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Almost Periodic Solutions of Neutral Delay Functional Differential Equations on Time Scales

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Abstract In this paper, based on the properties of almost periodic function and exponential dichotomy of linear system on time scales as well as Krasnoselskii's fixed point theorem, some sufficient conditions are established for the existence of almost periodic solutions of delayed neutral functional differential equations on time scales. Finally, an example is presented to illustrate the feasibility and effectiveness of the results.

Keywords Neutral differential equation · Almost periodic solution · Exponential dichotomy · Krasnoselskii's fixed point theorem · Time scale

Mathematics Subject Classification 34K14 · 34K40 · 34N05

1 Introduction

Neutral differential and difference equations arise in many areas of applied mathematics, such as population dynamics [1], stability theory [2], circuit theory [3], bifurcation analysis [4], and dynamical behavior of delayed network systems [5]. Also, qualitative analysis such as periodicity and almost periodicity of neutral differential and

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difference equations received more recently researchers' special attention due to their applications, see [6–8] and the references therein.

However, in the real world, there are many systems whose developing processes are both continuous and discrete. Hence, using the only differential equation or difference equation cannot accurately describe the law of their developments. Therefore, there is a need to establish correspondent dynamic models on new time scales.

The theory of calculus on time scales (see [9] and references cited therein) was initiated by Stefan Hilger in his Ph.D. thesis in 1988 [10] in order to unify continuous and discrete analysis, and it has a tremendous potential for applications and has recently received much attention since his foundational work, one may see [11–15]. Therefore, it is practicable to study that on time scales which can unify the continuous and discrete situations.

Motivated by the above, in the present paper, we focus on the following neutral delay functional differential equations on time scales:

$$x^{\Delta}(t) = A(t)x(t) + Q^{\Delta}(t, x_t) + G(t, x(t), x_t),\ t \in \mathbb{T}, \tag{1.1}$$

where $\mathbb{T}$ is an almost periodic time scale, A(t) is a nonsingular $n \times n$ matrix with continuous real-valued functions as its elements; the functions $Q : \mathbb{T} \times \mathbb{R}^n \to \mathbb{R}^n$ and $G : \mathbb{T} \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ are continuous with their arguments, respectively; $x_t \in C(\mathbb{T}, \mathbb{R}^n)$, and $x_t(s) = x(t + s)$, for all $s \in \mathbb{T}$.

Remark 1.1 The neutral differential and difference equations considered in [6–8] are the special cases of (1.1). To the best knowledge of the authors, there are few papers in literature dealing with the existence of almost periodic solutions of neutral delayed functional differential equations on time scales.

The purpose of this paper is to establish the existence of almost periodic solutions of (1.1) based on the properties of almost periodic function and exponential dichotomy of linear system on time scales as well as Krasnoselskii's fixed point theorem.

In this paper, for each $\phi = (\phi_1, \phi_2, \cdots, \phi_n)^T \in C(\mathbb{T}, \mathbb{R}^n)$, the norm of ϕ is defined as $\|\phi\| = \sup_{t\in\mathbb{T}} |\phi(t)|_0$, where $|\phi(t)|_0 = \sum_{i=1}^{n} |\phi_i(t)|$; and when it comes to that ϕ is continuous, delta derivative, delta integrable, and so forth, we mean that each element ϕ_i is continuous, delta derivative, delta integrable, and so forth.

2 Preliminaries

Let $\mathbb{T}$ be a nonempty closed subset (time scale) of $\mathbb{R}$. The forward and backward jump operators $\sigma, \rho : \mathbb{T} \to \mathbb{T}$ and the graininess $\mu : \mathbb{T} \to \mathbb{R}^+$ are defined, respectively, by

$$\sigma(t) = \inf\{s \in \mathbb{T} : s > t\},\ \rho(t) = \sup\{s \in \mathbb{T} : s < t\},\ \mu(t) = \sigma(t) - t.$$

A point $t \in \mathbb{T}$ is called left-dense if $t > \inf \mathbb{T}$ and $\rho(t) = t$, left-scattered if $\rho(t) < t$, right-dense if $t < \sup \mathbb{T}$ and $\sigma(t) = t$, and right-scattered if $\sigma(t) > t$. If

$\mathbb{T}$ has a left-scattered maximum m, then $\mathbb{T}^k = \mathbb{T}\backslash\{m\}$; otherwise $\mathbb{T}^k = \mathbb{T}$. If $\mathbb{T}$ has a right-scattered minimum m, then $\mathbb{T}_k = \mathbb{T}\backslash\{m\}$; otherwise $\mathbb{T}_k = \mathbb{T}$.

A function $f : \mathbb{T} \to \mathbb{R}$ is right-dense continuous provided it is continuous at right-dense point in $\mathbb{T}$, and its left-side limits exist at left-dense points in $\mathbb{T}$. If f is continuous at each right-dense point and each left-dense point, then f is said to be a continuous function on $\mathbb{T}$.

The basic theories of calculus on time scales, one can see [9].

A function $p : \mathbb{T} \to \mathbb{R}$ is called regressive provided $1 + \mu(t)p(t) \neq 0$ for all $t \in \mathbb{T}^k$. The set of all regressive and rd-continuous functions $p : \mathbb{T} \to \mathbb{R}$ will be denoted by $\mathcal{R} = \mathcal{R}(\mathbb{T}, \mathbb{R})$.

If r is a regressive function, then the generalized exponential function e_r is defined by

$$e_r(t, s) = \exp\left\{ \int_s^t \xi_{\mu(\tau)}(r(\tau))\Delta\tau \right\}$$

for all $s, t \in \mathbb{T}$, with the cylinder transformation

$$\xi_h(z) = \begin{cases} \frac{\text{Log}(1+hz)}{h}, & \text{if } h \neq 0, \\ z, & \text{if } h = 0. \end{cases}$$

Let $p, q : \mathbb{T} \to \mathbb{R}$ be two regressive functions, define

$$p \oplus q = p + q + \mu pq, \quad \ominus p = -\frac{p}{1 + \mu p}, \quad p \ominus q = p \oplus (\ominus q).$$

Lemma 2.1 *(see [9]) Assume that $p, q : \mathbb{T} \to \mathbb{R}$ be two regressive functions, then*

(i) $e_0(t, s) \equiv 1$ and $e_p(t, t) \equiv 1$;
(ii) $e_p(\sigma(t), s) = (1 + \mu(t)p(t))e_p(t, s)$;
(iii) $e_p(t, s) = \frac{1}{e_p(s,t)} = e_{\ominus p}(s, t)$;
(iv) $e_p(t, s)e_p(s, r) = e_p(t, r)$;
(v) $(e_{\ominus p}(t, s))^{\Delta} = (\ominus p)(t)e_{\ominus p}(t, s)$.

Lemma 2.2 *(see [9]) If $p \in \mathcal{R}$ be an $n \times n$-matrix-valued function on $\mathbb{T}$ and $a, b, c \in \mathbb{T}$, then*

$$[e_p(c, \cdot)]^{\Delta} = -p[e_p(c, \cdot)]^{\sigma} \textit{ and } \int_a^b p(t)e_p(c, \sigma(t))\Delta t = e_p(c, a) - e_p(c, b).$$

The definitions of almost periodic function and uniformly almost periodic function on time scales can be found in [16, 17].

In what follows, we need the following notation. For every real sequence $\alpha = (\alpha_n)$ and a continuous function $f : \mathbb{T} \to \mathbb{R}^n$, define $T_\alpha f = \lim_{n\to\infty} f(t+\alpha_n)$ if $\lim_{n\to\infty} f(t+\alpha_n)$ exists.

Lemma 2.3 *A function $f : \mathbb{T} \to \mathbb{R}^n$ is almost periodic if and only if f is continuous and for each $\alpha = (\alpha_n)$, there exists a subsequence α' of (α_n) such that $\mathbb{T}_{\alpha'} f = g$ uniformly on $\mathbb{T}$.*

Lemma 2.4 *Let $f : \mathbb{T} \to \mathbb{R}^n$ is an almost periodic function, then $f(t)$ is bounded and uniformly continuous on $\mathbb{T}$.*

The proofs of Lemma 2.3 and 2.4 are similar to the Theorem 3.13 in [18] and the Theorem 1.1 in [19], respectively. Hence, we omit it.

Lemma 2.5 *If $f : \mathbb{T} \times \mathbb{R}^n \to \mathbb{R}^n$ is an almost periodic function in t uniformly for $x \in \mathbb{R}^n$, then $f(t, x)$ is bounded on $\mathbb{T} \times D$, where D is any compact subset of $\mathbb{R}^n$.*

Proof For given $\varepsilon \leq 1$ and a compact subset $D \subset \mathbb{R}^n$, there exists a constant l, such that in any interval of length $l(\varepsilon, D)$, $f(t, x)$ is uniformly continuous on $[0, l(\varepsilon, D)] \times D$. Therefore, there exists a number $M > 0$, such that

$$|f(t, x)|_0 < M, \quad \text{for } (t, x) \in [0, l(\varepsilon, D)] \times D.$$

For any $t \in \mathbb{T}$, we can take $\tau \in E\{\varepsilon, f\} \cap [-t, -t + l(\varepsilon, D)]$, then we have $t + \tau \in [0, l(\varepsilon, D)]$. Hence, we can obtain

$$|f(t + \tau, x)|_0 < M, \quad \forall x \in D$$

and

$$|f(t + \tau, x) - f(t, x)|_0 < \varepsilon \leq 1, \quad \forall (t, x) \in \mathbb{T} \times D.$$

Hence, for any $(t, x) \in \mathbb{T} \times D$, we have

$$|f(t, x)|_0 \leq |f(t + \tau, x)|_0 + |f(t + \tau, x) - f(t, x)|_0 < M + 1.$$

That is, $f(t, x)$ is bounded on $\mathbb{T} \times D$. The proof is completed. □

Lemma 2.6 *If $f : \mathbb{T} \times \mathbb{R}^n \to \mathbb{R}^n$ is an almost periodic function in t uniformly for $x \in \mathbb{R}^n$, $\phi(t)$ is also an almost periodic function and $\phi(t) \subset S$ for all $t \in \mathbb{T}$, S is a compact subset of $\mathbb{R}^n$, then $f(t, \phi(t))$ is almost periodic.*

Proof For any real sequence α', we can find a subsequence $\alpha \subset \alpha'$. Assume that $\varphi(t)$ is an almost periodic function, $g(t, x)$ is an almost periodic function in t uniformly for $x \in \mathbb{R}^n$, we make that $T_\alpha f(t, x) = g(t, x)$ uniformly on $\mathbb{T}$ and $T_\alpha \phi(t) = \varphi(t)$ also uniformly on $\mathbb{T}$. Hence, $g(t, x)$ is uniformly continuous on $\mathbb{T} \times S$. For any $\varepsilon > 0$, there exists a positive number $\delta(\frac{\varepsilon}{2}) > 0$, $\forall x_1, x_2 \in S$, such that $|x_1 - x_2|_0 < \delta(\frac{\varepsilon}{2})$ implies $|g(t, x_1) - g(t, x_2)|_0 < \frac{\varepsilon}{2}$, for any $t \in \mathbb{T}$, and there exists a positive integer $N_0(\varepsilon) > 0$, when $n \geq N_0(\varepsilon)$, we have

$$|f(t + \alpha_n, x) - g(t, x)|_0 < \frac{\varepsilon}{2}, \quad \forall (t, x) \in \mathbb{T} \times S$$

and

$$|\phi(t+\alpha_n)-\varphi(t)|_0 < \delta(\frac{\varepsilon}{2}), \quad \forall t \in \mathbb{T}.$$

Moreover, $\phi(t+\alpha_n) \subset S$,$\varphi(t) \subset S$ for all $t \in \mathbb{T}$. Then, when $n \geq N_0(\varepsilon)$, it is easy to see that

$$\begin{aligned}
&|f(t+\alpha_n, \phi(t+\alpha_n)) - g(t, \varphi(t))|_0 \\
&\quad \leq |f(t+\alpha_n, \phi(t+\alpha_n)) - g(t, \phi(t+\alpha_n))|_0 + |g(t, \phi(t+\alpha_n)) - g(t, \varphi(t))|_0 \\
&\quad < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\end{aligned}$$

Hence, $T_\alpha f(t, \phi(t)) = g(t, \varphi(t))$ uniformly on $\mathbb{T}$. So $f(t, \phi(t))$ is an almost periodic function. The proof is completed. □

Lemma 2.7 *If $u : \mathbb{T} \to \mathbb{R}^n$ is an almost periodic function, then u_t is almost periodic.*

Proof It is clear that u_t is continuous for $t \in \mathbb{T}$. For any sequence $\alpha' = (\alpha'_n)$. Since $u(t)$ is an almost periodic function, then there exists a subsequence $\alpha = (\alpha_n)$ of (α'_n), such that

$$T_\alpha u(t) = \overline{u}(t) \tag{2.1}$$

uniformly for $t \in \mathbb{T}$. On the other hand, since $u(t)$ is an almost periodic function, it is uniformly continuous on $\mathbb{T}$. For any $\varepsilon > 0$, there exists a positive number $\delta(\varepsilon)$, such that $|t_1 - t_2| < \delta$ implies $|u(t_1) - u(t_2)|_0 < \varepsilon$. From (2.1), there exists a positive integer N, such that

$$|u(t+\alpha_n) - \overline{u}(t)|_0 < \varepsilon, \ t \in \mathbb{T},$$

when $n > N$, we have

$$|(u_t)_{\alpha_n} - \overline{u}_t|_0 = |u(t+\alpha_n+\theta) - \overline{u}(t+\theta)|_0 < \varepsilon.$$

Hence $u(t+\alpha_n)$ converges to $\overline{u}_t$ uniformly on $\mathbb{T}$. So u_t is almost periodic. The proof is completed. □

Definition 2.1 (see [16]) Let $x \in \mathbb{R}^n$ and $A(t)$ be an $n \times n$ rd-continuous matrix on $\mathbb{T}$, the linear system

$$x^\Delta(t) = A(t)x(t) \tag{2.2}$$

is said to admit an exponential dichotomy on $\mathbb{T}$, if there exist positive constants $\alpha > 0$, $k \geq 1$, projection P and the fundamental solution matrix $X(t)$ of (2.2) satisfying

$$\|X(t)PX^{-1}(\sigma(s))\| \leq ke_{\ominus\alpha}(t, \sigma(s)) \quad s, t \in \mathbb{T}, t \geq \sigma(s), \tag{2.3}$$

$$\|X(t)(I-P)X^{-1}(\sigma(s))\| \leq ke_{\ominus\alpha}(\sigma(s), t) \quad s, t \in \mathbb{T}, t \leq \sigma(s), \tag{2.4}$$

where $\|\cdot\|$ is a matrix norm on $\mathbb{T}$.

Remark 2.1 It is clear that when $A(t) = \mathrm{diag}(1, -1)$, (2.2) admits exponential dichotomy. More generally, in the case $A(t) \equiv A$, a constant matrix, (2.2) admits exponential dichotomy if and only if the eigenvalues of A have a nonzero real part.

Lemma 2.8 *Suppose (2.2) admits exponential dichotomy, that is, there exist constants $\alpha > 0, k \geq 1$, such that (2.3) and (2.4) hold. If $A(t + t_k)$ converges to $\overline{A}(t)$ uniformly on any compact subset of $\mathbb{T}$, then $\{X(t + t_k)PX^{-1}(\sigma(s) + t_k)\}$ and $\{X(t + t_k)(I - P)X^{-1}(\sigma(s) + t_k)\}$ converges to $\{\overline{X}(t)\overline{P}\,\overline{X}^{-1}(\sigma(s))\}$ and $\{\overline{X}(t)(I - \overline{P})\overline{X}^{-1}(\sigma(s))\}$ uniformly on any compact subset $\mathbb{T} \times \mathbb{T}$, respectively. Furthermore, the following inequalities hold:*

$$\|\overline{X}(t)\overline{P}\,\overline{X}^{-1}(\sigma(s))\| \leq ke_{\ominus\alpha}(t, \sigma(s)) \quad s, t \in \mathbb{T}, t \geq \sigma(s),$$
$$\|\overline{X}(t)(I - \overline{P})\overline{X}^{-1}(\sigma(s))\| \leq ke_{\ominus\alpha}(\sigma(s), t) \quad s, t \in \mathbb{T}, t \leq \sigma(s),$$

where $\overline{X}$ is the fundamental matrix solution of the following equation

$$x^{\Delta}(t) = \overline{A}(t)x. \tag{2.5}$$

Proof we first prove that $\{X(t_k)PX^{-1}(t_k)\}$ is convergent. From (2.3), we see that

$$\|X(t_k)PX^{-1}(t_k)\| \leq k.$$

Suppose $\{X(t_k)PX^{-1}(t_k)\}$ is not convergent. Then, we can find two subsequence:

$$\left\{X(t_{k_m})PX^{-1}(t_{k_m})\right\}, \ \left\{X(t_{k'_m})PX^{-1}(t_{k'_m})\right\},$$

such that

$$\lim_{m\to\infty} X(t_{k_m})PX^{-1}(t_{k_m}) = \overline{P}, \ \lim_{m\to\infty} X(t_{k'_m})PX^{-1}(t_{k'_m}) = \underline{P},$$

and $\overline{P} \neq \underline{P}$.
Then, from (2.3) we have

$$\|X(t + t_{k_m})PX^{-1}(\sigma(s) + t_{k_m})\| \leq ke_{\ominus\alpha}(t, \sigma(s)) \quad s, t \in \mathbb{T}, t \geq \sigma(s), \tag{2.6}$$

and

$$\|X(t + t_{k'_m})PX^{-1}(\sigma(s) + t_{k'_m})\| \leq ke_{\ominus\alpha}(t, \sigma(s)) \quad s, t \in \mathbb{T}, t \geq \sigma(s). \tag{2.7}$$

Assume that $X_{k_m}(t) and X_{k'_m}(t)$ are the fundamental matrix solutions of systems

$$x^{\Delta}(t) = A(t + t_{k_m})x, \ x^{\Delta}(t) = A(t + t_{k'_m})x$$

respectively, then $X(t+t_{k_m}) = X_{k_m}(t)X(t_{k_m}) and\ X(t+t_{k'_m}) = X_{k'_m}(t)X(t_{k'_m})$. Since $\{A(t+t_k\}$ converges to $\overline{A}(t)$ uniformly on any compact subset of $\mathbb{T}$, then $\{A(t+t_k)x\}$ converges to $\overline{A}(t)x$ uniformly on any compact subset of $\mathbb{T}\times\mathbb{R}^n$. It follows that $\{A(t+t_{k_m})x\}$ and $\{A(t+t_{k'_m})x\}$ converge to $\overline{A}(t)x$ uniformly on any compact subset of $\mathbb{T}\times\mathbb{R}^n$. So $X_{k_m}(t) and\ X_{k'_m}(t)$ converge to $\overline{X}(t)$ uniformly on any compact set of $\mathbb{T}$. Furthermore, it follows from (2.6), (2.7) that

$$\|X_{k_m}(t)X(t_{k_m})PX^{-1}(t_{k_m})X_{k_m}^{-1}(\sigma(s))\| \le ke_{\ominus\alpha}(t,\sigma(s)) \quad s,t\in\mathbb{T}, t\ge\sigma(s)$$

and

$$\|X_{k'_m}(t)X(t_{k'_m})PX^{-1}(t_{k'_m})X_{k'_m}^{-1}(\sigma(s))\| \le ke_{\ominus\alpha}(t,\sigma(s)) \quad s,t\in\mathbb{T}, t\ge\sigma(s).$$

Let $m\to\infty$, we have

$$\|\overline{X}(t)\overline{P}\,\overline{X}^{-1}(\sigma(s))\| \le ke_{\ominus\alpha}(t,\sigma(s)) \quad s,t\in\mathbb{T}, t\ge\sigma(s) \tag{2.8}$$

and

$$\|\overline{X}(t)\underline{P}\,\overline{X}^{-1}(\sigma(s))\| \le ke_{\ominus\alpha}(t,\sigma(s)) \quad s,t\in\mathbb{T}, t\ge\sigma(s). \tag{2.9}$$

Similarly, we can obtain

$$\|\overline{X}(t)(I-\overline{P})\,\overline{X}^{-1}(\sigma(s))\| \le ke_{\ominus\alpha}(\sigma(s),t) \quad s,t\in\mathbb{T}, t\le\sigma(s) \tag{2.10}$$

and

$$\|\overline{X}(t)(I-\underline{P})\,\overline{X}^{-1}(\sigma(s))\| \le ke_{\ominus\alpha}(\sigma(s),t) \quad s,t\in\mathbb{T}, t\le\sigma(s). \tag{2.11}$$

From (2.8)–(2.11), we see that (2.5) admits exponential dichotomy; both $\overline{P}$ and $\underline{P}$ are its projections. So $\overline{P}=\underline{P}$, which is a contradiction. Hence, $\{X(t_k)PX^{-1}(t_k)\}$ is convergent.

Let $\{X(t_k)PX^{-1}(t_k)\}\to\overline{P}$ as $k\to\infty$. Now assume that $X_k(t)$ is the fundamental matrix solution of the system $x^{\Delta}(t) = A(t+t_k)x$, then $X_k(t)$ converge to $\overline{X}(t)$ uniformly on any compact set of $\mathbb{T}$. It is easy to see that $\{X_k^{-1}(\sigma(s))\}$ converges to $\overline{X}^{-1}(\sigma(s))$ uniformly on any compact subset of $\mathbb{T}$. So $X(t+t_k)PX^{-1}(\sigma(s)+t_k)$ and $\{X(t+t_k)(I-P)X^{-1}(\sigma(s)+t_k)\}$ converges to $\overline{X}(t)\overline{P}\,\overline{X}^{-1}(\sigma(s))$ and $\overline{X}(t)(I-\overline{P})\overline{X}^{-1}(\sigma(s))$ uniformly on any compact subset $\mathbb{T}\times\mathbb{T}$, respectively. Furthermore, from (2.6) and (2.7) we have

$$\|X(t+t_k)PX^{-1}(\sigma(s)+t_k)\| \le ke_{\ominus\alpha}(t,\sigma(s)) \quad s,t\in\mathbb{T}, t\ge\sigma(s)$$

and

$$\|X(t+t_k)(I-P)X^{-1}(\sigma(s)+t_k)\| \le ke_{\ominus\alpha}(\sigma(s),t) \quad s,t\in\mathbb{T}, t\le\sigma(s).$$

That is,

$$\|X_k(t)X(t_k)PX^{-1}(t_k)X_k^{-1}(\sigma(s))\| \le ke_{\ominus\alpha}(t,\sigma(s)) \quad s,t \in \mathbb{T}, t \ge \sigma(s)$$

and

$$\|X_k(t)X(t_k)(I-P)X^{-1}(t_k)X_k^{-1}(\sigma(s))\| \le ke_{\ominus\alpha}(\sigma(s),t) \quad s,t \in \mathbb{T}, t \le \sigma(s).$$

Let $k \to \infty$, we obtain

$$\|\overline{X}(t)\overline{P}\,\overline{X}^{-1}(\sigma(s))\| \le ke_{\ominus\alpha}(t,\sigma(s)) \quad s,t \in \mathbb{T}, t \ge \sigma(s)$$

and

$$\|\overline{X}(t)(I-\overline{P})\,\overline{X}^{-1}(\sigma(s))\| \le ke_{\ominus\alpha}(\sigma(s),t) \quad s,t \in \mathbb{T}, t \le \sigma(s).$$

The proof is completed. □

Lemma 2.9 *(see [20]) Let M be a closed convex nonempty subset of a Banach space $(B, \|\cdot\|)$. Suppose that B and C map M into B, such that*

(1) $x, y \in M$, implies $Bx + Cy \in M$,
(2) C is continuous and $C(M)$ is contained in a compact set,
(3) B is a contraction mapping. Then, there exists $z \in M$ with $z = Bz + Cz$.

3 Main Results

Let $AP(\mathbb{T})$ be the set of all almost periodic functions on almost times scales $\mathbb{T}$, then $(AP(\mathbb{T}), \|\cdot\|)$ is a Banach space with the supremum norm given by $\|\psi\| = \sup_{t\in\mathbb{T}} |\psi(t)|_0$, where $|\psi(t)|_0 = \sum_{i=1}^{n} |\psi_i(t)|$.
Hereafter, we make the following assumptions:

(H_1) There exist positive numbers L_Q, L_G such that

$$|Q(t,\phi_t) - Q(t,\varphi_t)|_0 \le L_Q|\phi_t - \varphi_t|_0 \tag{3.1}$$

for all $t \in \mathbb{T}$, $\phi_t, \varphi_t \in AP(\mathbb{T})$, and

$$|G(t,u,\phi_t) - G(t,v,\varphi_t)|_0 \le L_G(|u-v|_0 + |\phi_t - \varphi_t|_0) \tag{3.2}$$

for all $t \in \mathbb{T}$, $(u,\phi_t), (v,\varphi_t) \in \mathbb{R}^n \times AP(\mathbb{T})$. (H_2) $A(t)$ is an almost periodic function, $Q(t,u_t)$ is an almost periodic function in t uniformly for $u_t \in AP(\mathbb{T})$, and $G(t,u,u_t)$ is also an almost periodic function in t uniformly for $u, u_t \in \mathbb{R}^n \times AP(\mathbb{T})$. (H_3) System (2.2) admits exponential dichotomy, that is, there exist constants $\alpha > 0$, $k \ge 1$, such that (2.3) and (2.4) hold.

Define a mapping Φ by

$$(\Phi u)(t) = Q(t, u_t) + \int_{-\infty}^{t} X(t)PX^{-1}(\sigma(s))G(s, u(s), u_s)\Delta s - \int_{t}^{+\infty} X(t)(I-P)X^{-1}(\sigma(s))G(s, u(s), u_s)\Delta s. \quad (3.3)$$

Lemma 3.1 *If u is an almost periodic function, then Φu is an almost periodic function.*

Proof For $u(t)$ is an almost periodic function, from (H_2), Lemma 2.4 to 2.7, then $Q(t, u_t) and G(t, u(t), u_t)$ are all almost periodic functions, so they are uniformly bounded on $\mathbb{T}$. Let $M_1 and M_2$ be positive numbers such that

$$\|Q(\cdot, u_\cdot)\| \le M_1, \quad \|G(\cdot, u(\cdot), u_\cdot)\| \le M_2.$$

Now, we prove that $(\Phi u)(t)$ is an almost periodic function. First, it is clear that $(\Phi u)(t)$ is continuous on $\mathbb{T}$. For any sequence $\alpha = (\alpha_n)$, since $Q(t, u_t) and G(t, u(t), u_t)$ are almost periodic functions, combining with Lemma 2.3 and 2.8, we can find a common subsequence of (α_n), we still denote it as (α_n), such that

$$T_\alpha Q(t, u_t) = Q_1(t), \quad T_\alpha G(t, u(t), u_t) = G_1(t) \quad (3.4)$$

uniformly for $t \in \mathbb{T}$ and

$$\lim_{k\to\infty} X(t+\alpha_k)PX^{-1}(\sigma(s)+\alpha_k) = \overline{X}(t)\overline{P}\,\overline{X}^{-1}(\sigma(s)), t \ge \sigma(s) \quad (3.5)$$

$$\lim_{k\to\infty} X(t+\alpha_k)(I-P)X^{-1}(\sigma(s)+\alpha_k) = \overline{X}(t)(I-\overline{P})\overline{X}^{-1}(\sigma(s)), t \le \sigma(s). \quad (3.6)$$

Then,

$$\begin{aligned}(\Phi u)(t+\alpha_k) &= Q(t+\alpha_k, u_{t+\alpha_k}) + \int_{-\infty}^{t+\alpha_k} X(t+\alpha_k)PX^{-1}(\sigma(s))G(s, u(s), u_s)\Delta s \\ &\quad - \int_{t+\alpha_k}^{+\infty} X(t+\alpha_k)(I-P)X^{-1}(\sigma(s))G(s, u(s), u_s)\Delta s \\ &= Q(t+\alpha_k, u_{t+\alpha_k}) + \int_{-\infty}^{t} X(t+\alpha_k)PX^{-1}(\sigma(s)+\alpha_k) \\ &\quad \times G(s+\alpha_k, u(s+\alpha_k), u_{s+\alpha_k})\Delta s \\ &\quad - \int_{t}^{+\infty} X(t+\alpha_k)(I-P)X^{-1}(\sigma(s)+\alpha_k) \\ &\quad \times G(s+\alpha_k, u(s+\alpha_k), u_{s+\alpha_k})\Delta s.\end{aligned}$$

From (3.4)–(3.6) and Lebesgue's control convergence theorem, we see that $(\Phi u)(t+\alpha_k)$ converges to

$$Q_1(t)+\int_{-\infty}^{t}\overline{X}(t)\overline{P}\,\overline{X}^{-1}(\sigma(s))G_1(s)\Delta s-\int_{t}^{+\infty}\overline{X}(t)(I-\overline{P})\overline{X}^{-1}(\sigma(s))G_1(s)\Delta s$$

uniformly for $t\in\mathbb{T}$. So, $(\Phi u)(t)$ is an almost periodic function. The proof is completed. □

In order to apply Krasnoselskii's theorm, we need to construct two mappings, one is a contraction and the other is compact. Let

$$(\Phi u)(t)=(Bu)(t)+(Cu)(t);$$

where $B, C: AP(\mathbb{T})\to AP(\mathbb{T})$ are given by

$$(Bu)(t)=Q(t,u_t), \tag{3.7}$$

$$(Cu)(t)=\int_{-\infty}^{t}X(t)PX^{-1}(\sigma(s))G(s,u(s),u_s)\Delta s-\int_{t}^{+\infty}X(t)(I-P)X^{-1}(\sigma(s))G(s,u(s),u_s)\Delta s. \tag{3.8}$$

Lemma 3.2 *(see [7]) The operator B is a contraction provided* $L_Q<1$.

Lemma 3.3 *The operator C is continuous and the image C(M) is contained in a compact set, where* $M=\{u\in AP(\mathbb{T}):\|u\|\le R\}$*, R is a fixed constant.*

Proof First, by (3.7), we have

$$\begin{aligned}\|(Cu)(\cdot)\| &\le \int_{-\infty}^{t}\|X(t)PX^{-1}(\sigma(s))\|\|G(\cdot,u(\cdot),u.)\|\Delta s\\ &\quad+\int_{t}^{+\infty}\|X(t)(I-P)X^{-1}(\sigma(s))\|\|G(\cdot,u(\cdot),u.)\|\Delta s\\ &\le \|G(\cdot,u(\cdot),u.)\|\bigg(\int_{-\infty}^{t}\|X(t)PX^{-1}(\sigma(s))\|\Delta s\\ &\quad+\int_{t}^{+\infty}\|X(t)(I-P)X^{-1}(\sigma(s))\|\Delta s\bigg)\\ &\le \|G(\cdot,u(\cdot),u.)\|\bigg(\int_{-\infty}^{t}ke_{\ominus\alpha}(t,\sigma(s))\Delta s+\int_{t}^{+\infty}ke_{\ominus\alpha}(\sigma(s),t)\Delta s\bigg).\end{aligned}$$

By Lemma 2.2, we can get

$$\int_{-\infty}^{t}ke_{\ominus\alpha}(t,\sigma(s))\Delta s+\int_{t}^{+\infty}ke_{\ominus\alpha}(\sigma(s),t)\Delta s\le k(\frac{1}{\alpha}-\frac{1}{\ominus\alpha}).$$

Therefore,

$$\|(Cu)(\cdot)\| \leq k(\frac{1}{\alpha} - \frac{1}{\ominus\alpha})\|G(\cdot, u(\cdot), u.)\|. \tag{3.9}$$

Now, we show that C is continuous. In fact, let $u, v \in AP(\mathbb{T})$, for any $\varepsilon > 0$, take $\delta = \varepsilon/[2kL_G(\frac{1}{\alpha} - \frac{1}{\ominus\alpha})]$, whenever $\|u - v\| < \delta$, we have

$$\begin{aligned}
&\|(Cu)(\cdot) - (Cv)(\cdot)\| \\
&\leq \int_{-\infty}^{t} \|X(t)PX^{-1}(\sigma(s))\| \|G(\cdot, u(\cdot), u.) - G(\cdot, v(\cdot), v.)\|\Delta s \\
&\quad + \int_{t}^{+\infty} \|X(t)(I - P)X^{-1}(\sigma(s))\| \|G(\cdot, u(\cdot), u.)) - G(\cdot, v(\cdot), v.)\|\Delta s \\
&\leq \int_{-\infty}^{t} ke_{\ominus\alpha}(t, \sigma(s))L_G(\|u(\cdot) - v(\cdot)\| + \|u. - v.\|)\Delta s \\
&\quad + \int_{t}^{+\infty} ke_{\ominus\alpha}(\sigma(s), t)L_G(\|u(\cdot) - v(\cdot)\| + \|u. - v.\|) \\
&\leq 2L_G\|u - v\|(\int_{-\infty}^{t} ke_{\ominus\alpha}(t, \sigma(s))\Delta s + \int_{t}^{+\infty} ke_{\ominus\alpha}(\sigma(s), t)\Delta s) \\
&\leq 2kL_G(\frac{1}{\alpha} - \frac{1}{\ominus\alpha})\|u - v\| \\
&< \varepsilon.
\end{aligned}$$

This proves that C is continuous.

For $M = \{u \in AP(\mathbb{T}) : \|u\| \leq R\}$. Now, we show that the image of $C(M)$ is contained in a compact set. In fact, let u_n be a sequence in M. In view of (3.2), we have

$$\begin{aligned}
\|G(\cdot, u(\cdot), u.)\| &\leq \|G(\cdot, u(\cdot), u.) - G(\cdot, 0, 0)\| + \|G(\cdot, 0, 0)\| \\
&\leq L_G(\|u\| + \|u.\|) + a \\
&\leq 2L_G R + a,
\end{aligned} \tag{3.10}$$

where $a = \|G(\cdot, 0, 0)\|$. From (3.9) and (3.10), we have

$$\|Cu_n(\cdot)\| \leq k(\frac{1}{\alpha} - \frac{1}{\ominus\alpha})(2L_G R + a) := L. \tag{3.11}$$

Next, we calculate $(Cu_n)^{\Delta}(t)$ and show that it is uniformly bounded. By a direct calculate, we have

$$\begin{aligned}
(Cu_n)^{\Delta}(t) = A(t)(Cu_n)(t) + X(t)PX^{-1}(\sigma(s))G(t, u_n(t), (u_n)_t) \\
-X(t)(I - P)X^{-1}(\sigma(s))G(t, u_n(t), (u_n)_t).
\end{aligned} \tag{3.12}$$

For $A(t)$ is an almost periodic function, then $A(t)$ is bounded. So, there exists a positive constant A_0, such that $\|A\| \leq A_0$. Together with (3.10), (3.11), and (3.12) implies

$$\begin{aligned}\|(Cu_n)^{\Delta}(\cdot)\| &\leq A_0 L + (ke_{\ominus\alpha}(t,\sigma(s)) + ke_{\ominus\alpha}(\sigma(s),t))\|G(\cdot,u_n(\cdot),(u_n).)\| \\ &\leq A_0 L + (k+k)(2RL_G + a) \\ &\leq A_0 L + 2k(2RL_G + a).\end{aligned}$$

Thus, the sequence (Cu_n) is uniformly bounded and equi-continuous. Hence, by the Arzela-Ascoli theorem, *C*(*M*) is compact. The proof is completed. □

Theorem 3.1 *Assume that* $(H_1) - (H_3)$ *hold. Let* $a = \|G(\cdot,0,0)\|, b = \|Q(\cdot,0)\|$. *Let* R_0 *be a positive constant satisfies*

$$L_Q R_0 + b + k(\frac{1}{\alpha} - \frac{1}{\ominus\alpha})(2L_G R_0 + a) \leq R_0. \tag{3.13}$$

Then, (1.1) has an almost periodic solution in $M = \{u \in AP(\mathbb{T}) : \|u\| \leq R_0\}$.

Proof Define $M = \{u \in AP(\mathbb{T}) : \|u\| \leq R_0\}$. By Lemma 3.3, the mapping C defined by (3.7) is continuous and CM is contained in a compact set. By lemma 3.2, the mapping B defined by (3.7) is a contraction and it is clear that $B : AP(\mathbb{T}) \to AP(\mathbb{T})$.

Next, we show that if $u, v \in M$, we have $\|Bu + Cv\| \leq R_0$. In fact, let $u, v \in M$ with $\|u\|, \|v\| \leq R_0$. Then

$$\begin{aligned}\|(Bu)(\cdot) + (Cv)(\cdot)\| &\leq \|Q(\cdot,u.) - Q(\cdot,0)\| + \|Q(\cdot,0)\| \\ &\quad + \int_{-\infty}^{t} \|X(t)PX^{-1}(\sigma(s))\| \cdot \|G(\cdot,v(\cdot),v.)\|\Delta s \\ &\quad + \int_{t}^{+\infty} \|X(t)(I-P)X^{-1}(\sigma(s))\| \cdot \|G(\cdot,v(\cdot),v.)\|\Delta s \\ &\leq L_Q\|u\| + b + k(\frac{1}{\alpha} - \frac{1}{\ominus\alpha})(2L_G R + a) \\ &\leq L_Q R_0 + b + k(\frac{1}{\alpha} - \frac{1}{\ominus\alpha})(2L_G R_0 + a) \\ &\leq R_0.\end{aligned}$$

Thus, $Bu+Cv \in M$. Hence all the conditions of Krasnoselskii's theorem are satisfied. Hence there exists a fixed point $z \in M$, such that z = Bz + Cz. By Lemma 2.9, (1.1) has an almost periodic solution. The proof is completed. □

Theorem 3.2 *Assume that* $(H_1) - (H_3)$ *hold. If*

$$L_Q + 2kL_G\left(\frac{1}{\alpha} - \frac{1}{\ominus\alpha}\right) < 1, \tag{3.14}$$

then, (1.1) has a unique almost periodic solution.

Proof Let the mapping Φ be given by (3.3). For $u, v \in AP(\mathbb{T})$, in view of (3.3), we have

$$\begin{aligned}
&\|(\Phi u)(\cdot) - (\Phi v)(\cdot)\| \\
&\quad \le \|Q(\cdot, u.) - Q(\cdot, v.)\| \\
&\qquad + \int_{-\infty}^{t} \|X(t)PX^{-1}(\sigma(s))\| \|G(\cdot, u(\cdot), u.) - G(\cdot, v(\cdot), v.)\| \Delta s \\
&\qquad + \int_{t}^{-\infty} \|X(t)(I-P)X^{-1}(\sigma(s))\| \|G(\cdot, u(\cdot), u.) - G(\cdot, v(\cdot), v.)\| \Delta s \\
&\quad \le L_Q\|u - v\| + L_G(\|u - v\| + \|u. - v.\|) \cdot (\int_{-\infty}^{t} ke_{\ominus\alpha}(t, \sigma(s))\Delta s \\
&\qquad + \int_{t}^{+\infty} ke_{\ominus\alpha}(\sigma(s), t)\Delta s) \\
&\quad \le L_Q\|u - v\| + 2L_G\|u - v\|k(\frac{1}{\alpha} - \frac{1}{\ominus\alpha}) \\
&\quad = (L_Q + 2kL_G(\frac{1}{\alpha} - \frac{1}{\ominus\alpha}))\|u - v\|.
\end{aligned}$$

This completes the proof by invoking the contraction mapping principle. □

Remark 3.1 If the conditions of the main result of [7] hold, then (2.2) admits exponential dichotomy with projection $P = I$, hence system (1.1) has an almost periodic solution. So our main result greatly improves the main result of [7].

4 An Example

For small positive ε_1 and ε_2, we consider the perturbed Van Der Pol equation

$$x^{\Delta\Delta} + (\varepsilon_2 x^2 - 1)x^{\Delta} + x - \varepsilon_1(\sin t\, x_t^2)^{\Delta} - \varepsilon_2 \cos t = 0, \tag{4.1}$$

where x_t is defined by $x_t(\theta) = x(t + \theta)$ for $t, \theta \in \mathbb{T}$ is nonnegative, continuous and almost periodic function. Using the transformation $x_1^{\Delta} = x_2$, we can transform the above equation to

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}^{\Delta} = \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 0 \\ \varepsilon_1 \sin t x_{1t}^2 \end{pmatrix}^{\Delta} + \begin{pmatrix} 0 \\ \varepsilon_2 \cos t - \varepsilon_2 x_2 x_1^2 \end{pmatrix},$$

that is, $A = \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}$, $Q(t, x_t) = \begin{pmatrix} 0 \\ \varepsilon_1 \sin t x_{1t}^2 \end{pmatrix}$, $G(t, x(t), x_t) = \begin{pmatrix} 0 \\ \varepsilon_2 \cos t - \varepsilon_2 x_2 x_1^2 \end{pmatrix}$.

Since the real part of the eigenvalues of A is nonzero, by Remark 2.1, we see that $x^{\Delta}(t) = A(t)x(t)$ admits exponential dichotomy. Let $\phi(t) = (\phi_1(t), \phi_2(t))$ *and* $\varphi(t) = (\varphi_1(t), \varphi_2(t))$. Define $M = \{u \in AP(\mathbb{T}) : \|u\| \le R_0\}$, where R_0 is a positive constant.

Then for $\phi, \varphi \in M$, we have

$$\|Q(\cdot, \phi) - Q(\cdot, \varphi)\| \leq 2\varepsilon_1 R_0 \|\phi - \varphi\|,$$

and

$$\begin{aligned}
&\|G(\cdot), \phi(\cdot), \phi.) - G(\cdot, \varphi(\cdot), \varphi.)\| \\
&\leq \varepsilon_2 \sup_{t \in \mathbb{T}} \left| (\phi_2(t)(\phi_1(t) + \varphi_1(t)), \varphi_1^2(t)) \begin{pmatrix} \phi_1(t) - \varphi_1(t) \\ \phi_2(t) - \varphi_2(t) \end{pmatrix} \right| \\
&\leq 2\varepsilon_2 R_0^2 \|\phi - \varphi\|.
\end{aligned}$$

Hence, let $L_Q = 2\varepsilon_1 R_0$, $L_G = \varepsilon_2 R_0^2$, $a = \|G(t, 0, 0)\| = \varepsilon_2$ and $b = \|Q(t, 0)\| = 0$. Thus, inequality (3.13) becomes

$$2\varepsilon_1 R_0^2 + k\varepsilon_2 (\frac{1}{\alpha} - \frac{1}{\ominus\alpha})(2R_0^3 + 1) \leq R_0,$$

which is satisfied for small ε_1 and ε_2. By Theorem 3.1, (4.1) has an almost periodic solution.

Moreover,

$$2\varepsilon_1 R_0 + 2k\varepsilon_2 R_0^2 (\frac{1}{\alpha} - \frac{1}{\ominus\alpha}) < 1$$

is also satisfied for small ε_1 and ε_2. By Theorem 3.2, (4.1) has a unique almost periodic solution.

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