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On Starlike Functions Connected with k-Fibonacci Numbers

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Abstract We present a new subclass $\mathcal{SL}^k$ of starlike functions which is related to a shell-like curve. The coefficients of such functions are connected with k-Fibonacci numbers $F_{k,n}$ defined recurrently by $F_{k,0} = 0$, $F_{k,1} = 1$ and $F_{k,n} = kF_{k,n} + F_{k,n-1}$ for $n \geq 1$, where k is a given positive real number. We investigate some basic properties for the class $\mathcal{SL}^k$.

Keywords Univalent function · Starlike function · Subordination · k-Fibonacci number

Mathematics Subject Classification 30C45

1 Introduction

Let $\mathbb{D} = \{z : |z| < 1\}$ denote the unit disc. Let $\mathcal{A}$ be the class of all analytic functions f in the open unit disc $\mathbb{D}$ with normalization $f(0) = 0$, $f'(0) = 1$ and let $\mathcal{S}$ denote the subset of $\mathcal{A}$ which is composed of univalent functions. We say that f is subordinate to F in $\mathbb{D}$, written as $f \prec F$, if and only if $f(z) = F(\omega(z))$ for some analytic function ω, $\omega(0) = 0$, $|\omega(z)| < 1$, $z \in \mathbb{D}$. The idea of subordination was used for defining many classes of functions studied in geometric function theory. Let us recall

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$$\mathcal{S}^*[\varphi] := \left\{ f \in \mathcal{A} : \frac{zf'(z)}{f(z)} \prec \varphi(z),\ z \in \mathbb{D} \right\}, \tag{1.1}$$

$$\mathcal{K}[\varphi] := \left\{ f \in \mathcal{A} : \left[1 + \frac{zf''(z)}{f'(z)} \right] \prec \varphi(z),\ z \in \mathbb{D} \right\}, \tag{1.2}$$

where φ is analytic in $\mathbb{D}$ with $\varphi(0) = 1$. For $\varphi(z) = (1+z)/(1-z)$ we obtain the well-known classes $\mathcal{S}^*$, $\mathcal{K}$ of starlike and convex functions, respectively. A lot of classes of functions have been defined by exchanging the function φ in (1.1) or in (1.2) by other functions giving very often an interesting image of the unit circle. If $\varphi(z) = (1 + (1-2\alpha)z)/(1-z)$, $\alpha < 1$, then $\varphi(\mathbb{D})$ is the half plane $\mathfrak{Re}(w) > \alpha$, and the sets (1.1), (1.2) become the classes $\mathcal{S}^*(\alpha)$ of starlike or $\mathcal{K}(\alpha)$ of convex functions of order α, respectively, introduced in [14]. If $\varphi(z) = (1 + Az)(1 + Bz)$, $-1 < B < A \le 1$, then $\varphi(\mathbb{D})$ is a disc, and the classes (1.1), (1.2) become the classes considered in [6,7]. The class of strongly starlike functions of order β, $0 < \beta \le 1$, see [20] is obtained from (1.1) with $\varphi(z) = ((1+z)/(1-z))^{\beta}$. Then $\varphi(\mathbb{D})$ is an angle. If

$$\varphi(z) = 1 + \frac{2}{\pi^2} \left(\log \frac{1+\sqrt{z}}{1-\sqrt{z}} \right)^2,$$

then $\varphi(\mathbb{D})$ is a parabolic region, and the set (1.2) is a class of the so-called uniformly convex function introduced in [5,11,15]. Close related classes, connected with a hyperbola or with an ellipse were considered in [8–10]. If $\varphi(z) = \sqrt{1+z}$, where the branch of the square root is chosen in order that $\sqrt{1} = 1$, then $\varphi(\mathbb{D})$ is interior of the right loop of the Lemniscate of Bernoulli and the class (1.1) becomes a class considered in [17,19]. The function

$$\varphi(z) = \left(\frac{1+z}{1 + (1-b)/bz} \right)^{1/\alpha}$$

in (1.1) forms a class considered in [13]. In the above and in other not cited here cases the function φ is a convex univalent function. In [12] Ma and Minda proved some general results for classes (1.1) and (1.2), where φ is assumed to be univalent, $\varphi(\mathbb{D})$ is assumed to be symmetric with respect to real axis and starlike with respect to $\varphi(0) = 1$. The problems in the classes defined by (1.1) and by (1.2) become much more difficult if the function φ is not univalent. In [18] the second author defined the class $\mathcal{SL}$ of shell-like functions as the set of functions $f \in \mathcal{A}$ satisfying the condition that

$$\frac{zf'(z)}{f(z)} \prec \widetilde{p}(z), \quad z \in \mathbb{D},$$

where

$$\widetilde{p}(z) = \frac{1 + \tau^2 z^2}{1 - \tau z - \tau^2 z^2},\ \tau = \frac{1-\sqrt{5}}{2} \approx -0.618,\ z \in \mathbb{D}.$$

The class $\mathcal{SL}$ is a subclass of the class of starlike functions $\mathcal{S}^\star$. The name attributed to the class $\mathcal{SL}$ is motivated by the shape of the curve

$$\mathcal{C} = \left\{ \widetilde{p}(e^{it}) : t \in [0, 2\pi) \setminus \{\pi\} \right\},$$

which is a shell-like curve. Furthermore, the coefficients of shell-like functions are connected with well-known Fibonacci numbers F_n defined as

$$F_0 = 0,\ F_1 = 1 \quad \text{and} \quad F_{n+1} = F_n + F_{n-1} \quad \text{for} \quad n \geq 1. \tag{1.3}$$

More recently, a lot of new studies have been done about several classes of functions related to shell-like curves connected with Fibonacci numbers (see [1,2] and [16]).

Motivated by the above studies, we define new subclasses $\mathcal{SL}^k$ of the class $\mathcal{S}^\star$, where k is any positive real number. The coefficients of such functions are connected with k-Fibonacci numbers. For $k = 1$, we obtain the class $\mathcal{SL}$ of shell-like functions.

For any positive real number k, the k-Fibonacci numbers $F_{k,n}$ are defined recurrently by

$$F_{k,0} = 0,\ F_{k,1} = 1 \quad \text{and} \quad F_{k+1,n} = kF_{k,n} + F_{k,n-1} \quad \text{for} \quad n \geq 1. \tag{1.4}$$

The Fibonacci numbers defined in (1.3) are obtained from (1.4) for $k = 1$. It is known that the nth k-Fibonacci number is given by

$$F_{k,n} = \frac{(k - \tau_k)^n - \tau_k^n}{\sqrt{k^2 + 4}}, \tag{1.5}$$

where $\tau_k = \frac{k-\sqrt{k^2+4}}{2}$ (see [3] and [4] for more details about k-Fibonacci numbers).

2 The Class $\mathcal{SL}^k$

Definition 2.1 Let k be any positive real number. The function $f \in \mathcal{S}$ belongs to the class $\mathcal{SL}^k$ if satisfies the condition that

$$\frac{zf'(z)}{f(z)} \prec \widetilde{p}_k(z),\ z \in \mathbb{D},$$

where

$$\widetilde{p}_k(z) = \frac{1 + \tau_k^2 z^2}{1 - k\tau_k z - \tau_k^2 z^2} = \frac{1 + \tau_k^2 z^2}{1 - (\tau_k^2 - 1)z - \tau_k^2 z^2},\ \tau_k = \frac{k - \sqrt{k^2 + 4}}{2},\ z \in \mathbb{D}. \tag{2.1}$$

Theorem 2.1 *The image of the unit circle of the function* $\widetilde{p}_k(z)$ *defined in* (2.1) *is the curve* $\mathcal{C}_k$ *with equation*

$$x = \frac{k\sqrt{k^2+4}}{2\left[k^2+2-2\cos\theta\right]}, \quad y = \frac{\left(4\cos\theta - k^2\right)\sin\theta}{2\left[k^2+2-2\cos\theta\right]\left[1+\cos\theta\right]}, \quad \theta \in [0, 2\pi)\setminus\{\pi\}. \tag{2.2}$$

Proof The proof follows by some straightforward calculations. □

Recall that the curve which is called conchoid of Sluze has the following equation

$$a(x-a)\left(x^2+y^2\right) + \lambda^2 x^2 = 0, \tag{2.3}$$

where $a > 0$ and $\lambda > 0$. For $\lambda = 2a/k$, the conchoid of Sluze (2.3) becomes the curve

$$x^3 + (x-a)y^2 + \left(\frac{4-k^2}{k^2}\right)ax^2 = 0. \tag{2.4}$$

For $k = 1$, this curve is the trisectrix of Maclaurin.

We can find the corresponding Cartesian equation of the curve $\mathcal{C}_k$ with Eq. (2.2) as

$$\left[(8+2k^2)x - k\sqrt{k^2+4}\right]y^2 = \left(\frac{\sqrt{k^2+4}}{k} - 2x\right)\left(\sqrt{k^2+4}x - k\right)^2. \tag{2.5}$$

If we rewrite (2.5) in the following form

$$\left(\frac{k\sqrt{k^2+4}}{k^2+4} - x\right)^3 + \frac{4-k^2}{k^2}\cdot\frac{k\sqrt{k^2+4}}{2(k^2+4)}\left(\frac{k\sqrt{k^2+4}}{k^2+4} - x\right)^2$$
$$+\left[\left(\frac{k\sqrt{k^2+4}}{k^2+4} - x\right) - \frac{k\sqrt{k^2+4}}{2(k^2+4)}\right]y^2 = 0,$$

then the image of the unit circle under the function $\widetilde{p}_k$ is translated into a curve with Eq. (2.4), where

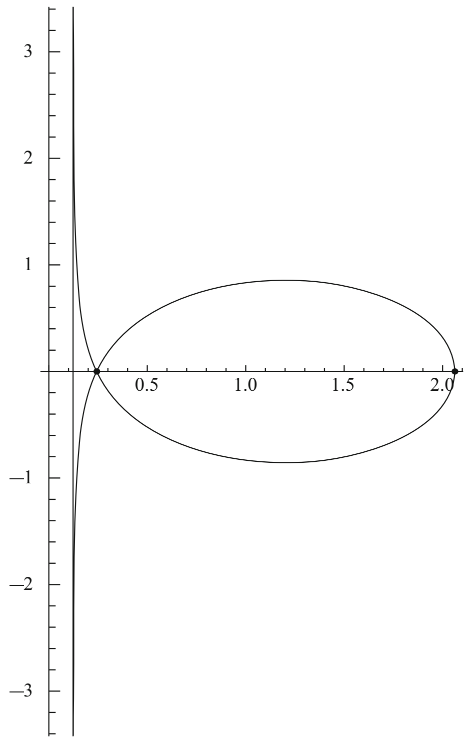
$$a = \frac{k\sqrt{k^2+4}}{2(k^2+4)} = \frac{1 - \frac{2\tau_k\left(1-k\sqrt{k^2+4}\right)}{k-\sqrt{k^2+4}}}{2(k^2+4)}.$$

Therefore, the curve $\mathcal{C}_k$ has a shell-like shape and symmetric with respect to the real axis, see Fig. 1.

For $k < 2$, note that we have

$$\widetilde{p}\left(e^{\pm i\arccos\left(\frac{k^2}{4}\right)}\right) = \frac{k\sqrt{k^2+4}}{k^2+4},$$

Fig. 1 The curve $\mathcal{C}_k$ for $k = \frac{1}{2}$



and so the curve $\mathcal{C}_k$ intersects itself on the real axis at the point $\frac{k\sqrt{k^2+4}}{k^2+4}$. Thus, $\mathcal{C}_k$ has a loop intersecting the real axis at the points $e = \frac{k\sqrt{k^2+4}}{k^2+4}$ and $f = \frac{\sqrt{k^2+4}}{2k}$. For $k \geq 2$, the curve $\mathcal{C}_k$ has no loops and it is like a conchoid.

Corollary 2.1 *For each $k > 0$, $\mathcal{SL}^k \subset \mathcal{S}^*(\alpha_k)$, where $\alpha_k = \frac{k\sqrt{k^2+4}}{2(k^2+4)} = \frac{k(k-2\tau_k)}{2(k^2+4)}$, that is, $f \in \mathcal{SL}^k$ is starlike of order α_k.*

The function $\widetilde{p}_k$ defined in (2.1) is not univalent in $\mathbb{D}$. For example, we have $\widetilde{p}_k(0) = \widetilde{p}(\frac{-k}{2\tau_k}) = 1$ and $\widetilde{p}(1) = \widetilde{p}(\tau_k^4) = \frac{\sqrt{k^2+4}}{2k}$. We can give the following theorem.

Theorem 2.2 *For each $k > 0$, the function $\widetilde{p}_k$ is univalent in the disc $\mathbb{D}_{r_k} = \{z : |z| < r_k\}$, where*

$$r_k = \frac{2-\sqrt{k^2+4}}{k\tau_k} = \frac{k^2-2k+4+(k-2)\sqrt{k^2+4}}{2k} \tag{2.6}$$

and it is not univalent in the disc $\mathbb{D}_{r_k}$ for each $r \geq r_k$.

Proof Suppose that $\widetilde{p}_k(z) = \widetilde{p}_k(w)$ for some $z, w \in \mathbb{D}$. After some calculations we have

$$\tau_k\,(z - w)\left(w - \frac{2\tau_k z + k}{k\tau_k^2 z - 2\tau_k}\right) = 0. \tag{2.7}$$

We see that the function

$$g_k(z) = \frac{2\tau_k z + k}{k\tau_k^2 z - 2\tau_k}$$

maps a circle $|z| = r < 2/(k\tau_k)$ onto a circle centred at $m = -\frac{2k(1+\tau_k^2 r^2)}{\tau_k(4-k^2\tau_k^2 r^2)}$ and of radius $\rho = \frac{r(k^2+4)}{4-k^2\tau_k^2 r^2}$ with the diameter from $g_k(-r)$ to $g_k(r)$. Therefore, g_k maps the circle $|z| = r_k$ onto a circle with the diameter from the point $g_k(r_k) = r_k$ to the point $g_k(-r_k)$. We have $g_k(-r_k) > g_k(r_k) = r_k$ for all k because the function $g_k(x)$, $x \in \mathbb{R}$ has negative derivative for all real x. Therefore, if $|w| \le r_k$ and $|z| \le r_k$, then the third factor in (2.7) is equal to 0 for $w = z - r_k$ only. Consequently, we see that (2.7) is not satisfied when $|w| < r_k$ and $|z| < r_k$, which proves that in the disc (2.6) the function $\widetilde{p}_k(z)$ is univalent.

On the other hand, the derivative of the function $\widetilde{p}_k(z)$ is

$$\widetilde{p}_k'(z) = \frac{(z - r_k)\left(z - \frac{2+\sqrt{k^2+4}}{k\tau_k}\right)}{\left(1 - k\tau_k z - \tau_k^2 z^2\right)^2}.$$

The function $\widetilde{p}_k'(z)$ vanishes at the point $z = r_k$ and hence we see that the function $\widetilde{p}_k(z)$ fails to be univalent for $|z| \ge r_k$. □

Theorem 2.3 *Let $(F_{k,n})$ be the sequence of k-Fibonacci numbers defined in* (1.4). *If*

$$\widetilde{p}_k(z) = \frac{1 + \tau_k^2 z^2}{1 - k\tau_k z - \tau_k^2 z^2} = 1 + \sum_{n=1}^{\infty} p_n z^n,$$

then we have

$$p_n = (F_{k,n-1} + F_{k,n+1})\tau_k^n, \quad n = 1, 2, 3, \ldots . \tag{2.8}$$

Proof Let us denote $u = \tau_k z$, $|u| < |\tau_k|$. Using the equations $\tau_k (k - \tau_k) = -1$ and $2\tau_k - k = -\sqrt{k^2 + 4}$, we have

$$\begin{aligned}\widetilde{p}_k(z) &= \frac{1 + \tau_k^2 z^2}{1 - k\tau_k z - \tau_k^2 z^2} = \frac{1 + u^2}{1 - ku - u^2} = \left(u + \frac{1}{u}\right) \frac{u}{1 - ku - u^2} \\ &= \left(u + \frac{1}{u}\right) \frac{1}{\sqrt{k^2 + 4}} \left(\frac{1}{1 + \frac{u}{\tau_k}} - \frac{1}{1 + \frac{u}{k - \tau_k}}\right) \\ &= \left(u + \frac{1}{u}\right) \frac{1}{\sqrt{k^2 + 4}} \sum_{n=1}^{\infty} (-1)^n \left[\left(\frac{u}{\tau_k}\right)^n - \left(\frac{u}{k - \tau_k}\right)^n\right] \\ &= \left(u + \frac{1}{u}\right) \sum_{n=1}^{\infty} \frac{(k - \tau_k)^n - \tau_k^n}{\sqrt{k^2 + 4}} u^n.\end{aligned}$$

Now by the Eq. (1.5), we find

$$\begin{aligned}\widetilde{p}_k(z) &= \left(u + \frac{1}{u}\right) \sum_{n=1}^{\infty} F_{k,n} u^n \\ &= 1 + \sum_{n=1}^{\infty} \left(F_{k,n-1} + F_{k,n+1}\right) u^n \\ &= 1 + \sum_{n=1}^{\infty} \left(F_{k,n-1} + F_{k,n+1}\right) \tau_k^n z^n,\end{aligned}$$

and hence we obtain (2.8). □

Theorem 2.4 *A function $f \in \mathcal{S}$ belongs to the class $\mathcal{SL}^k$ if and only if there exists a function q, $q \prec \widetilde{p}_k(z) = \frac{1+\tau_k^2 z^2}{1-k\tau_k z-\tau_k^2 z^2}$ such that*

$$f(z) = z \exp \int_0^z \frac{q(\zeta) - 1}{\zeta} d\zeta, \quad z \in \mathbb{D}. \tag{2.9}$$

Proof Let $f \in \mathcal{SL}^k$. Then by definition

$$\frac{zf'(z)}{f(z)} = \widetilde{p}_k(\omega(z)), \ |\omega(z)| < 1, \quad z \in \mathbb{D}. \tag{2.10}$$

If we take $q(z) = \widetilde{p}(\omega(z))$, we see that the Eq. (2.10) is equivalent to (2.9). □

For $\widetilde{p}_k(z) = \frac{1+\tau_k^2 z^2}{1-k\tau_k z-\tau_k^2 z^2}$, the formula (2.9) gives $f_0(z) = \frac{z}{1-k\tau_k z-\tau_k^2 z^2}$. Hence the function f_0 belongs to the class $\mathcal{SL}^k$ and it is extremal function for several problems in this class.

Theorem 2.5 *If $f(z)=z+\sum_{n=2}^{\infty}a_nz^n$ belongs to the class $\mathcal{SL}^k$, then we have*

$$|a_n|\le|\tau_k|^{n-1}F_{k,n},\tag{2.11}$$

where $\left(F_{k,n}\right)$ is the sequence of k-Fibonacci numbers and $\tau_k=\frac{k-\sqrt{k^2+4}}{2}$. Equality holds in (2.11) for the function $f_0(z)=\frac{z}{1-k\tau_kz-\tau_k^2z^2}$.

Proof Let $f\in\mathcal{SL}^k$, $f(z)=\sum_{m=0}^{\infty}a_mz^m$, $a_0=0$, $a_1=1$. By the definition of the class $\mathcal{SL}^k$, there exists a function ω, $|\omega(z)|<1$ for $z\in\mathbb{D}$ such that

$$\frac{zf'(z)}{f(z)}=\frac{1+\tau_k^2\omega^2(z)}{1-k\tau_k\omega(z)-\tau_k^2\omega^2(z)}.$$

We get

$$zf'(z)-f(z)=k\tau_k\omega(z)zf'(z)+\tau_k^2\omega^2(z)\left[zf'(z)+f(z)\right],$$
$$\sum_{m=1}^{\infty}(m-1)a_mz^m=k\tau_k\omega(z)\sum_{m=1}^{\infty}ma_mz^m+\tau_k^2\omega^2(z)\sum_{m=1}^{\infty}(m+1)a_mz^m$$

and so

$$\sum_{m=1}^{n}(m-1)a_mz^m+\sum_{m=n+1}^{\infty}c_mz^m=k\tau_k\omega(z)\sum_{m=1}^{n-1}ma_mz^m+\tau_k^2\omega^2(z)\sum_{m=1}^{n-2}(m+1)a_mz^m.$$

For $n\ge2$, we find

$$\begin{aligned}&\left|\sum_{m=1}^{n}(m-1)a_mz^m+\sum_{m=n+1}^{\infty}c_mz^m\right|^2\\
&=\left|k\tau_k\omega(z)\sum_{m=1}^{n-1}ma_mz^m+\tau_k^2\omega^2(z)\sum_{m=1}^{n-2}(m+1)a_mz^m\right|^2\\
&\le\left|k\tau_k\sum_{m=1}^{n-1}ma_mz^m+\tau_k^2\omega(z)\sum_{m=1}^{n-1}ma_{m-1}z^{m-1}\right|^2\\
&\le\sum_{m=1}^{n-1}\left|k\tau_kma_mz^m+\tau_k^2\omega(z)ma_{m-1}z^{m-1}\right|^2\\
&\le\sum_{m=1}^{n-1}\left(\left|k\tau_kma_mz^m\right|^2+\left|\tau_k^2ma_{m-1}z^{m-1}\right|^2+2\left|k\tau_k^3m^2a_ma_{m-1}z^{2m-1}\right|\right).\end{aligned}$$

Integrating the both sides of this inequality around $z = re^{im\varphi}$ and taking limit $r \to 1^-$ we obtain

$$\sum_{m=1}^{n}(m-1)^2|a_m|^2 + \sum_{m=n+1}^{\infty}|c_m|^2 \le k^2\tau_k^2\sum_{m=1}^{n-1}m^2|a_m|^2 + \tau_k^4\sum_{m=1}^{n-1}m^2|a_{m-1}|^2 + 2k|\tau_k|^3\sum_{m=1}^{n-1}m^2|a_m||a_{m-1}|$$

and hence we find

$$(n-1)^2|a_n|^2 \le \sum_{m=1}^{n-1}\left\{k^2\tau_k^2m^2-(m-1)^2\right\}|a_m|^2 + \sum_{m=1}^{n-1}\tau_k^4m^2|a_{m-1}|^2 + \sum_{m=1}^{n-1}2k|\tau_k|^3m^2|a_m||a_{m-1}| \tag{2.12}$$

The inequality (2.11) holds for $n = 1$. Assume that the estimation (2.11) holds for all natural numbers less or equal to n. Then from (2.12) and from (2.11) we have

$$\begin{aligned} & n^2|a_{n+1}|^2 \\ & \le \sum_{m=1}^{n}\left\{k^2\tau_k^2m^2-(m-1)^2\right\}|a_m|^2+\tau_k^4\sum_{m=1}^{n}m^2|a_{m-1}|^2+2k|\tau_k|^3\sum_{m=1}^{n}m^2|a_m||a_{m-1}| \\ & \le \sum_{m=1}^{n}\left\{k^2\tau_k^2m^2-(m-1)^2\right\}\left\{|\tau_k|^{m-1}F_{k,m}\right\}^2 + \tau_k^4\sum_{m=1}^{n}m^2\left\{|\tau_k|^{m-2}F_{k,m-1}\right\}^2 \\ & +2k|\tau_k|^3\sum_{m=1}^{n}m^2\left\{|\tau_k|^{m-1}F_{k,m}\right\}\left\{|\tau_k|^{m-2}F_{k,m-1}\right\} \\ & = \sum_{m=1}^{n}\left[\left\{m\tau_k^m\left(kF_{k,m}+F_{k,m-1}\right)\right\}^2-(m-1)^2\left\{|\tau_k|^{m-1}F_{k,m}\right\}^2\right] \\ & = \sum_{m=1}^{n}\left[\left\{m\tau_k^mF_{k,m+1}\right\}^2-(m-1)^2\left\{|\tau_k|^{m-1}F_{k,m}\right\}^2\right] \\ & = n^2|\tau_k|^{2n}\left\{F_{k,n+1}\right\}^2. \end{aligned} \tag{2.13}$$

In this way we have proved by induction the inequality (2.11) for all $n \in \mathbb{N}$. □

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