Bull. Malays. Math. Sci. Soc. (2015) 38(1):259–269
DOI 10.1007/s40840-014-0017-9

BULLETIN *of the* MALAYSIAN MATHEMATICAL SCIENCES SOCIETY
http://math.usm.my/bulletin

ORIGINAL PAPER

Homothetic Motions and Surfaces in $\mathbf{E}^4$

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Received 17 September 2012/ Revised 19 December 2012 / Published online: 16 October 2014

Abstract In this paper, we determine a surface M by means of homothetic motion in $\mathbb{R}^4$ and reparametrize this surface M with bicomplex numbers. Also, by using curves and surfaces which are obtained by homothetic motion, we give some special subgroups of the Lie group P.

Keywords Lie group · Bicomplex number · Surfaces in Euclidean space · Homothetic motion

Mathematics Subject Classification 43A80 · 30G35 · 53A05

1 Introduction

A one-parameter homothetic motion of a rigid body in Euclidean n-space is given analytically by

$$X' = h(t)A(t)X + C(t) \tag{1}$$

in which X' and X are the position vectors of the same point with respect to the rectangular coordinate frames of the fixed space R' and the moving space R, respectively. A is an orthonormal $n \times n$ matrix, C is a translation vector, and h is the homothetic

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Communicated by Rosihan M. Ali, Dato'.

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scale of the motion. Also h, A, and C are continuously differentiable function of a real parameter t. If we take an arbitrary position vector of a curve instead of the point X at one-parameter homothetic motion equation which is given by (1), we obtain a surface.

In mathematics, a Lie group is a group which is also a differentiable manifold with the property that the group operations are differentiable. A manifold M carrying n linearly independent non-vanishing vector fields is called parallelisable and a Lie group is parallelisable. The spheres that admit the structure of a Lie group are the 0-sphere S^0 (real numbers with absolute value 1), the circle S^1 (complex numbers with absolute value 1), the 3-sphere S^3 (the set of quaternions of unit form), and S^7. For even $n > 1$ S^n is not a Lie group because it can not be parallelisable as a differentiable manifold. Thus, S^n is parallelisable if and only $n = 0, 1, 3, 7$.

Özkaldı and Yaylı [7] showed that a hyperquadric P in $\mathbb{R}^4$ is a Lie group by using bicomplex number product. They determined some special subgroups of this Lie group P, by using the tensor product surfaces of Euclidean planar curves.

In this paper, we determine a homothetic motion by using a rotation matrix which is given by Moore [5] and obtain a surface M by means of this homothetic motion in $\mathbb{R}^4$. If we take as the homothetic scale $h(t) = 1$ and the translation vector $C(t) = 0$, we obtain a rotational surface [5,9]. Even if, in special cases, we get some tensor product surfaces by means of this homothetic motion [3,7]. We reparametrize this surface M with bicomplex number product and addition. To establish group structure on the surface is quite difficult. How should we choose the position vector of the curve at homothetic motion given by (2) that the surface M be a Lie subgroup of the hyperquadric P. In this study, we answer this question and by using surface M which is obtained by homothetic motion, we determine some special Lie subgroups of this Lie group P. Furthermore, we mention C^∞ -action of the Lie group P onto the manifold $\mathbb{R}^4$ and define an action of $\mathbb{R}$ on P by using orthonormal matrix at homothetic motion. Also, we determine a Lie subgroup of P with this action and give some results.

2 Preliminaries

Bicomplex number is defined by the basis $\{1, i, j, ij\}$ where i, j, ij satisfy $i^2 = -1$, $j^2 = -1$, $ij = ji$. Thus, any bicomplex number x can be expressed as $x = x_1 1 + x_2 i + x_3 j + x_4 ij$, $\forall x_1, x_2, x_3, x_4 \in \mathbb{R}$. We denote the set of bicomplex numbers by C_2. For any $x = x_1 1 + x_2 i + x_3 j + x_4 ij$ and $y = y_1 1 + y_2 i + y_3 j + y_4 ij$ in C_2 the bicomplex number addition is defined as

$$x + y = (x_1 + y_1) + (x_2 + y_2)\, i + (x_3 + y_3)\, j + (x_4 + y_4)\, ij.$$

The multiplication of a bicomplex number $x = x_1 1 + x_2 i + x_3 j + x_4 ij$ by a real scalar λ is defined as

$$\lambda x = \lambda x_1 1 + \lambda x_2 i + \lambda x_3 j + \lambda x_4 ij.$$

So the set of bicomplex numbers C_2 is a vector space over the real numbers with above the addition and scalar multiplication operations

Let us denote the bicomplex number product over C_2 by $\times$. The bicomplex number product is given by

$$
\begin{aligned}
x \times y = {} & (x_1 y_1 - x_2 y_2 - x_3 y_3 + x_4 y_4) + (x_1 y_2 + x_2 y_1 - x_3 y_4 - x_4 y_3)\, i \\
& + (x_1 y_3 + x_3 y_1 - x_2 y_4 - x_4 y_2)\, j + (x_1 y_4 + x_4 y_1 + x_2 y_3 + x_3 y_2)\, ij.
\end{aligned}
$$

the set of bicomplex numbers C_2 is a real algebra with above the bicomplex number product operation $\times$.

We can consider any bicomplex number as 4×4 real matrix as follows:

$$
Q = \left\{ \begin{pmatrix} x_1 & -x_2 & -x_3 & x_4 \\ x_2 & x_1 & -x_4 & -x_3 \\ x_3 & -x_4 & x_1 & -x_2 \\ x_4 & x_3 & x_2 & x_1 \end{pmatrix}; \qquad x_i \in \mathbb{R}, \qquad 1 \leq i \leq 4 \right\}.
$$

The set Q together with matrix addition and scalar matrix multiplication is a real vector space. Furthermore, the vector space together with matrix product is an algebra [7].

The transformation

$$
g : C_2 \to Q
$$

given by

$$
g\left(x = x_1 1 + x_2 i + x_3 j + x_4 ij\right) = \begin{pmatrix} x_1 & -x_2 & -x_3 & x_4 \\ x_2 & x_1 & -x_4 & -x_3 \\ x_3 & -x_4 & x_1 & -x_2 \\ x_4 & x_3 & x_2 & x_1 \end{pmatrix}
$$

is one to one and onto. Morever $\forall x, y \in C_2$ and $\lambda \in \mathbb{R}$, we have

$$
\begin{aligned}
g(x + y) &= g(x) + g(y) \\
g(\lambda x) &= \lambda g(x) \\
g(xy) &= g(x)\, g(y)\,.
\end{aligned}
$$

Thus the algebras C_2 and Q are isomorphic.

Let $x \in C_2$. Then x can be expressed as $x = (x_1 + x_2 i) + (x_3 + x_4 i)\, j$. In that case, there is three different conjugations for bicomplex numbers as follows:

$$
\begin{aligned}
x^{t_1} &= [(x_1 + x_2.i) + (x_3 + x_4.i)\, j]^{t_1} = (x_1 - x_2.i) + (x_3 - x_4.i)\, j \\
x^{t_2} &= [(x_1 + x_2.i) + (x_3 + x_4.i)\, j]^{t_2} = (x_1 + x_2.i) - (x_3 + x_4.i)\, j \\
x^{t_3} &= [(x_1 + x_2.i) + (x_3 + x_4.i)\, j]^{t_3} = (x_1 - x_2.i) - (x_3 - x_4.i)\, j.
\end{aligned}
$$

And we can write

$$xx^{t_1} = \left(x_1^2 + x_2^2 - x_3^2 - x_4^2\right) + 2\left(x_1x_3 + x_2x_4\right) j$$
$$xx^{t_2} = \left(x_1^2 - x_2^2 + x_3^2 - x_4^2\right) + 2\left(x_1x_2 + x_3x_4\right) i$$
$$xx^{t_3} = \left(x_1^2 + x_2^2 + x_3^2 + x_4^2\right) + 2\left(x_1x_4 - x_2x_3\right) ij.$$

3 Homothetic Motions and Surfaces in E^4

In this section, we define a surface by using the homothetic motion as follows:

$$\varphi(t,s) = h(t)\begin{pmatrix} \cos t & -\sin t & 0 & 0 \\ \sin t & \cos t & 0 & 0 \\ 0 & 0 & \cos t & -\sin t \\ 0 & 0 & \sin t & \cos t \end{pmatrix}\begin{pmatrix} \alpha_1(s) \\ \alpha_2(s) \\ \alpha_3(s) \\ \alpha_4(s) \end{pmatrix} + \begin{pmatrix} C_1(t) \\ C_2(t) \\ C_3(t) \\ C_4(t) \end{pmatrix}, \qquad (2)$$

where $h(t)$ is the homothetic scale of the motion, $C(t) = (C_1(t), C_2(t), C_3(t), C_4(t))$ is the translation vector and $\alpha(s) = (\alpha_1(s), \alpha_2(s), \alpha_3(s), \alpha_4(s))$ is a profile curve.

Now, we can reparametrize this surface by using bicomplex number product and addition.

Proposition 1 *Let $\varphi : M \to E^4$ be an immersion of a surface M in the Euclidean 4-space. If M is a surface in E^4 given by the parametrization (2), then M can be reparametrized by $\varphi(t,s) = \beta(t) \times \alpha(s) + C(t)$, where "$\times$" bicomplex product, "$+$" bicomplex addition, $\beta(t) = (h(t)\cos t, h(t)\sin t, 0, 0)$, $\alpha(s) = (\alpha_1(s), \alpha_2(s), \alpha_3(s), \alpha_4(s))$ are the curves and $C(t) = (C_1(t), C_2(t), C_3(t), C_4(t))$ is the translation vector.*

Proof We can consider the curves β, α and the translation vector C as bicomplex numbers. Then we can rewrite them as follows:

$$\beta(t) = h(t)\cos t + (h(t)\sin t)\, i$$
$$\alpha(s) = \alpha_1(s) + \alpha_2(s)i + \alpha_3(s)j + \alpha_4(s)ij$$
$$C(t) = C_1(t) + C_2(t)\, i + C_3(t)\, j + C_4(t)\, ij.$$

By using the bicomplex product and addition, we obtain $\varphi(t,s) = \beta(t) \times \alpha(s) + C(t)$. □

Corollary 1 *Let $M_{\beta(t)}$ be the matrix representation of bicomplex $\beta(t) = h(t)\cos t + (h(t)\sin t)\, i$. Then we get the surface M given by the parametrization (2) as $\varphi(t,s) = M_{\beta(t)}\alpha(s) + C(t)$.*

The surface M given by the parametrization (2) is reparametrized as bicomplex product of two curves in four dimensional Euclidean space. Now, we can reparametrize this surface M as bicomplex product of a curve and a surface

Corollary 2 *Let $\varphi : M \to E^4$ be an immersion of a surface M in the Euclidean 4-space and M be a surface given by the parametrization (2). Then, the surface M can be reparametrized by $\varphi(t,s) = \gamma(t) \times r(t,s) + C(t)$, where $\gamma(t) = (\cos t, \sin t, 0, 0)$ is a circle, $r(t,s) = h(t)\alpha(s)$ is a surface and $C(t) = (C_1(t), C_2(t), C_3(t), C_4(t))$ is the translation vector.*

Corollary 3 *Let $M_{\gamma(t)}$ be the matrix representation of bicomplex $\gamma(t) = \cos t + (\sin t)\, i$. Then the surface M given by the parametrization (2) can be written as $\varphi(t,s) = M_{\gamma(t)} r(t,s) + C(t)$.*

Remark 1 Let M be a surface in E^4 given by the parametrization (2). In particular, if we take as the homothetic function $h(t) = 1$ and the translation vector $C(t) = 0$, we obtain a rotation surface given by Moore [5].

Remark 2 Let M be a surface in E^4 given by the parametrization (2). In particular, if we take as the homothetic function $h(t) = 1$ and the translation vector $C(t) = 0$ and the profile curve $\alpha(s) = (r(s)\cos s, 0, r(s)\sin s, 0)$, we obtain a rotation surface which is called Vranceanu surface [9].

4 Lie Groups, C^∞ Action of the Lie Groups and Some Special Lie Subgroups

4.1 Lie Groups

In this subsection, by Theorem 1, we mention that the hyperquadric P is a Lie group with bicomplex number product.

$$P = \{x = (x_1, x_2, x_3, x_4) \neq 0; \qquad x_1x_4 = x_2x_3\}.$$

We can consider P as the set of bicomplex numbers as follows:

$$P = \{x = x_1 1 + x_2 i + x_3 j + x_4 ij \;\; x_1x_4 = x_2x_3,\; x \neq 0\}.$$

Also, let the matrix representation of P be given by

$$\tilde{P} = \left\{ M_x = \begin{pmatrix} x_1 & -x_2 & -x_3 & x_4 \\ x_2 & x_1 & -x_4 & -x_3 \\ x_3 & -x_4 & x_1 & -x_2 \\ x_4 & x_3 & x_2 & x_1 \end{pmatrix};\; x_1x_4 = x_2x_3,\; x \neq 0 \right\}.$$

Theorem 1 *([7]) The set of P together with the bicomplex number product is a Lie group*

Proof $\tilde{P}$ is a differentiable manifold and at the same time a group with group operation given by matrix multiplication. The group function

$$. : \tilde{P} \times \tilde{P} \to \tilde{P}$$

defined by $(x, y) \to x.y$ is differentiable. So (P, .) can be made a Lie group so that g is a isomorphism. □

Let us denote the set of all unit bicomplex numbers on P by P_1. We can consider P_1 as follow:

$$P_1 = \left\{x \in P\,; \qquad \|x\|_{t_3} = 1\right\}.$$

And we denote $\tilde{P}_1$ the matrix form of the group P_1

$$\tilde{P}_1 = \left\{x \in \tilde{P}\,;\ \|x\|_{t_3} = 1\right\}.$$

P_1 is a subgroup of P with the group operation of bicomplex multiplication

Lemma 1 *([7]) P_1 is 2-dimensional Lie subgroup of P.*

Remark 3 S^3 is a Lie group with the quaternion multiplication. We can write the set P_1 as $P_1 = P \cap S^3$ and P_1 is a Lie group with bicomplex multiplication. Even though P_1 is a subset of the sphere S^3 and P_1 is a Lie group, P_1 is not a Lie subgroup of S^3.

4.2 $\mathbf{C}^\infty$ Action of the Lie Groups

In this subsection, we mention $C^\infty-$actions of the Lie groups $\tilde{P}$ and $\tilde{P}_1$ onto the manifold $\mathbb{R}^4$ and we define an action of $\mathbb{R}$ on $\tilde{P}$ and $\tilde{P}_1$ Lie groups. We give some special Lie subgroups of P and P_1 by means of the action of $\mathbb{R}$ on $\tilde{P}$ and $\tilde{P}_1$.

Let us consider the mapping

$$\theta : \tilde{P} \times \mathbb{R}^4 \to \mathbb{R}^4$$

for any $A \in \tilde{P}$ and $X \in \mathbb{R}^4$ given by

$$(A, X) \to \theta(A, X) = AX.$$

Theorem 2 *([7]) The mapping θ, defined above, is a $C^\infty-$action of the Lie group $\tilde{P}$ onto the manifold $\mathbb{R}^4$. This action is transitive and effective.*

Theorem 3 *Let $f : \mathbb{R} \to \tilde{P}$ be the mapping which sends every $t \in \mathbb{R}$ to*

$$t \to f(t) = e^{bt} \begin{pmatrix} \cos t & -\sin t & 0 & 0 \\ \sin t & \cos t & 0 & 0 \\ 0 & 0 & \cos t & -\sin t \\ 0 & 0 & \sin t & \cos t \end{pmatrix},$$

and the mapping ψ be given by

$$\psi : \mathbb{R}\times\tilde{P} \to \tilde{P}$$
$$(t, A) \to \psi(t, A) = Af(t).$$

Then the mapping ψ is an action of $\mathbb{R}$ on $\tilde{P}$.

Proof Let the mapping $\psi : \mathbb{R}\times\tilde{P} \rightarrow \tilde{P}$ be given by

$$\psi(t, A) = Af(t)$$

Since f is a homomorphism, it can be easily seen that ψ satisfies

(i) $\psi(0, A) = A$
(ii) $\psi(t_1 + t_2, A) = \psi(t_1, \psi(t_2, A))$

Hence, the mapping ψ is an action of $\mathbb{R}$ on $\tilde{P}$. □

Corollary 4 *The image of the mapping f determines a one-parameter Lie-subgroup of P.*

Proof Since the mapping $f : \mathbb{R} \rightarrow \tilde{P}$ is a homomorphism, homomorphic image $H = f(\mathbb{R})$ is a subgroup of $\tilde{P}$ and since g is a isomorphism, H is a spiral curve as $\alpha(t) = e^{bt}(\cos t, \sin t, 0, 0)$ in P. So, it is a one-parameter Lie-subgroup of P. □

Corollary 5 *Let $\psi : \mathbb{R}\times\tilde{P} \rightarrow \tilde{P}$ be an action of $\mathbb{R}$ on $\tilde{P}$. The infinitesimal generator associated with the mapping ψ is $X_x = (bx_1 - x_2, bx_2 + x_1, bx_3 - x_4, bx_4 + x_3)$ and $\alpha(t) = e^{bt}(\cos t, \sin t, 0, 0)$ is an integral curve of X_x.*

Proof Let $\psi : \mathbb{R}\times\tilde{P} \rightarrow \tilde{P}$ be an action of $\mathbb{R}$ on $\tilde{P}$. The infinitesimal generator at $x \in \tilde{P}$ is given by

$$X_x = \dot{\psi}(0, x) = (bx_1 - x_2, bx_2 + x_1, bx_3 - x_4, bx_4 + x_3),$$

where $\dot{\psi}(0, x) = \left(\frac{\partial\psi}{\partial t}(t, x)\right)_{t=0}$. It can be easily seen that $\alpha(t) = e^{bt}(\cos t, \sin t, 0, 0)$ is an integral curve of X_x. □

Corollary 6 *$\theta_{\tilde{P}_1} : \tilde{P}_1 \times \mathbb{R}^4 \rightarrow \mathbb{R}^4$ defines a C^∞ action of $\tilde{P}_1$ on $\mathbb{R}^4$.*

Proof Since $\tilde{P}_1$ is a Lie-subgroup of $\tilde{P}$ and the inclusion map $i : \tilde{P}_1 \rightarrow \tilde{P}$ is C^∞. The restriction $\theta_{\tilde{P}_1} = \theta \circ i : \tilde{P}_1 \times \mathbb{R}^4 \rightarrow \mathbb{R}^4$ defines a C^∞ action of $\tilde{P}_1$ on $\mathbb{R}^4$. □

Theorem 4 *Let $f_1 : \mathbb{R} \rightarrow \tilde{P}_1$ be the mapping which sends every $t \in \mathbb{R}$ to*

$$t \rightarrow f_1(t) = \begin{pmatrix} \cos t & -\sin t & 0 & 0 \\ \sin t & \cos t & 0 & 0 \\ 0 & 0 & \cos t & -\sin t \\ 0 & 0 & \sin t & \cos t \end{pmatrix},$$

and the mapping ψ_1 be given by

$$\begin{aligned} \psi_1 &: \mathbb{R}\times\tilde{P}_1 \rightarrow \tilde{P}_1 \\ (t, A) &\rightarrow \psi_1(t, A) = Af_1(t). \end{aligned}$$

Then, the mapping ψ_1 is an action of $\mathbb{R}$ on $\tilde{P}_1$.

Proof Since f_1 is a homomorphism, it can be easily seen that ψ_1 satisfies

(i) $\psi_1(0, A) = A$
(ii) $\psi_1(t_1 + t_2, A) = \psi_1(t_1, \psi_1(t_2, A))$

Hence, the mapping ψ_1 is an action of $\mathbb{R}$ on $\tilde{P}_1$. □

Corollary 7 *The image of the mapping f_1 determines a one-parameter Lie-subgroup of P_1.*

Proof Since the mapping $f_1 : \mathbb{R} \to \tilde{P}_1$ is a homomorphism, homomorphic image $H_1 = f_1(\mathbb{R})$ is a subgroup of $\tilde{P}_1$ and since g is a isomorphism, H_1 is a circle as $\alpha(t) = (\cos t, \sin t, 0, 0)$ in P_1. So, it is a one-parameter Lie-subgroup of P_1. □

Corollary 8 *Let $\psi_1 : \mathbb{R} \times \tilde{P}_1 \to \tilde{P}_1$ be an action of $\mathbb{R}$ on $\tilde{P}_1$. The infinitesimal generator associated with the mapping ψ_1 is $X_x = (-x_2, x_1, -x_4, x_3)$ and $\alpha(t) = (\cos t, \sin t, 0, 0)$ is an integral curve of X_x.*

Proof Let $\psi_1 : \mathbb{R} \times \tilde{P}_1 \to \tilde{P}_1$ be an action of $\mathbb{R}$ on $\tilde{P}_1$. The infinitesimal generator at $x \in \tilde{P}_1$ is given by

$$X_x = \dot{\psi}_1(0, x) = (-x_2, x_1, -x_4, x_3),$$

where $\dot{\psi}_1(0, x) = \left(\frac{\partial \psi_1}{\partial t}(t, x)\right)_{t=0}$. It can be easily seen that $\alpha(t) = (\cos t, \sin t, 0, 0)$ is an integral curve of X_x. □

Corollary 9 *f_1 induces an action on $S^3 = \left\{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4;\ x_1^2 + x_2^2 + x_3^2 + x_4^2 = 1\right\}$.*

Proof

$$\bar{\psi}(t, x_1, x_2, x_3, x_4) = \begin{pmatrix} x_1 \cos t - x_2 \sin t, x_1 \sin t + x_2 \cos t, \\ x_3 \cos t - x_4 \sin t, x_3 \sin t + x_4 \cos t \end{pmatrix}.$$

It is obvious that $\bar{\psi}$ is an action on S^3. □

4.3 Some Special Lie Subgroups

Özkaldı and Yaylı showed that a hyperquadric P in $\mathbb{R}^4$ is a Lie group using bicomplex number product. Also, they determined some special subgroups of Lie group P, using the tensor product surfaces of Euclidean planar curves [7].

Our aim in this subsection is to determine some special subgroups of this Lie group P using the surface M which is obtained with homothetic motion. In this case, how should we choose the position vector of the curve at homothetic motion given by (2) that the surface M be a Lie subgroup of the hyperquadric P. We answer this question. If we take the profile curve $\alpha(s) = (\alpha_1(s), \alpha_2(s), \alpha_3(s), \alpha_4(s))$ such that $\alpha_1(s)\alpha_4(s) = \alpha_2(s)\alpha_3(s)$ and the translation vector $C(t) = 0$, then the surface M is given by the parametrization (2) is subset of P.

Theorem 5 *Let γ be a curve which is obtained using the homothetic motion with the homothetic function $h(t) = e^{at}$ and the profile curve $\alpha(t) = e^{bt}(\cos t, \sin t, 0, 0)$ where a, b are real constants. Then curve γ is a one-parameter subgroup in a Lie group P.*

Proof We can write the curve γ as follows:

$$\gamma(t) = e^{at}\begin{pmatrix} \cos t & -\sin t & 0 & 0 \\ \sin t & \cos t & 0 & 0 \\ 0 & 0 & \cos t & -\sin t \\ 0 & 0 & \sin t & \cos t \end{pmatrix}\begin{pmatrix} e^{bt}\cos t \\ e^{bt}\sin t \\ 0 \\ 0 \end{pmatrix}$$
$$= e^{(a+b)t}(\cos 2t, \sin 2t, 0, 0).$$

It can be easily seen that

$$\gamma(t_1) \times \gamma(t_2) = \gamma(t_1 + t_2)$$

for all $t_1, t_2 \in \mathbb{R}$. Hence, $(\gamma(t), \times)$ is one-parameter Lie subgroup of $(P, \times)$. □

Remark 4 From Corollary (4), we know that $\alpha(t) = e^{bt}(\cos t, \sin t, 0, 0)$ is a one-parameter Lie subgroup of P. In Theorem (5), we show that the trajector of the curve α under the homothetic motion is a one-parameter Lie subgroup of P too.

Theorem 6 *Let γ be a curve which is obtained using the homothetic motion with the homothetic function $h(t) = e^{at}$ and the profile curve $\alpha(t) = e^{bt}(\cos t, 0, \sin t, 0)$ where a, b are real constants. Then, the curve γ is a one-parameter Lie-subgroup in Lie group P.*

Proof We can write the curve γ as follows:

$$\gamma(t) = e^{at}\begin{pmatrix} \cos t & -\sin t & 0 & 0 \\ \sin t & \cos t & 0 & 0 \\ 0 & 0 & \cos t & -\sin t \\ 0 & 0 & \sin t & \cos t \end{pmatrix}\begin{pmatrix} e^{bt}\cos t \\ 0 \\ e^{bt}\sin t \\ 0 \end{pmatrix}$$
$$= e^{(a+b)t}\left(\cos^2 t, \sin t\cos t, \sin t\cos t, \sin^2 t\right).$$

It can be easily seen that

$$\gamma(t_1) \times \gamma(t_2) = \gamma(t_1 + t_2)$$

for all $t_1, t_2 \in \mathbb{R}$. Hence $(\gamma(t), \times)$ is one-parameter Lie subgroup of $(P, \times)$. □

Remark 5 The above curve γ can be expressed as tensor product of two spirals with the same parameter, that is, let $\beta : \mathbb{R} \to E^2$, $\beta(t) = e^{at}(\cos t, \sin t)$ and $\delta(t) = e^{bt}(\cos t, \sin t)$ be two spirals. Then the curve γ can be written as $\gamma(t) = \beta(t) \otimes \delta(t)$.

Corollary 10 *Let γ be a curve which is obtained using the homothetic motion with the homothetic function $h(t) = 1$ and the profile curve $\alpha(t) = (\cos t, \sin t, 0, 0)$. Then, the curve γ is a one-parameter Lie-subgroup in Lie group P_1.*

Proof For $h(t) = 1$ and the profile curve $\alpha(t) = (\cos t, \sin t, 0, 0)$, we get

$$\gamma(t) = (\cos 2t, \sin 2t, 0, 0)$$

Since $\|\gamma(t)\|_{t_3} = 1$, it follows that $\gamma(t) \subset P_1$. So it is a one-parameter Lie subgroup in Lie group P_1. □

Corollary 11 *Let γ be a curve which is obtained using the homothetic motion with the homothetic function $h(t) = 1$ and the profile curve $\alpha(t) = (\cos t, 0, \sin t, 0)$. Then the curve γ is a one-parameter Lie subgroup in Lie group P_1.*

Proof For $h(t) = 1$ and the profile curve $\alpha(t) = (\cos t, 0, \sin t, 0)$, we get

$$\gamma(t) = \left(\cos^2 t, \sin t \cos t, \sin t \cos t, \sin^2 t\right)$$

Since $\|\gamma(t)\|_{t_3} = 1$, it follows that $\gamma(t) \subset P_1$. So, it is a one-parameter Lie-subgroup in Lie group P_1. □

Remark 6 The above curve γ can be expressed as tensor product of two circles with the same parameter, that is, let $\beta : \mathbb{R} \to E^2$, $\beta(t) = (\cos t, \sin t)$ and $\delta(t) = (\cos t, \sin t)$ be circles. Then, the curve γ can be written as $\gamma(t) = \beta(t) \otimes \delta(t)$.

Theorem 7 *Let M be a surface which is obtained using the homothetic motion with the homothetic function $h(t) = e^{at}$ and the profile curve $\alpha(s) = e^{bs}(\cos s, 0, \sin s, 0)$. Then, the surface M is a 2-dimensional Lie-subgroup of P.*

Proof

$$\begin{aligned}\varphi(t,s) &= e^{at}\begin{pmatrix}\cos t & -\sin t & 0 & 0\\ \sin t & \cos t & 0 & 0\\ 0 & 0 & \cos t & -\sin t\\ 0 & 0 & \sin t & \cos t\end{pmatrix}\begin{pmatrix}e^{bs}\cos s\\ 0\\ e^{bs}\sin s\\ 0\end{pmatrix}\\ &= e^{at+bs}(\cos s\cos t, \cos s\sin t, \sin s\cos t, \sin s\sin t).\end{aligned}$$

Since $\varphi(t,s)$ is both a subgroup and submanifold of Lie group P, we obtain that $\varphi(t,s)$ is a 2-dimensional Lie subgroup of P. □

Remark 7 The above surface M can be expressed as tensor product surface of two spirals, that is, let $\beta : \mathbb{R} \to E^2$, $\beta(s) = e^{as}(\cos s, \sin s)$ and $\delta(t) = e^{bt}(\cos t, \sin t)$ be two spirals. Then the surface M can be written as $\varphi(t,s) = \beta(s) \otimes \delta(t)$.

Corollary 12 *Let M be a Vranceanu surface with the profile curve $\alpha(s) = e^{bs}(\cos s, 0, \sin s, 0)$. Then the surface M is a 2-dimensional Lie subgroup of P.*

Proof If we take as $a = 0$ in Theorem (7), we obtain a rotation surface which is called Vranceanu surface in E^4. Then, Vranceanu surface is a 2-dimensional Lie subgroup of P. □

Corollary 13 *Clifford torus is a 2-dimensional Lie-subgroup of* P_1.

Proof By using the homothetic function $h(t) = 1$ and the profile curve $\alpha(s) = (\cos s, 0, \sin s, 0)$, we obtain a rotation surface which is called Clifford Torus. This surface is product of two plane circles with the same radius, that is,

$$\varphi(t, s) = (\cos s \cos t, \cos s \sin t, \sin s \cos t, \sin s \sin t).$$

Since $\|\varphi(t, s)\|_{t_3} = 1$, $\varphi(t, s)$ is subset of P_1. Hence, Clifford Torus is a 2-dimensional Lie subgroup of P_1. □

Remark 8 The Clifford Torus is a subset of S^3 and it is a Lie group with bicomplex number product, but it is not a Lie subgroup of S^3. Also, since the Clifford Torus is a Lie group, it is parallelisable.

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