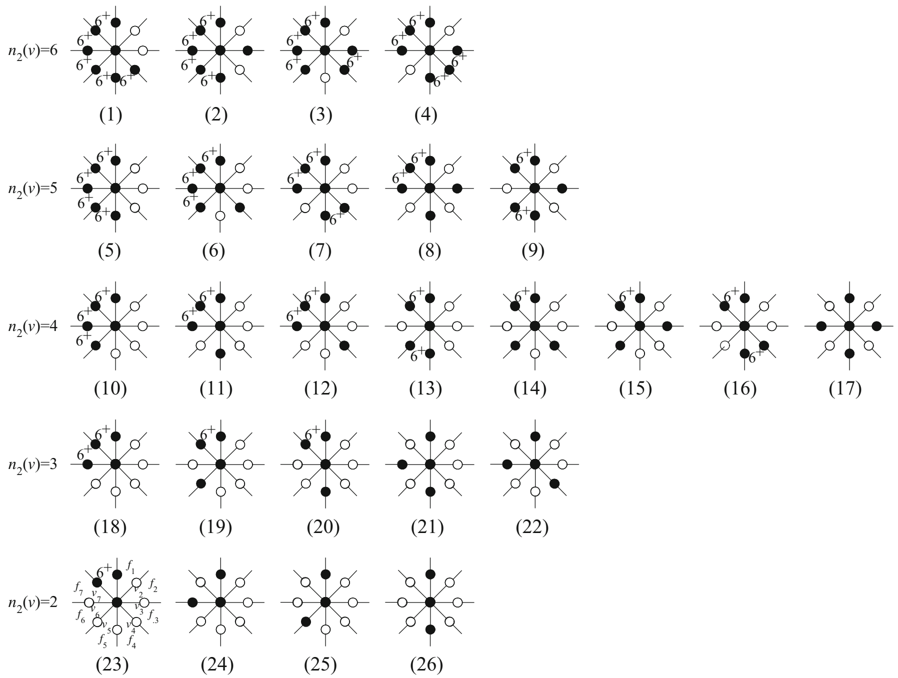

Fig. 5 $n_2(v) \le 6$

Fig. 5(10), we have $f_{6^+}(v) \ge 3$ and $ch'(v) \ge 10-1\times4-\frac{21}{4} = \frac{3}{4} > 0$ by Claim (d). For Fig. 5(11) and (13), we have $f_{6^+}(v) \ge 2$ and $ch'(v) \ge 10-1\times4-\frac{3}{2}-4 = \frac{1}{2} > 0$. For Fig. 5(12) and (16), we have $f_{6^+}(v) \ge 2$ and $ch'(v) \ge 10-1\times4-\frac{11}{4}\times2 = \frac{1}{2} > 0$. For Fig. 5(14) and (15), we have $f_{6^+}(v) \ge 1$ and $ch'(v) \ge 10-1\times4-\frac{3}{2}\times2-\frac{11}{4} = \frac{1}{4} > 0$. For Fig. 5 (17), we have $ch'(v) \ge 10-1\times4-\frac{3}{2}\times4 = 0$. For Fig. 5(18), we have $f_{6^+}(v) \ge 2$ and $ch'(v) \ge 10-1\times3-\frac{13}{2} = \frac{1}{2} > 0$ by Claim (e). For Fig. 5(19), we have $f_{6^+}(v) \ge 1$ and $ch'(v) \ge 10-1\times3-\frac{3}{2}-\frac{21}{4} = \frac{1}{4} > 0$. For Fig. 5(20), we have $f_{6^+}(v) \ge 1$ and $ch'(v) \ge 10-1\times3-\frac{11}{4}-4 = \frac{1}{4} > 0$. For Fig. 5(21), we have $ch'(v) \ge 10-1\times3-\frac{3}{2}\times2-4 = 0$. For Fig. 5(22), we have $ch'(v) \ge 10-1\times3-\frac{3}{2}-\frac{11}{4}\times2 = 0$. For Fig. 5(23), we have $f_{6^+}(v) \ge 1$. Suppose $d(f_2) = d(f_3) = d(f_4) = d(f_5) = d(f_6) = 3$. Then $\min\{d(v_3), d(v_4), d(v_5), d(v_6)\} \ge 4$. If $d(v_2) = d(v_6) = 3$, then $d(f_1) \ge 5$ and $d(f_7) \ge 5$ by Lemma 3, so $ch'(v) \ge 10-1\times2-\frac{3}{2}\times2-\frac{5}{4}\times2-1\times1-\frac{1}{3}\times2 = \frac{5}{6} > 0$. If f_2, f_3, f_4, f_5 and f_6 are incident with no 3^--vertex, then $ch'(v) \ge 10-1\times2-\frac{5}{4}\times5-\frac{3}{4}\times2 = \frac{1}{4} > 0$. For Fig. 5(24), we have $ch'(v) \ge 10-1\times2-\frac{3}{2}-\frac{13}{2} = 0$. For Fig. 5(25), we have $ch'(v) \ge 10-1\times2-\frac{11}{4}-\frac{21}{4} = 0$. For Fig. 5(26), we have $ch'(v) \ge 10-1\times2-4\times2 = 0$. □

Hence we complete the proof of the theorem.

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